Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

Dr. Thiruvalloor Eesamaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology Madras

Lecture No 39

Characterizing Non-Normality at a Point by the Zooming Process and the Proof of Zalcman's Lemma

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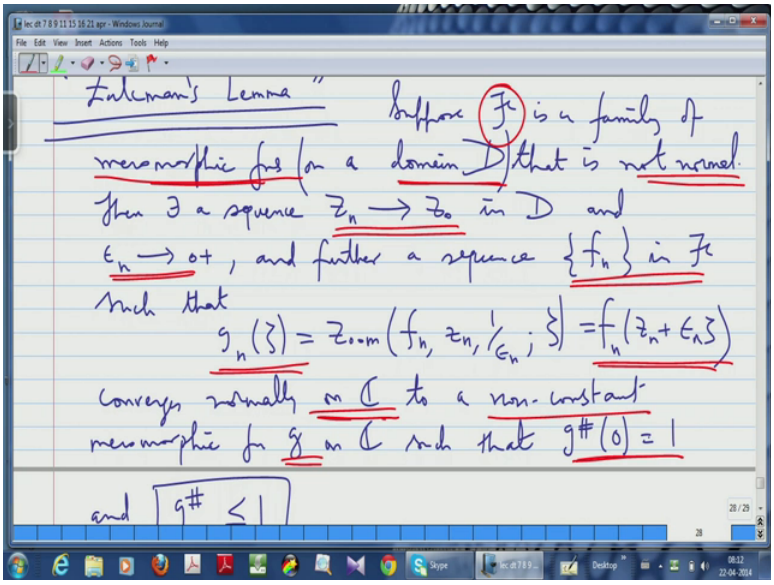

Alright so let us continue with the proof of Zalcman's Lemma, so basically what this Lemma is about, it is about characterising a non-normality of a family okay. So I begin with a family script F of Meromorphic functions okay and am assuming that the family is defined on this domain D alright and I assume that the family is not normal okay then as the non-normality will manifest at some point okay the family normally if it is not normal at least 1 point in the domain and how do you get that non-normal point that is exactly about Zalcman's Lemma is all about.

So you see what it says is that you can find a sequence of points in the domain converging with this point Z not which is the so-called which is the point of non-normality and you can find these decreasing sequence of radii, positive radii so that you know if you take…and you will be able to find…see the fact that the family is not normal means what? It means that the family is not sequentially compact, normally sequentially compact. I mean our definition of normal is normally sequentially compact and that is the correct version of compactness for us okay when you are looking at a family of analytic functions or Meromorphic functions the correct version of compactness is normal sequentially compactness that is given every sequence you should find a normally convergence subsequence and when you say a family is not normal what you are saying is that you are saying that there is a sequence or which you cannot find any normally convergence of sequence okay and you have to… and Zalcman's Lemma ratio that you can find such a sequence and that is the sequence here f n you can find this sequence.

It is a non-normal family in fact the sequence itself forms are non-normal family, so that if you take the members of the sequence and then you take the corresponding zoomed functions okay, so g n is the zooming of f n centred at Z n and with the magnification factor 1 by Epsilon n okay. Then this zoom family converges normally on the whole complex plane to a non-constant Meromorphic function g okay and the point is that the non-constant C of the Meromorphic function reflects the fact that it is spherical derivatives is not 0 because the moment the spherical derivatives of a Meromorphic function is 0 it means it has to be constant right, so this non-constant C of g as a Meromorphic function is further you know fixed by this fact that the spherical derivative at the origin is 1 and all the spherical derivatives are bounded by 1 okay.

So this is Zalcman's Lemma so the point about this Lemma is that the family…if a family of Meromorphic functions on a domain is not normal it gives you a non-normal point Z not and it gives you a non-normal sequence in the family which violates normality in a neighbourhood of Z not that is the whole point alright and I have explained to you that what happens if the family were normal, if the family were normal what would happen is that no matter what Z not you choose and the sequence Z n you choose like this and you choose these any radii Epsilon in going to 0 okay.

The zoom function will always converge normally to a constant Meromorphic function I mean to a constant function okay covert so the normality of the family will tell you that always the zoom functions will be constant and the non-normality of the family is reflected by being able to find a sequence for which the zoom functions not converge to normally to a constant function but actually they converge normally were non-constant Meromorphic functions okay. So it is the limit function that matters, if it is normal glow zoom function will always converge to a constant, if it is not normal I can find a situation where the zoom functions converge to a non-constant Meromorphic functions that is the whole point okay.

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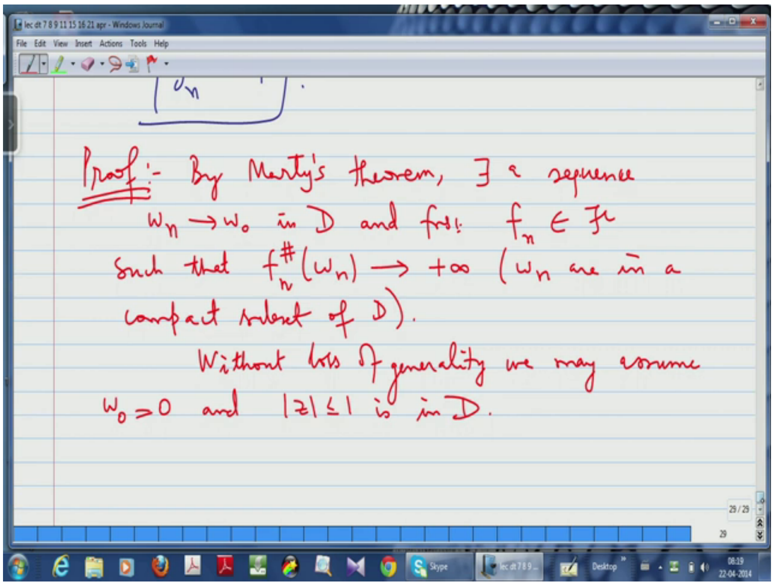

So let us see a proof of this the proof is tricky, so as I have mentioned in the reference material the text book that I am following is that of (())(6:04) okay and the proof as is mentioned that is tricky as you will see, so we will make a couple of reductions the 1st thing is that you know what is given to me is that the family is not normal okay and now you know we have already proof Marty's theorem which is a characterisation of normality namely it says that a family is normal if and only if you know if you take the family of spherical derivatives is normally uniformly bounded okay.

