Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

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Lecture No 37

Local Analysis of Normality and the Zooming Process - Motivation for Zalcman's Lemma

Alright so what we did in the last lecture was you know try to explain how the notion of normal family makes sense for a domain which includes the point at infinity okay and the technique has that we always follow is that you make a two-piece definition okay you make one definition for a domain punctured at infinity which will be a domain in the usual plane and then to take care of a neighbourhood of infinity what you do is that you invert the variable Z to 1 by w and instead of considering Z equal to infinity you have to consider w equal to 0, so you take you consider a neighbourhood of 0 okay which is again it is also a domain in the usual complex plane okay, so therefore you are able to treat the case of normality at infinity okay.

So with that we saw that both Montel's theorems and Marty's theorem hold good for domain which probably include which may include the point at infinity okay that is domain in the extended complex plane and of course I also told you that it is a matter of terminology that a family being normally sequentially compact that is sequentially compact with respect to normal convergence that is sometimes abbreviated to simply normal family okay, so now what we are going to do is you know as I told you we have to… We have come very close to the proof of the Picard theorem which was the main aim of this all these lectures okay and but before that there is something called there is a theorem which is called as Zalcman's Lemma which we have to use okay and see this is got to do with characterisation of non-normality okay, so in order to you know motivate this Zalcman's Lemma okay what we are going to do is we are going to try to analyse what happens to a normal family at the point okay.

So let me make this definition we define normality of a family at a single point to be normality in a small neighbourhood of that point in an open disk containing that point okay, so this is just like definition of analyticity, so you say a function is analytic if it is differentiable not only at a point but in a small neighbourhood of that point, so in the same way normality is also defined at a point by requiring the property in a small open disk surrounding that point or an nonempty open set containing that point okay.

Well of course any open set contains that point is nonempty, so the reason for this definition is that normality is a local property okay. If you have for example a cover of your space which is countable and if your domain can be covered by countably many open sets and on each of these open sets if your family is normal then on the whole domain also the family will be normal because you can use…Because of accountability you can use a diagonalization argument okay, so normality is a local property okay.

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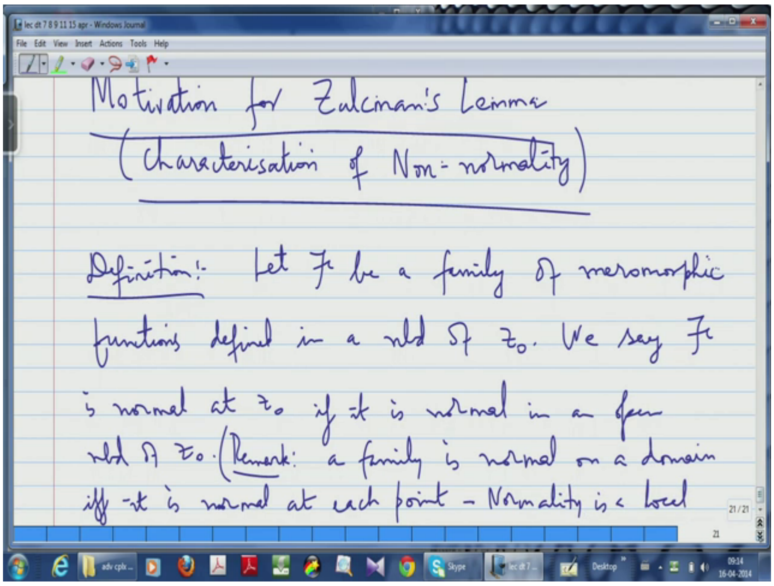

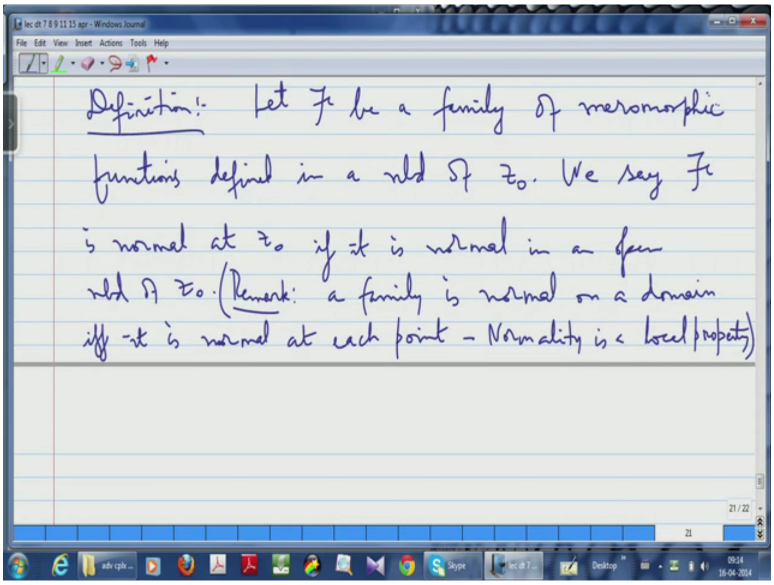

So the 1st thing I am going to start with is so I will put the heading as motivation…let me use a different color, motivation for Zalcman's Lemma which is actually a characterisation of non-normality, so what we will do is we will make the following definition, let script F be a family of Meromorphic functions defined in a neighbourhood of a point Z naught okay and of course this point Z naught could also be the point at infinity but let us assume that Z naught is a point in the plane, the case of you know how to treat the case at infinity okay, so suppose this is a family of Meromorphic function defined in a neighbourhood of Z naught that means it is defined in an open disk containing Z naught okay.

