Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

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Lecture No 38

Characterizing Normality at a Point by the Zooming Process and the Motivation for Zalcman's Lemma

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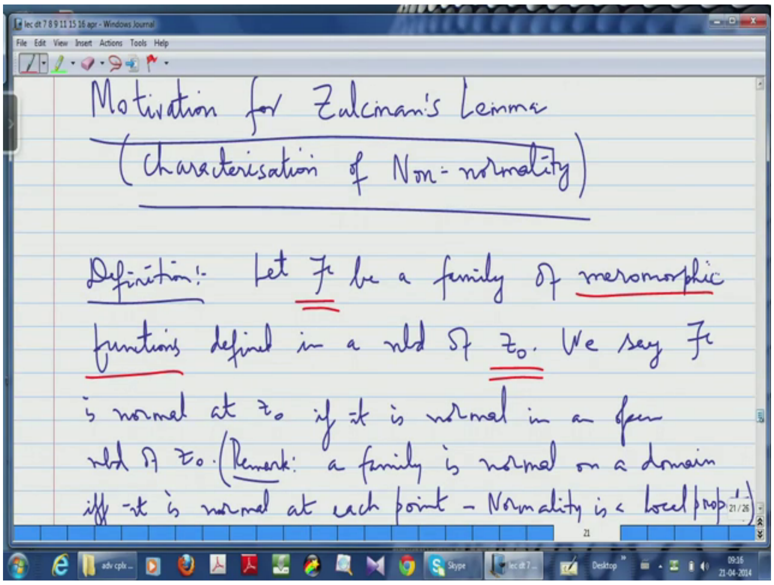

Alright so you see I am continuing with this motivation for Zalcman's Lemma alright and… So let me recall what we were doing, so let me use a different color. So basically you are trying to understand the behaviour of a normal family at that point okay that is the so understanding that is you know the key to understanding the statement of Zalcman's Lemma okay, how does a normal family of Meromorphic functions how does it behave at a point, at a given point? So you know the important outcome of this analysis is that you know you can characterise normality at a point and normality at the point is defined as normality in the neighbourhood of that point in some neighbourhood open disk surrounding that point okay and then so you see this means that normality can be defined locally and it is in fact a local property like analyticity okay.

So we start with a family script F these are Meromorphic functions defined in a neighbourhood of the point Z not, I am assuming the point Z naught is in the complex plane it could have been also a point in the extended complex plane namely it could have been the point at infinity okay. All arguments will work but you will have to make the right modifications so you know how to do that because we know how to deal with the point at infinity and taking in finite limits and things like that okay.

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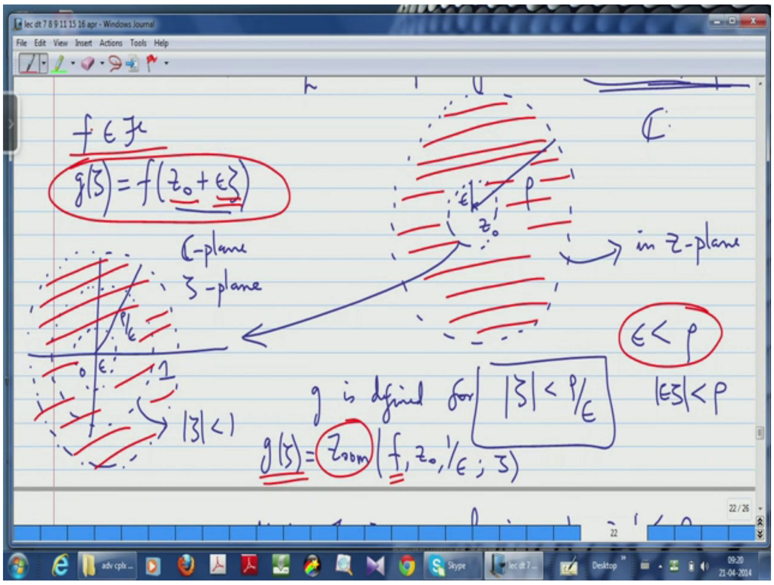

So alright so what I told you last time is that we start with this family script F and what you can do is given a function small f and script F okay you want to study that function in neighbourhood of the point Z naught okay so what you do is that you know you take this neighbourhood of this point which is actually it is a disk centred at Z naught radius rho okay you take this neighbourhood and I am also assuming that the boundary of that disk is in the domain where the function are defined okay, so and the reason why I am including the boundary is because I want a compact set okay.