So recall if you take a family of analytic functions okay the condition that that family is normal that it is normally sequentially compact is by Montel's theorem equivalent to the family being normally uniformly bounded and if you consider Meromorphic functions you get the analog as theorems which is Marty's theorem with says that the condition for normality is that a family of spherical derivatives is normally uniformly bounded. So spherical derivatives being normally uniformly bounded is equivalent to normality of the family, so the family is not normal you have a violation of the bounded normal uniform boundedness of spherical derivatives and what does that mean?

It means that there is a compact set on which this spherical derivatives can become unbounded, so this means by Marty's theorem you can actually find a sequence of points okay and functions such that the corresponding actions at those points its spherical derivatives go to infinity plus infinity okay, so that is the 1st step so let me write this by Marty's theorem there exist a sequence of w n tending to w naught in D and functions f n in the family f such that f n hash of w n goes to plus infinity and at w n are in a compact subset of D okay. So I can find is just because of Marty's theorem because Marty's theorem says that you know normality is equivalent to the spherical derivatives being uniformly bounded on compact subsets okay fine.

Now we will make a couple of reduction what we will do is for convenience we will assume that you know w naught is actually the origin okay you assume w naught is the origin and how can we do that? You can do that by simply translating the domain so that you make w naught the origin, so you translate the domain by minus w naught you will get a new domain and you look at the functions there, the translated functions. So without loss of generality what you can do is, you can assume that w naught is the origin okay that is one thing and the 2nd thing is that you can also assume that the moment you assume w naught is the origin, so the origin is the point of D okay then of course there is a small disk surrounding the origin which is also at D because after all D is an open set and by using a scaling transformation you assume that the unit disk along with the boundaries also at D okay.

So these are you scale the domain I mean you scaled the domain and you translate the domain so that you can assume without loss of generality that the compact set you are looking at where you got this sequence w n is actually the unit disk okay along with the boundary unit circle and the sequence actually converges to the origin okay, so we will make these assumptions without any loss of generality, so let me write that down. Without loss of generality we may assume w naught equal to 0 and mod Z less than or equal to 1 is in D okay, so for this all you have to do is that you have 2 translate D by minus w naught and then you have 2 scaled be suitably so that the unit disk which is a neighbourhood of w naught equal to 0 is inside D alright fine, so you can do this. So you see my picture is now like this so here is my I think I will have to… okay so let me go down.

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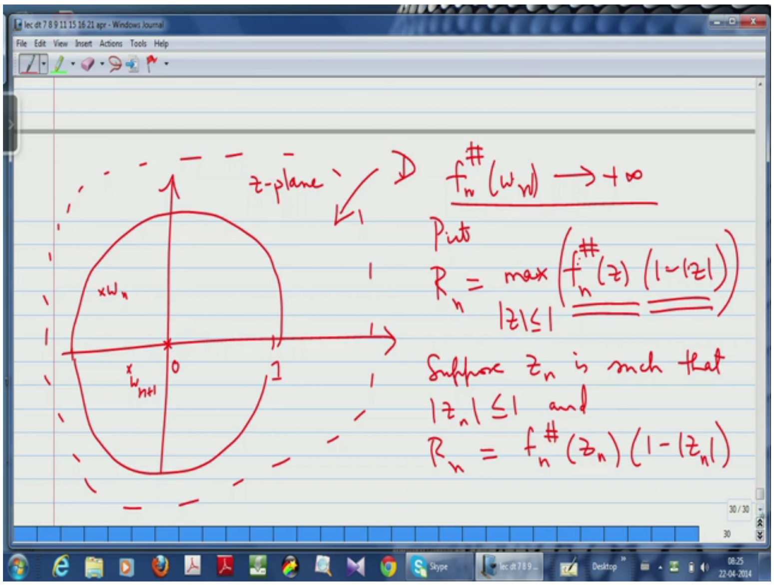