The disk along with the boundary forms a compact set is a closed and bounded set okay and well so suppose I want to study a particular function small f in the family script F its behaviour very close to the point Z naught then what I do is I zoom into the point Z naught and try to study the behaviour of the function and how do I do this zooming, I do this zooming in this way so I define this zoomed function here okay g so given f I zoom f to get a new function g okay and what is this zooming? You are zooming the function f centred at Z naught and the zoom factor is 1 by Epsilon where Epsilon is supposed to be small positive quantity, so that 1 by Epsilon is large.

In fact Epsilon is to be taken less than rho okay and then you define this new function you define this new function which is for this f you define this function g which is a zoomed function. g is just you take f look at its behaviour in a small disk around Z naught and then you zoomed the behaviour by a factor 1 by Epsilon okay. So what you do is that so this is how the zooming is defined the zoom function g of Zeta is f of Z naught plus Epsilon Zeta which means that actually what you have done is you are actually translating Z naught to the origin okay and then what happens is that this whole disk centred at Z naught radius rho along with the boundary translates to disk centred at the origin with radius rho by Epsilon okay.

So 1 by Epsilon is the scaling factor and mind you should think of it like this, Epsilon is small so 1 by Epsilon is large, so rho by Epsilon is greater than rho so you have actually zoomed okay and so the behaviour of the zoom function is being studied here okay and in this dish and I am calling this zoomed function g, so this is a very simple thing and the point is that the difference between f and g is just you know it is a bilinear transformation, it is actually… It is translation and it involves translation and it involves scaling okay. So what you have done is you have taken the variable Zeta your scale did by Epsilon okay you multiplied it by Epsilon mind you Epsilon is a positive real quantity and then you have translation by Z naught okay so the g and f are of the same type of action okay.

G is analytic if and only if f is analytic, g is Meromorphic if and only if f is Meromorphic and if g has a pole at a certain point then f will also have a pole of the same order at the corresponding point okay and conversely. So g at f are literally the same function except that you have made a change of variable alright but the point is that g is close up look of f you are looking at f very close in a neighbourhood of Z naught that is the whole point okay. Now what you do is that this is what you do if you have a single function but what you could have done is? You could have done this to a sequence of functions.

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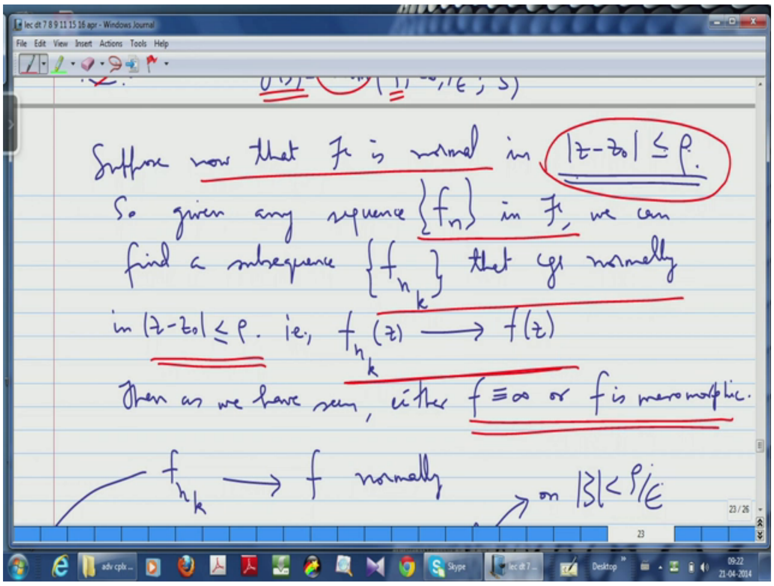

So the pointers if I knew that this family script F is normal then suppose you assume that the family is normal in this closed disk which means it is actually it is normally in an open set which contains this closed disk including the boundary okay and suppose this family is normal you know normality is the correct notion of compactness that we require. Normality actually means normally sequentially compact, it means that given any sequence you can find a subsequence which converges normally okay. So converges normally means converges uniformly on compact subsets, so assuming that the family script F is normal and I want to study the behaviour of the family at a given point Z naught okay.