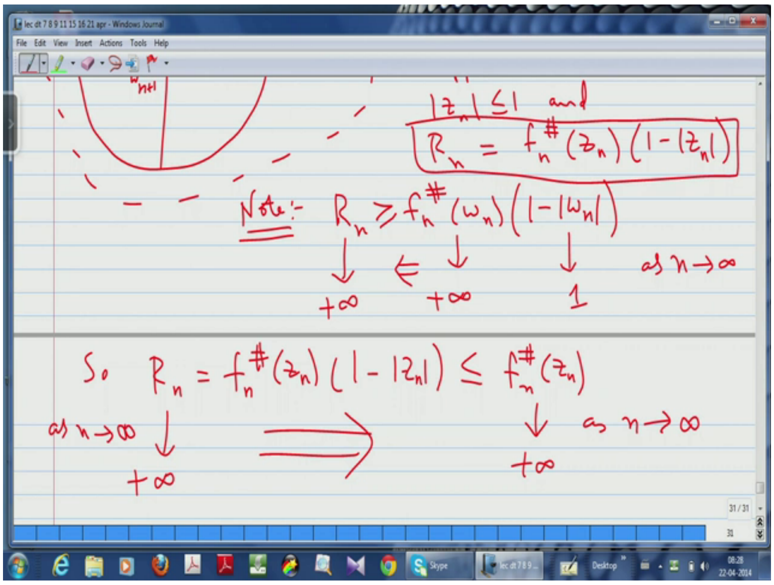

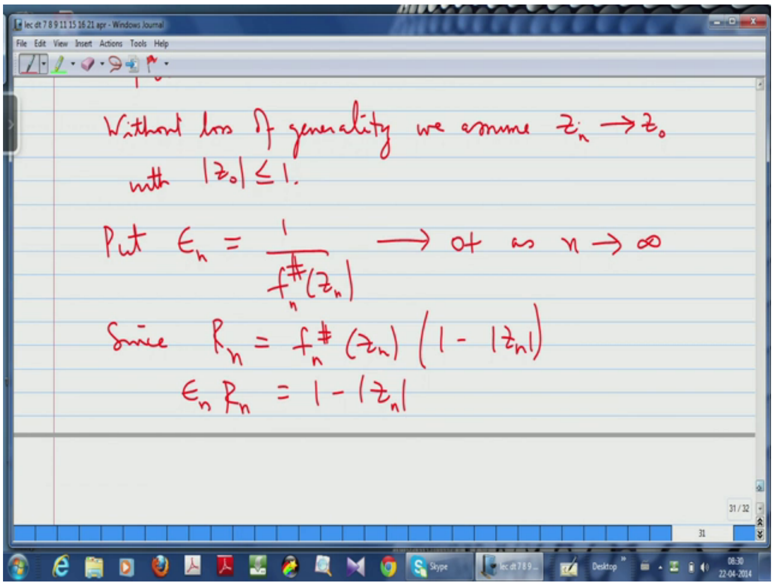

So here is my picture, so I have so this is the complex plane and I have this, this is the origin so this is unit disk, this is unit disk and well this is inside D, so you know your domain D is contains the unit disk, so this is D and well this is the Z plane okay and of course there are these so there is the sequence of points w n , w n plus 1 that is tending towards the origin okay and what you are given is that okay fine. So I have this, now what you do is that you… So the tricky thing is that so you know what I am looking for? I am looking for I have to extract this sequence Z n okay which goes to Z not and I have to extract this sequence of functions which you know which has this property that the zoomed functions okay they converge to a non-constant a Meromorphic function.

So you see the trick is the following, so what you do is you put R n to be the maximum over mod Z less than or equal to 1 of f n hash of Z and then you multiplied by 1 minus mod Z okay, so you do this. So this is the tricky part, the trick is you see what is given to you is these functions f n hash they are spherical derivatives of those…f n hash is just the spherical derivatives of f n and of course you know spherical derivatives is continuous mind you, the spherical derivatives of Meromorphic function is continuous function.

It is continues nonnegative real valued function okay it is a positive function at worst it can be 0 alright which is what happens if the function is a constant okay but the point is that mind you the spherical derivatives has no problem at poles when the Meromorphic function as poles there is no problem with the spherical derivatives unlike the usual derivatives, the usual derivative is not defined at a pole because it is a singular point but a spherical derivative is defined at a pole and we have already seen that if it is a pole of higher-order then the spherical derivative is 0 if it is a pole of order 1 namely a simple pole then the spherical derivative is 2 divided by modulus of the residue at that pole okay.

So this spherical derivatives is a nice continuous function okay non-negative real valued function and you are looking at this function on this domain mod Z less than or equal to 1 which is a compact set, so if you are looking at a continuous function on a compact set, continues real valued function on a compact set you know the function is of course it will be uniformly continuous and it will attain its maximum and minimum therefore this maximum is well-defined okay and the point is that you see what is given to me is that these f n hash they become larger and larger okay at points which are getting closer and closer to the origin.

See as an tends to infinity w n converges to 0 okay that means as n tends to infinity w n goes closer and closer to 0 and what is f n hash of w n that is going to infinity that means f n hash attains larger values closer and closer to the origin as n becomes large alright therefore…so you know what one does is that it could happen that the maximum values of f n hash could also be taken close to the boundary but if you go close to the boundary this quantity becomes very small, if you go closer to the boundary of the unit disk, the quantity 1 minus mod Z will become very small and that will offset this the value of f n hash at that point okay so heuristically this is the reason for multiplying by 1 minus mod Z okay instead of just considering the maximum of f n hash (())(16:01).

So mind you 1 minus mod Z is also a continuous real valued function, nonnegative real valued function inside the unit disk, so there is no problem about it okay, so the product is of course continues real valued function so it has a maximum okay. Now you have to make a series of observations, the 1st thing is suppose Z n is such that mod Z n is (())(16:35) to 1 and R n is attain at Z n, so R n is f n hash of Z n times 1 minus mod Z n okay. So R n which is the maximum is attained at some Z n okay, so look at that Z n and this is the Z n that I actually want or probably a subsequence of that as you will see. See the 1st thing is note that you see R n is greater than or equal to you know f n hash of w n into 1 minus mod w n this happens because R n mind you is the maximum of f n hash of Z into 1 minus mod Z, so if you put Z equal to w n, so the maximum value will always be greater than any of the other values.

So I will get this but then you see as n tends to infinity you see this goes to 1 okay because of w n tends to 0 and this fellow goes to infinity okay because that is the original assumption. The f n hash the spherical derivatives go to infinity okay that is how we pick the sequence w n because it was violating normality, while letting the conditions of Marty's theorem okay. So you see what is happening is that this will tell you that you know R n will tend to plus infinity, so this R n are becoming bigger and bigger and bigger okay that is something that you have to understand first. Now you look at this so you know if you look at this definition of Z n okay what it will tell you is that he f n hash of Z n will also go to infinity because you see if you take R n this is f n hash of z n times 1 minus mod z n and this is certainly you know greater than this is less than or equal to f n hash Z n because you know after all 1 minus mod Z n is less than or equal to 1 okay.

So this is going to plus infinity as n tends to infinity will imply that the f n hash of Z n will also go to plus infinity okay. So this implies that this goes to infinity, plus infinity as n tends to infinity okay, so what you have done is? You have got this from the sequence w n which goes to 0 to w naught you have cooked of this other sequence z n okay and the point is that the spherical derivatives at the Z n also go to infinity, plus infinity is like the spherical derivatives at the w n go to plus infinity okay but the point is that of course the sequence Z n that you have got that need not be convergent it is just the sequence okay but anyway it is a sequence inside the unit disk and you know the unit disk is compact sequentially compact therefore there is a convergence of sequence therefore without loss of generality can assume that this sequence of Z n is actually convergent okay.

So we will make their assumption without loss of generality we assume Z n converges to Z not you know Z not also of course in the unit disk because unit disk is closed okay. Of course when I say unit disk I am also including the boundary is not the open unit disk okay. Fine so we have gotten hold of the sequence actually alright and now the point is that we have…so you know what is our aim? Our aim is you have to get this sequence of functions and you have to get this sequence of points such that and then you have to get a certain sequence of radii okay such that the zoom functions they converge to a non-constant Meromorphic functions.

So where do you get those sequence of decreasing radii okay and that comes very simply, so what you do is you do put Epsilon n to be 1 by f n hash of Z n okay and then this will of course go to 0 as it will go to 0 plus as n tends to infinity that is because the f n hash of Z n is going to plus infinity alright, so this will serve as the zooming radii, so now everything is in place we have gotten what we want and so let me write this down since R n is f n hash of Z n times 1 minus mod z n what you will get is that? You will get Epsilon n R n is equal to 1 minus mod Z n okay because Epsilon n is just defined to be 1 by f n hash of Z n and now what you do is that you do the following thing.

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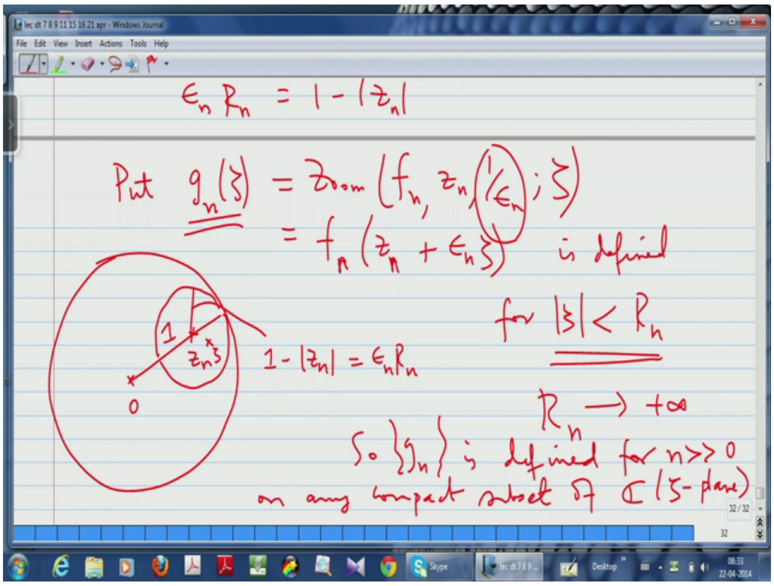