So if I take a sequence f n in this family then because of normality you can find a subsequence f n k that will converge normally on this disk in fact it will converge uniformly on that this because that is a closed disk okay. Normal convergence means it is uniform on compact subsets and this disk centred at Z naught radius rho is you know is a compact subset and again at the back of your mind you should remember that you know this argument will work given if Z naught is a point at infinity, the only thing is that if Z naught is a point at infinity you will have to think of this you should write this disk you know if you want with the spherical metric or you must invert the variable and look at the neighbourhood of 0 okay, so you can deal with the case when Z naught it is the point at infinity also okay but in any case given the sequence f n I have the subsequence which converges normally, so let me call the limit function as f and then we have already seen this whenever you take a normal limit of Meromorphic functions the limit is either identically infinity or it is Meromorphic okay.

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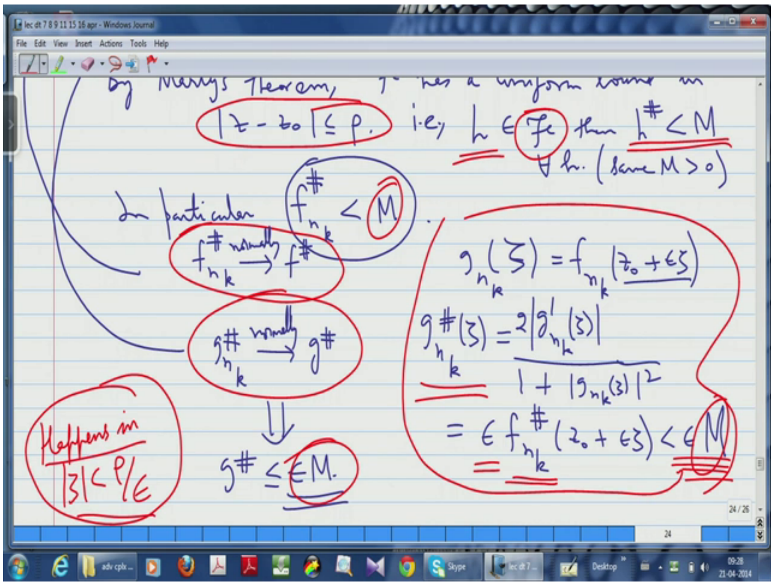

And now look at what happens to the zoomed functions, so if f n k converges to f normally then the zoomed functions for the sequence will converge to the zoomed function of the limit okay, so this is obvious okay because the differences is just a change of variable given by bilinear transformation consisting of a scaling and the translation okay, so this g n k will converge normally to g okay and all this is happening mind you because you have zooming factor is 1 by Epsilon it is all happening here, it is happening in the open disk centred at the origin radius rho by Epsilon which is greater than rho okay and well this is where it is happening. Now you see the point is that whenever a family of Meromorphic functions converges then the family of spherical derivatives will also converge okay.

Now this is just saying that you know the taking the spherical derivative will preserve the convergence in the normal normal convergence okay. So f n k converges to f normally, so what will happen is that the you will get this which is that the spherical derivatives of f n k is converge normally to the spherical derivatives of f okay and you already know that because f n k converges normally to F, g n k converges normally to g alright and therefore the spherical derivatives of g n k hash they will converge normally to g hash okay but the point is that you see the origin family the family script F you have assumed is normal.

So this family script F is normal and Marty's theorem tells you that there are families normal if and only if the spherical derivatives are normally uniformly bounded okay and since we are already looking at a compact set okay there is a uniform bound for all the spherical derivatives of the functions in the family mind you, you have to use the spherical derivatives because there are functions of family are not analytic functions they are Meromorphic functions, so you cannot talk about usual derivatives at a poles okay but then you can talk about spherical derivatives at a pole.