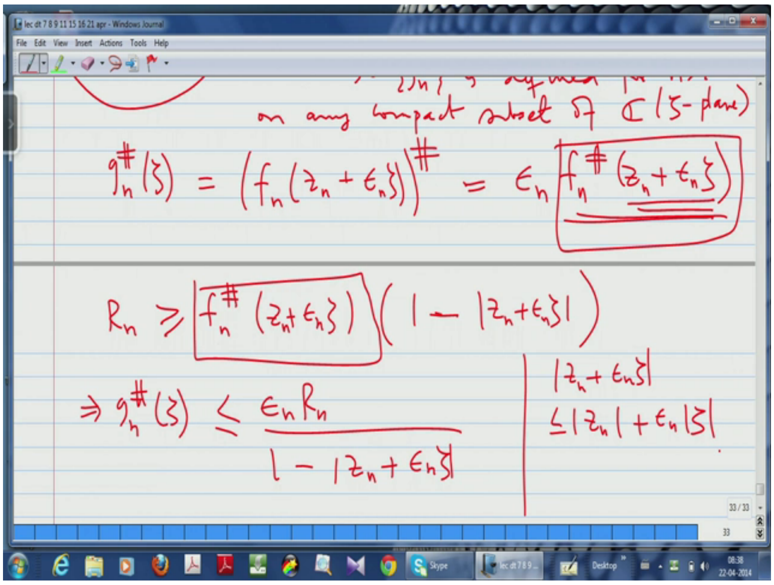

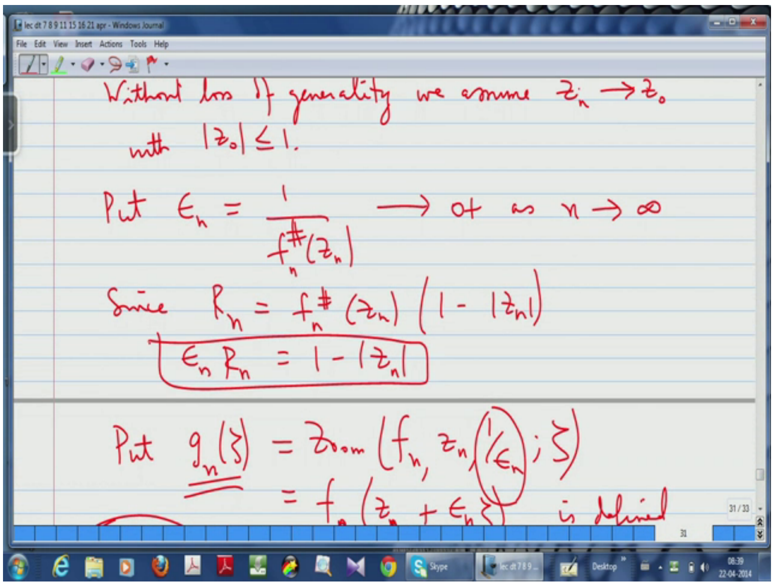

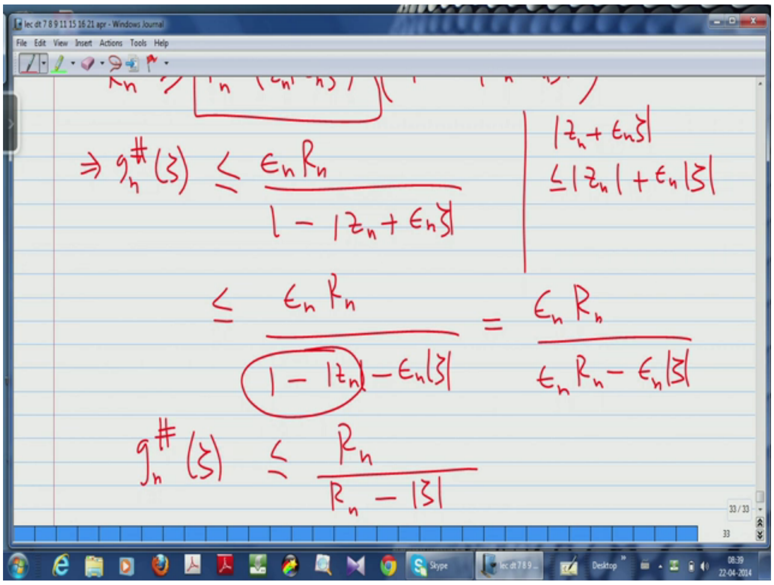

You put g you take the zoom functions, so you take g n of Zeta to be well f n so it is just you zoom the function f n centred at Z n with the magnification factor 1 by Epsilon n and use the variable Zeta, so this is going to be f n of Z n plus Epsilon n times Zeta, so this is the zoom functions this is the family of zoom functions okay mind you we have to find the family of zoom functions which converge normally to non-constant Meromorphic function this will be that family okay and where is this defined you see you know this is defined for mod Zeta less than R n you see what is happening is that, so the diagram is like this you have this unit disk this is the origin, this is one and then you have Z n somewhere here okay and then if you take the small disk centred at Z n its radius will be 1 minus mod z n, this radius will be 1 minus mod Z n but this one minus mod Z n is as I have written above it is just Epsilon n R n okay.

So if I think of a variable Zeta here okay then you know the maximum distance of Z n to Zeta can be Epsilon n R n okay and that means that the maximum value of Zeta can be up to R n because I have reduced you know I have actually used the scaling factor 1 by Epsilon n okay. So but the point is look at these functions g n, the zoom functions. The zoom functions are defined on mod Zeta less than R n and mind you R n tends to plus infinity, so what it means is that as before the zoom functions are eventually defined on any compact subset of the plane okay, so g n is defined for n sufficiently large on any compact subset of the complex plane and here of course the complex plane you are looking at is the Zeta plane mind you your brought in this new variable, the zoomed variable Zeta okay.

So now the fact is that g n does the job that is all you have to verify and how does one do that? Mind you we want to show that you know g n converges normally to a non-constant Meromorphic function okay that is what you want to show that the whole point. Now again use Marty's theorem of course g n are also Meromorphic because g n are just you know obtained from f n by translation and scaling okay g n is just f n translated, see you take the variable of f n okay and you know you translate that we will by minus Z n and then you divide by scale it by 1 by Epsilon and you will get f n okay. So f n have been obtained from g n by a translation and scaling, so g n are also Meromorphic okay and well and what am I trying to show?

I am trying to show that the g n converge normally but again I can apply Marty's theorem to show that the g n converge normally I will have to only show that the g n are normally uniformly bounded I mean the spherical derivatives of the g n are normally uniformly bounded okay. So that is what I check okay and that is just an estimate, so how do I check that? See you will see that g n hash so you know what will happen is g n hash if you calculate g n hash of Zeta this is spherical derivative of g n of Zeta, mind you this is…so I will have to take the spherical derivatives of f n of Z n plus Epsilon n Zeta, so this is the spherical derivative I have to take okay but then taking the spherical derivative you know will be the same as taking the spherical derivative of f n and then I will get a multiplication factor Epsilon n.

You know the spherical derivatives becomes smaller for the zoom functions in the spherical derivative it become smaller by the inverse of the zooming factor. The zooming factors 1 by Epsilon n, so the inverse of zooming factor is Epsilon n okay and so you know this is just change rule of differentiation, so this is Epsilon n times f n hash of Z n plus Epsilon n times Zeta this is what you get alright and now you see what you must understand is that now I have this inequality because you know f n hash z n times 1 minus mod Z n is R n and that is the maximum value okay.

So recall that we have this definition of R n here. R n is f n hash z n times 1 minus mod z n and mind you that is the maximum value of this quantity RN is actually the maximum value of f n hash of Z multiplied by 1 minus mod Z okay therefore what we can see is that R n is certainly going to be greater than or equal to the value of f n hash times 1 minus mod Z for any mod Z for any Z in the unit disk okay, so for Z I will put this so I can put Z n plus Epsilon n Zeta and here I will get Z n plus Epsilon n Zeta, so this is correct okay because in fact when you put Zeta equal to 0 the value on the right is actually R n, R n is the maximum value okay. So you have this but you see now I can use this to get a…so this is the quantity here and this is the quantity that is appearing here okay which multiplied by Epsilon is g n hash of Zeta.

So I can use this to get a bound for g n hash of Zeta, so what will I get? I will get g n hash of Zeta is equal to Epsilon n times this but this thing and this rectangle but this thing in this rectangle is less than or equal to R n by 1 minus mod Z n plus Epsilon n Zeta, so I will get this is less than or equal to Epsilon n R n by 1 minus mod Z n plus Epsilon n Zeta alright but then you see (())(29:47) inequality mod Z n plus Epsilon n Zeta is less than or equal to mod Z n plus Epsilon n mod Zeta okay and therefore what I will get is that this is also less than or equal to Epsilon n R n by 1 minus mod Z n minus Epsilon n mod Zeta I will get this right and now mind you go back…so here is why this proof is tricky, this one minus mod Z n mind you is Epsilon n R n okay that has to be trickily used, so this one minus mod Z n I can put Epsilon n R n then you can see this Epsilon n is coming out both the numerator and denominator and gets cancelled so you see I get Epsilon n R n by Epsilon n R n minus Epsilon n mod Zeta and this becomes that is less than or equal to RN by R n minus mod Zeta okay and you see…so this is what? This is the estimate for the spherical derivative of g n right.

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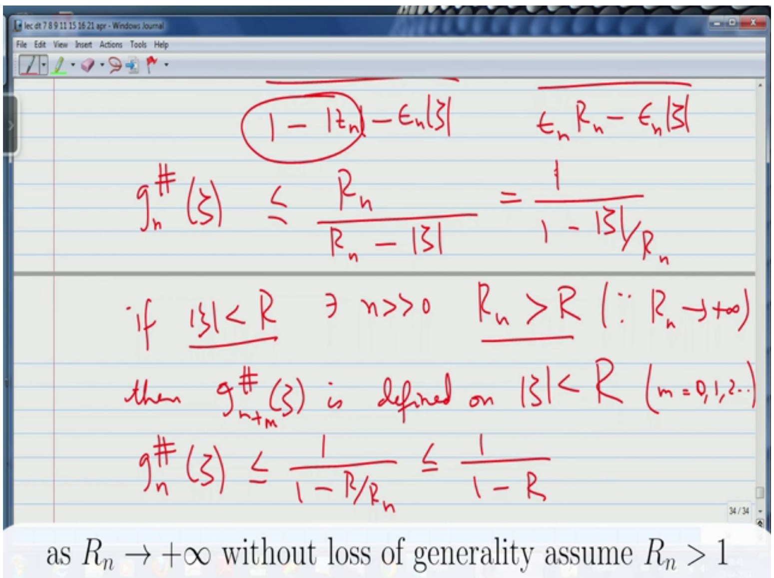

And see the point is that if mod Zeta is less than say some R okay there exist a n large enough such that R n is going to be greater than R okay because after all the R n tend to plus infinity okay, so beyond a certain stage all the R n are greater than R, so that means that you know so I can divide by R n and let me put equal to here I will get 1 by 1 minus mod zeta by R n and if mod Zeta is less than R and R n is greater than R okay then g n is defined on mod Zeta less than R okay then g n hash zeta is defined on mod zeta less than R okay because mod Zeta less than R is contained in mod Zeta less than R n and mod Zeta less than R n is the domain of g n okay.

So g n hash is defined and in fact it is not only for n it is also for higher values of n okay, so you know let me write g n hash of n plus m, m is equal to 1, 0, 1, 2 and so on, so all these g n are defined okay and the point is in any case this quantity you get this estimate g n hash of zeta is bounded by 1 by 1 minus R by R n which is bounded by 1 by 1 minus R okay. So and finally I have gotten this 1 by 1 minus R without any condition on the subscript n small n and that is the uniform bound for g n hash beyond a certain stage alright and that is it, now Marty's theorem will tell you that g n hash to therefore it has to converge normally okay.

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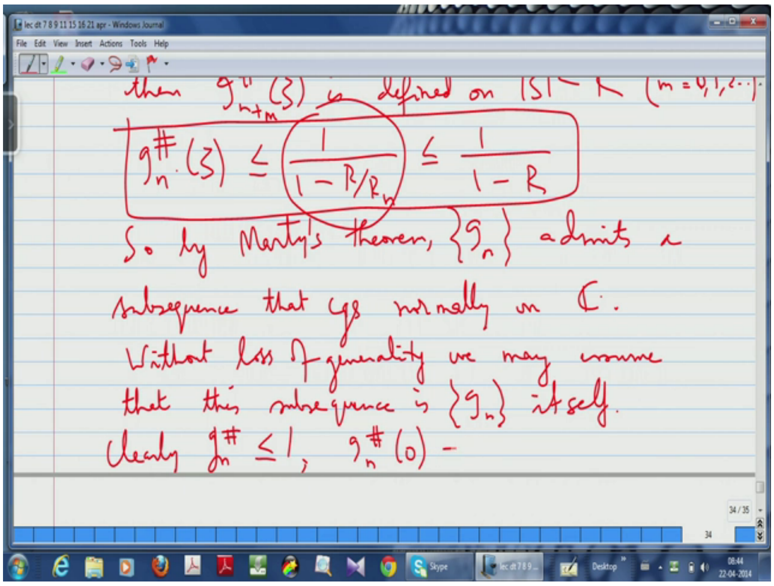

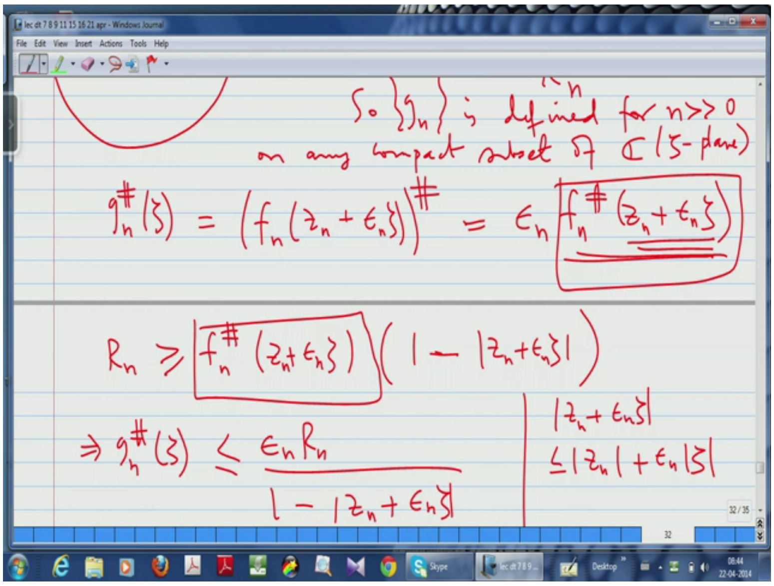

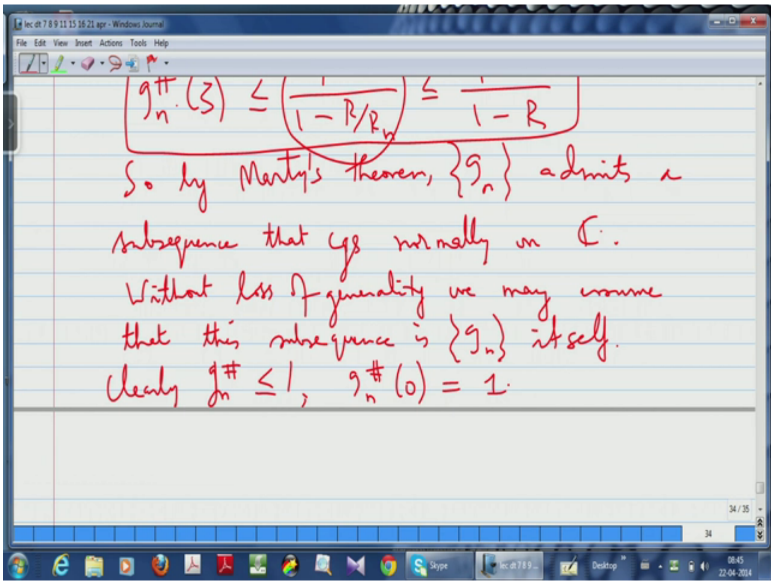

So by Marty's theorem g n admits a subsequence that converges normally on the whole complex plane okay and without loss of generality you may assume that the subsequence is g n itself okay, so after all I am interested in a convergence of sequence and what I have got is a sequence which I know admits a convergence of sequence, so without loss of generality I replace the sequence by the convergence of sequence okay if I do not do this then I will have to use a double subscript okay but it really does not matter but now this g n does the job because you see what happens is that 1st of all this tells you this bound on g n hashes, so what you do is now you let R n to tent to infinity okay then R by R n will go to 0 okay.