So by Marty's theorem mind you tells you that normal sequential compactness is equivalent to normal uniform bounded of the spherical derivatives okay and it is a generalisation of the Montel's theorem or analytic functions which says that you know the normal sequential compactness is equivalent to uniform boundedness, normal uniform boundedness of the original family okay but the point is when you go to Marty's theorem you go to from the uniform boundedness of the normal uniform boundedness of the given family you switch to the normal uniform boundedness of the family of spherical derivatives okay and that is the quantum jump that you make from Montel's theorem for analytic functions to you when you go to Marty's theorem which is for Meromorphic functions, so there is this uniform bound m which works for all functions in the family script F.

So it will work also for these f n k, it is bound for not the functions but it is bound for their derivatives okay spherical derivatives, so all the spherical derivatives of the f n k are bounded by m and then well what happens is therefore now if you take the limit, so there is a chain rule that works in here, there is a chain rule, the spherical derivatives of the zoom function g n k hash is Epsilon times the spherical derivatives of the f n k the spherical derivatives of g n k is Epsilon times spherical derivatives of f n k because this… So the idea is that you are zooming by a factor 1 by Epsilon okay you are going closer to the point by you are zooming into the point by a factor 1 by Epsilon okay but then this spherical derivative becomes smaller okay.

The spherical derivatives get multiplied by the factor Epsilon mind you Epsilon is a small quantity, so multiplying by Epsilon makes the quantity smaller okay whereas 1 by Epsilon is a zooming factor, so as you zoom closer your spherical derivative is going to become smaller okay so that is what is happening. So for the zoom function if the original function have this bound m then the zoomed function have the bound to Epsilon m and then because the zoom function g the spherical derivatives of the zoom functions converges normally to the spherical derivatives of the limit function which is zoom limit function what it will tell you is that the limit function also if you take its spherical derivatives is bounded by Epsilon times m okay.

Now the point is that you know all this happens in this domain which is the disk centred at the origin radius rho by Epsilon alright and mind you, you must remember that as I make smaller and smaller and smaller I am zooming closer and closer to the point Z naught but then the zoomed function are in larger and larger disk centred at the origin because rho by Epsilon becomes larger and larger as Epsilon becomes smaller. So now the point is that you repeat this game okay by changing the zoom factor okay, so what you do is instead of taking single Epsilon you take a sequence of Epsilons which go to 0 okay and of course from the right namely that you take sequence of positive Epsilons and go to 0 okay.

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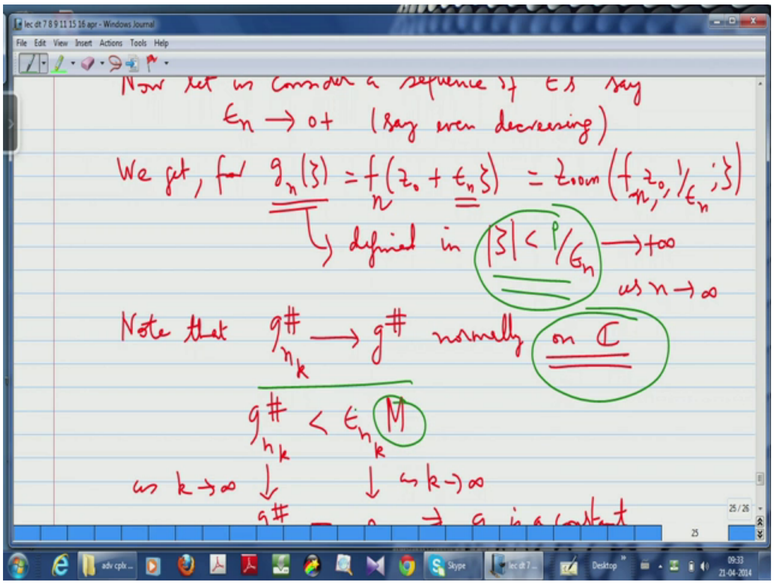

So if you repeat the game with that what you do is that now you know the nth function g n okay and well I think I have not written it correctly last time, the nth the function g n is the zooming of the nth function f n by the factor 1 by Epsilon n centred at Z naught with the variable Zeta, so there should have been an f n here alright. So GN is the zooming of f n alright but the only thing is that now the zooming factor is also dependent on M, so f n is zoomed by a factor 1 by Epsilon m and I am calling it as g m okay.