So you let n tend to infinity then R n goes to infinity R by R n goes to 0 and this quantity goes to 1 okay and that will tell you that all the g n hash they are all bounded by 1 okay so clearly you get all the g n hash are all bounded by 1 that is one condition and what is the other thing. What about g n hash of 0? If you calculate the g n hash of 0, g n hash of 0 is going to be what? So let us go back to what we have here, so go to this formula here you put 0 is equal to 0, g n hash of 0 as Epsilon n f n hash of Z n but you see Epsilon n f n hash of Z n is 1 because Epsilon n is actually 1 by f n hash of Z n, so g n hash of 0 is actually one okay. See these are all little tricks that I mean they are all there okay you have to look at them okay that is the reason why this proof is tricky.

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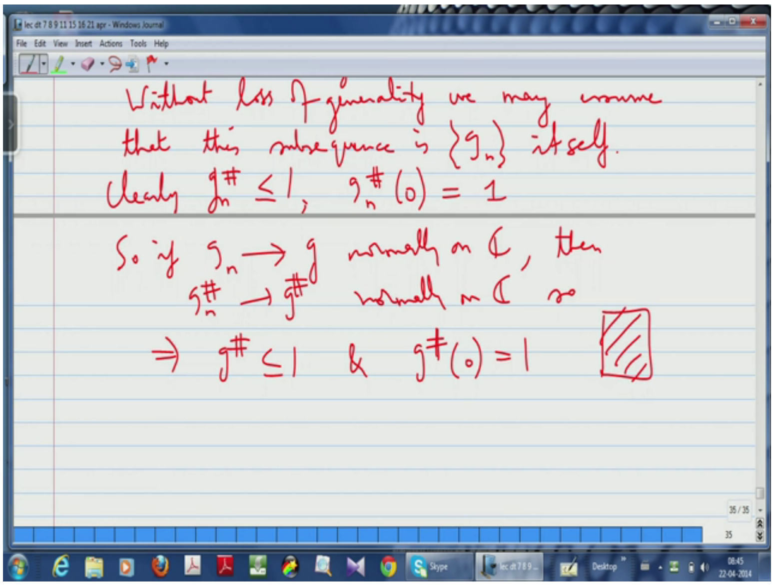

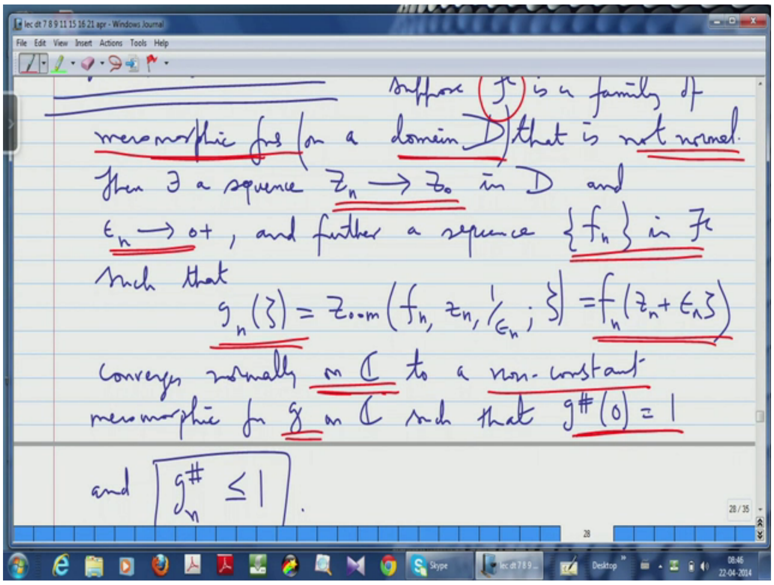

So this is actually 1 okay so if the g n converge to g normally on C then the g n hash converges to g hash normally on C, so what this means is that since the g n hash are all bounded by 1 the limit g hash also bounded by 1 and since all the g n hash at 0 are equal to 1, g hash also at 0 will be equal to 1 just by properties of limits and you have done with the proof of Zalcman's Lemma okay and we have used Marty's theorem that is the whole point right.

Now what I want you to understand is that ash in this Zalcman's Lemma basically you have condition for non-normality of a family okay and the fact is that the converse of Zalcman's Lemma is also true namely if you have a family script F such that you are able to find a sequence Z n tending to Z not and a sequence of radii Epsilon n and also sequence of functions such that the zoom family converges normally to a non-constant Meromorphic function then the original family has to be not normal. It has to be in fact normality will be actually you know normality will be violated at the point Z not. Z not is the point where a normality of the family is violated okay and in what sense…the point is that at Z not the spherical derivatives become as you approach Z not through by a Z n spherical derivatives becomes unbounded and the unboundedness of the spherical derivatives is the same as non-normality because that is Marty's theorem okay.