Just what we did some time ago was keeping all this Epsilon n the same Epsilon but now I am making the Epsilon n smaller as n becomes larger okay. Well so if by doing this what happens is mind you if I make the Epsilon n smaller than the domains on which the zoom function work these domains becomes bigger okay so you know I get these domains okay mod Zeta lesser than I think there was a rho there mod Zeta less than rho by Epsilon n and these are increasingly larger disk as Epsilon n goes to 0 these disk cover the whole plane okay and in fact is that means the n times to infinity you are getting your you will cover the whole plane in particular you will cover any compact subset of the plane.

So what we can say is that these zoom functions g n you know you can talk about normal convergence on the whole plane okay, so actually what happens is that the...anyway as before the f n k will converge to f okay and therefore the f n k hash will converge to f hash where hash then the spherical derivatives and then the corresponding zoom functions g n k hash will converge to g hash all these convergences are all normal okay but the only thing is that this when you come to the zoom functions this normal convergence is on all of C, it is on the whole complex plane because now you have covered the whole complex plane you have covered every compact subset of the complex plane by a sufficiently large disk centred at the origin where that means all the GN k beyond a certain k they are all defined on any compact subset beyond a certain stage okay.

So and so you have this…and the point is that you know the GN k if you take the spherical derivatives they are bounded by Epsilon n k times the bound for the original function which is m okay and therefore this limiting argument will tell you that the limit function g if you take its spherical derivative it has to be 0 and therefore you get a constant function on all of C okay so you see the point is therefore you know as you go closer and closer and closer to point and look at the zoom functions okay then the zoom functions they tend to become constant that is what it means, the zoom functions becomes constant.

So to sum of all this all I am saying is that you take a normal family you take a point where the family is normal and then what happens is that as you take any sequence of in that family you can always find a convergence subsequence if you study it at that point okay then the zoomed function will you know they will tend to a constant function that is the whole point okay and Zalcman's Lemma is all about you know being able to find zoom functions which converges to a non-constant Meromorphic functions okay. So something opposite to this happens, so what I want to do 1st is that I need to 1st of all tell you that I can repeat this argument with not just one point Z naught but I can even take the sequence of points tending to Z naught okay.

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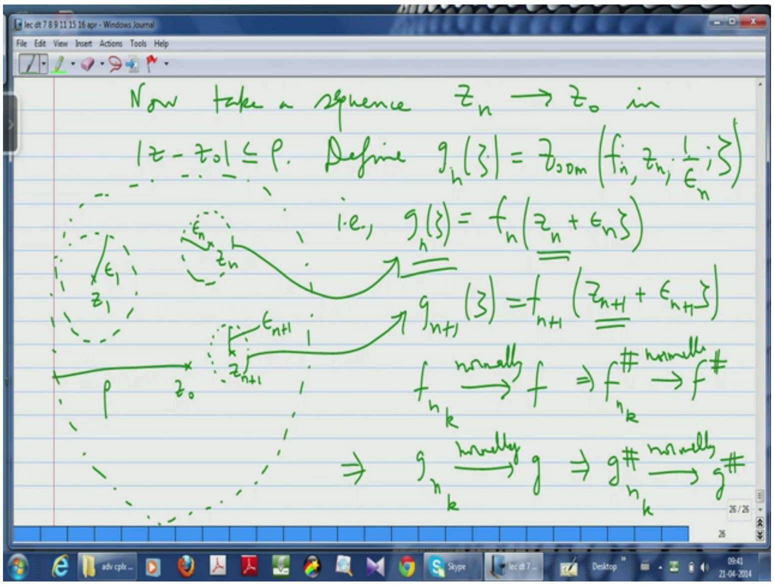

So now what we do is now take sequence Z n going to Z naught say the same disk mod Z minus Z naught less than or equal to rho if you want okay. Well for that matter if you take any sequence Z n going to Z naught the sequence beyond a certain stage will lie in that disk okay but I am assuming this whole sequence of lies in that those disk centred at Z naught radius rho okay and what you do is now you zoom at each Z n okay. See all this time we were zooming only at Z naught but now what you do is you zoom at each Z m, so you define g n of Zeta to be the zooming of you zoom the function f n but now centred at Z n and you know the zooming factor is 1 by Epsilon n and the new variable is Zeta, so this means that g n of Zeta is well it is f n of Z n plus Epsilon n times Zeta.

So this is the thing but then I have to worry about one has to worry about where this will… What about the domains of the GN okay you will have to worry about that but point is that essentially as n tends to infinity you are coming closer and closer to Z not, so the domains of g n is going to again cover the whole complex plane okay, so one can write out the details for that. So you know the picture is like this, the pictures is that you know so I have you know I have this Z naught and you know there is so there is a Z n and then I have this…so inside Z naught there is this big disk with radius rho where I am concentrating my attention and of course even the boundary is included though I am putting a dotted line.

Well and what I am doing is at Z n I am taking a smaller disk with radius Epsilon n okay and then you know as Z n are tending to Z not, so well Z n plus 1 is closer to Z naught if you want and there is and then I am taking a much smaller disk and the radius of that disk is Epsilon m plus 1 okay so you know I am coming I am taking smaller and smaller disk above points which are going closer and closer and closer to Z naught and the point is that on this I have the zoom function okay and then if I go to Z n plus 1 I have another zoom function it is g n plus 1 of Zeta and this is f n plus 1 centred at Z n plus 1. So it is Epsilon n plus 1 Zeta so this is just zooming of f n plus 1 centred at Z n plus 1 with a factor 1 by Epsilon and plus 1 and the variable Zeta okay.

So I have this zoom function okay and well of course you know I will have to assume that the Epsilon n are chosen in such a way that the disk centred at Z n radius Epsilon n is within my big disk centred at Z naught with radius rho okay but that will happen eventually even if you did not assume it beyond certain stage it has to happen okay but we assume therefore with lot of generality that this happens a diagram is like we have already shown it, so you know for example so I am assuming that if you take Z 1 then I am taking the Epsilon 1 that I am taking is such that the disk centred at Z 1 radius Epsilon 1 lies inside this big disk centred at Z naught radius rho okay and the point is that if this does not work I can replace the Epsilon (()) (25:50) by some Epsilon I primes which I can calculate okay.

So the condition I want is that you know the distance of Z naught where any point of the disk should not be greater than rho okay should be less than rho that is the condition I want. So well let us assumed that the picture is like this alright we do the same calculations as before you see f n you know you have this f n k they converges normally to f on this is normally and well what happens is that you will therefore get the spherical derivatives will converge to f normally and you will also get that the zoom functions will also converge normally to g and so will their spherical derivatives okay.

So you will get all this as before mind you the only thing now is that you have also changed the center of zooming okay. The centres of the zooming are all different now okay but that does not change the calculation for the spherical derivative okay because spherical derivative or the zoom functions with respect to Zeta and as far as Zeta is concerned the Z n are all constant okay.

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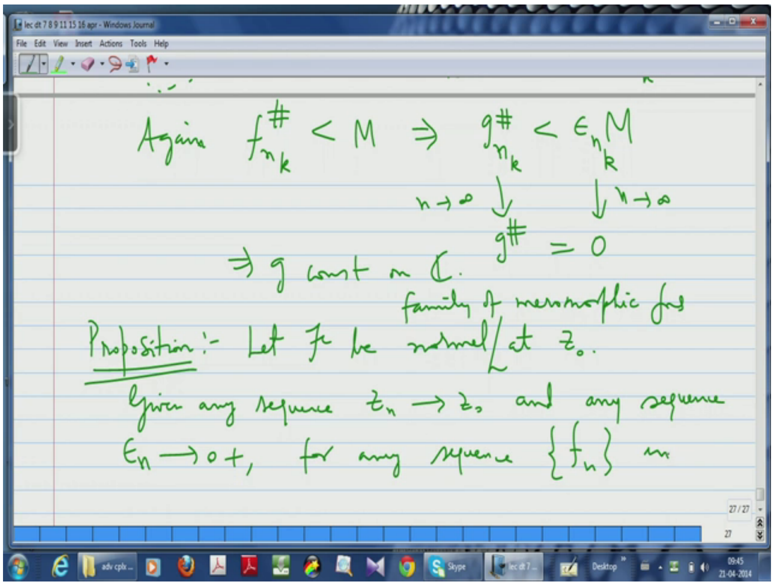