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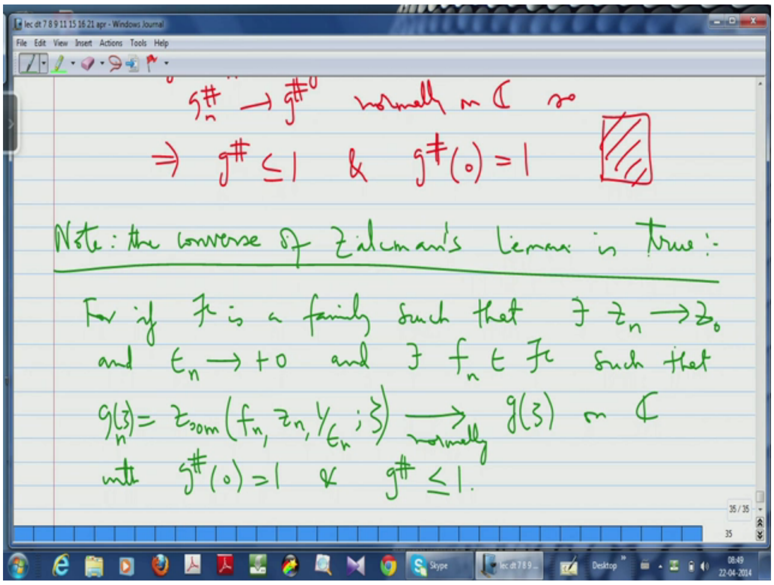

So let me probably say give this as a note that the converse of Zalcman's Lemma is true and why is that so, for if f is a family such that there exist sequence Z n going to Z not with and there is a sequence of radii going to 0 with the zoom functions and you have family of functions and there exist a sequence f n family of functions at f such that the zoom functions g n which is zooming of f n centred at Z n, the magnification factor 1 by Epsilon and in the new variable Zeta suppose this goes normally converges to g Zeta on the complex plane with g hash of 0 equal to 1 and g hash is always less than or equal to 1 suppose this happens okay then the family cannot be normal and why is that true that is very simple because you see then the script F is not normal. Why is that true?

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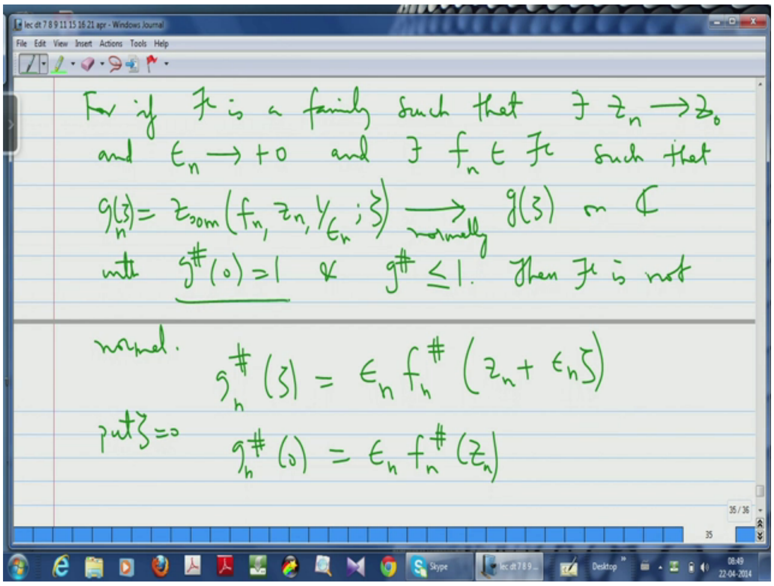

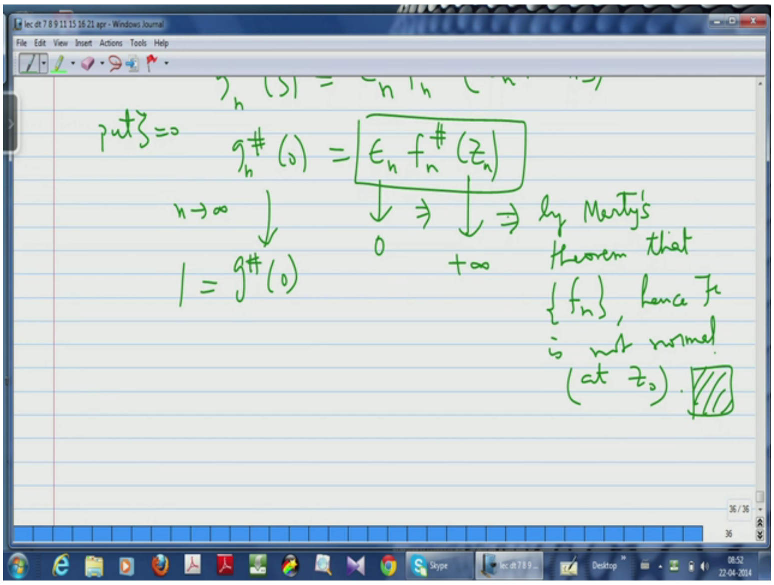

It is very simple because you know g n hash of Zeta as we just calculate we have seen is Epsilon n times f n hash of you know Z n plus Epsilon n Zeta you know this and put Zeta equals to 0 what you will get is g n hash of 0 is equal to Epsilon n into f n hash of Z n okay but you see g n hash of 0 g hash of 0 is 1 and g n hash goes to g okay, so g n hash goes to g hash of 0 which is one okay. So if you take any mind you all this Epsilon n are going to 0 okay, so this happens as an tends to infinity okay, so what will it mean? See you have a product of 2 quantities one of them is going to 0 but the product is bounded that means the other has to go to infinity, so what you will get is that so this implies that this to go to plus infinity okay and what does that mean?

It means that you have violated the sequence f n has violated the conditions of Marty’s theorem you have found functions whose spherical derivatives are going to plus infinity. Spherical derivatives are not bounded and where is this happening? See is f n hash spherical derivatives of f n at Z n is becoming larger and larger and larger going to plus infinity and the Z n are approaching Z not okay and mind you Z n are all approaching Z naught, so what is happening is that if you look at a compact neighbourhood of Z not an open disk closed disk containing Z not you see that on that compact neighbourhood okay these f n hash are not going to be bounded uniformly because they are going to plus infinity and now Marty’s theorem will tell you therefore that this even the f n that sequence itself as a family is not normal okay, so that it implies non-normality.

So the converse of Zalcman's Lemma is also true, so Zalcman's Lemma is actually an if and only if condition okay but the beautiful thing about the Lemma is that you if a family is not normal the Lemma is able to guarantee the existence of non-normal point and non-normal sequence at that point okay you get both the point and sequence that violates normality okay. So what is now left is that I will have to use this to prove Picard theorem and we will do that in the coming lectures, so let me write this here, so this implies by Marty’s theorem that f n, hence is not normal at Z not okay, so I will stop here.