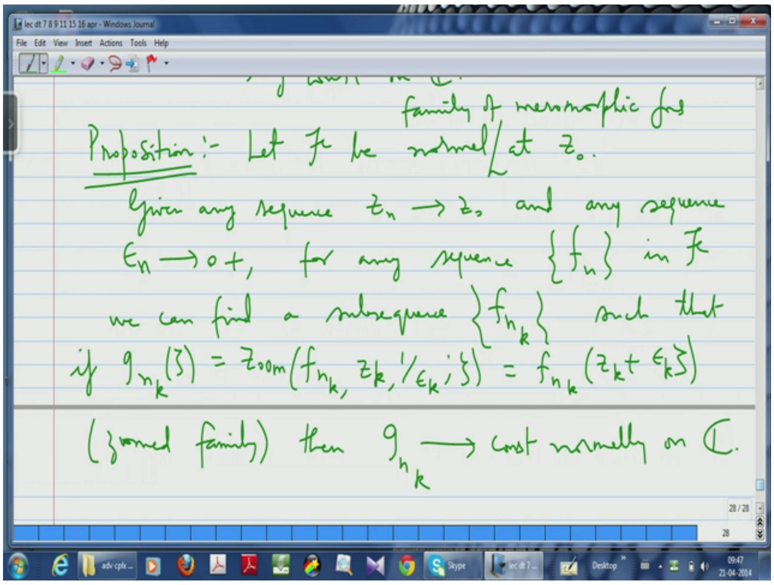

So well again what you will get is that you will get that again you will get that you know the original functions f n k their spherical derivatives of bounded by m and this will tell you that the zoom functions their spherical derivatives are going to be bounded by Epsilon the corresponding scaling factor times m I mean not a scaling factor, the corresponding Epsilon times m and if you take a limit as n tends to infinity you will get that the limit zoom function which is zoom function of the limit function, that will have spherical derivative is 0 okay that is because the Epsilon n k tends to 0 and as a fixed constant, positive constant and as before the moment you know that the spherical derivatives of Meromorphic function is 0 it has to be constant.

So this will tell you that g is constant on the complex plane mind you this constant could have very well-being the constant value infinity that is all out okay because we are in the context of Meromorphic functions but the whole point is and now you have got this important characterisation, you take a point Z naught where a family is normal okay then what happens is that you take any sequence of point Z n going to Z naught and you take a sequence of radii which is going to 0 the Epsilon n okay then given any sequence in the family you can find a subsequence such that if you take the subsequence and zoom it with respect to the sequence, the zoom functions converge uniformly on compact subsets of C that is normally on C to a constant function.

This is a behave of a normal family at a point okay a normal family in the neighbourhood of a point okay and the amazing thing is that you whole point is that this characterises normal families okay this is by this you can actually characterise normal families okay. So let me write this down and the fact that you can characterise normal families like this is actually the philosophy behind Zalcman's Lemma and the way it works is it tells you when this is contradicted, so Zalcman's Lemma will tell you that such a thing will not happen if the family is not normal okay.

So let me write this, so proposition is well let script F be normal at Z naught which means that it is a normal family in an open disk containing Z naught and of course all my families of functions I Meromorphic okay. So let me write here normal family of Meromorphic functions at Z naught okay given any sequence Z n tending to Z naught and any sequence of radii Epsilon n tending to 0 plus okay for any sequence f n in the family script F okay we can find a subsequence f n k such that the zoom sequence of f n k with respect to Z n and Epsilon n that zoom sequence converges to a constant function normally on all of the complex plane okay g n k of Zeta zooming of f n k with respect to center Z N, z k scaling factor 1 by Epsilon k new variable Zeta which is by definition you just take f n k of Z k is Epsilon k times Zeta this is the zooming okay.

Now says that if this is the zoomed family then g n k converges to a constant normally on the whole complex plane, so this is the characterisation of… this is how normal families behave and the big deal about…so this is normality at a point okay this is normality at a point Z naught which means which by definition is normality in a small open disk containing Z naught that is what it means okay and if you want to cover this to normality on a whole domain then you know you have to say family is normal on a domain if it is normal at each point of the domain okay. You can define it point wise, so now let me take Zalcman's Lemma which will tell you and you will appreciate (())(34:01) about but you know with all this background that I have giving you, you will see that Zalcman's Lemma is something natural to expect okay.

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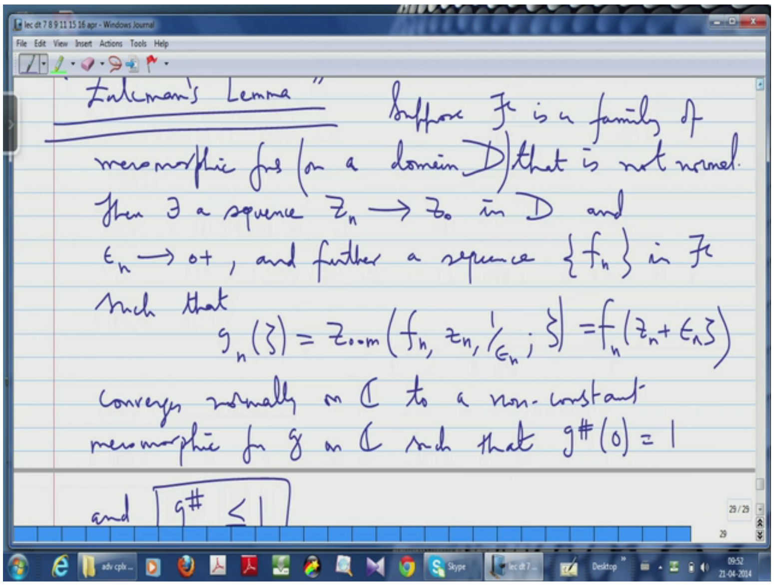

So here is Zalcman's Lemma and well it is call Zalcman's Lemma but it is actually a theorem. So what is a Lemma? The Lemma is about non-normality okay, see I will tell you in a very short words, what we just saw is that if you have a normal family then the zoom functions go to a constant okay. Zalcman's Lemma tells you that if you have an family which is not normal you can get hold of zoomed family of functions which will go to a non-constant Meromorphic function that is all that is the whole point okay. So let me write that of course you know you should always keep saying this in as little words as you can.

So that you grab the main idea and then of course when you write statements you will have to be you have to bring in lot of notation, you have to worry about a lot of notation and then of course proves are even most slightly complicated but the point is that you should always be able to zoom out you know and say things in just few words because that is how you will map it in your memory and remember it. So you do not lose track of the idea, so here is the Lemma suppose script F is a family of Meromorphic functions on a domain D that is not normal.

So here is a non-normal family then what happens is that there exist sequence Z n converging to Z naught in D and Epsilon n going to 0 plus with and further a sequence f n in a family script F such that the zoom functions g n of Zeta is equal to the zooming of f n centred at Z n scaling factor 1 by Epsilon n new variable Zeta namely f n of Z n plus Epsilon n Zeta converges normally on C to a non-constant that is the whole point to a non-constant Meromorphic functions g on the whole complex plane such that a fact that you know it is a non-constant Meromorphic function so you do not expect the spherical derivative to be 0, so the spherical derivatives at the origin will be 1 and all the spherical derivatives is bounded by 1 okay, so this is Zalcman's Lemma. It tells you that the behaviour is exactly opposite to what you saw of a normal family okay, so we will try to prove this in the next lecture.