We say the family script F is normal at Z naught if it is normal in a neighbourhood of Z naught in an open neighbourhood of Z naught okay, so well that is the definition and so the question is suppose a family… And of course a family is normal on a domain if it is normal at every point okay. Further family is normal on a domain if and only if it is normal at each point, so I should not say further in fact this is a remark, so this is a remark that you can check. It just says that the notion of normality is a local property, normality is a local property. Fine so here is the question, the question as I now have a family defined in a neighbourhood of a point okay it is a family of Meromorphic functions and it is given to me that this family is normal okay and I want to analyse how the family behaves at that point okay, so the clue to Zalcman's Lemma are at least an understanding or motivation for Zalcman's Lemma is trying to understand this okay.

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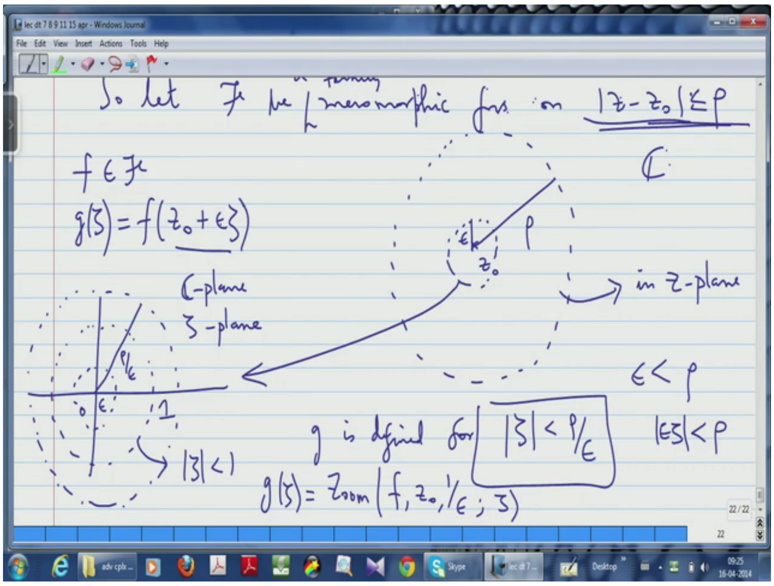

So what we will do is let us so I will call this as local analysis of normal families, so let script F be Meromorphic let script F be a family of Meromorphic functions on mod Z minus Z naught less than or equal to rho okay, so you say mod Z minus Z naught less than or equal to rho is actually a closed set, it is a compact set okay that is provided Z naught is a point in the usual complex plane in Z naught is infinity then you know you have to rewrite this as… You should not use the Euclidean metric what you have to use the spherical metric okay but I am assuming for simplicity that Z naught is a point in the plane okay and this is the closed end bounded set I am taking a compact set because I can choose such rho certainly it is a family of Meromorphic functions at the point Z naught so it is defined in the neighbourhood of Z naught, so it is defined on a small enough disk which contains Z naught and if you take a small enough radius then the closed disk centred at Z naught of that radius will also be in the domain where the Meromorphic functions is defined where the family is defined.

So I am taking without loss of (())(9:31) this closed set because the reason is I want to compact set so that I can you know I can get uniform convergence, whenever I talk about convergence you know uniform convergence it happens only on compact sets its normal convergence and whenever I am talking about uniform boundedness it happens only on compact sets, so everything I need a compact set that is the reason why I am doing this okay, so what it means is that a family is defined on bigger open set which contains this closed set okay. Now you see what we want to do is we want to really so you know let me draw a diagram, so here is Z naught alright and here is this there is this disk centred at Z naught radius rho.

So this is the this centred at Z naught radius rho and the point is you give me function small f in the family script F okay and what I want to really do is I want to really analyse this function in that small disk okay I want to analyse this function, this function in this small disk and how do I do it? So what I do is you know I have to look closer at the small disk, so what I actually do is zoom in okay and how do I zoom in? So it is by scaling okay, so what I do is that I put I define g for every function f in the family I define this g and what is this G? This g is define with the new variable Zeta and this is defined as f of Z naught plus Epsilon times Zeta I put this condition I make this definition, so what is happening is that this is in the Z plane, so the variable here is Z so this is also the complex plane and then I have another complex plane okay and this is the Zeta plane.

So the variable here is Zeta alright and what I do is I take this unit disk okay I take this unit disk, this is unit disk mod Zeta less than 1 okay I will take this disk and now I am defining this function g as a function of Zeta okay and what is (())(11:54) it is actually you know you have just magnified the behaviour of … The value of g at Zeta is the value of f at Z naught plus Epsilon times Zeta, so what it means is that you know if you start with… And of course I am assuming that Epsilon is less than rho okay, so you know if I take this disk of radius one centred at the origin with variable Zeta then if I multiply by Epsilon, if I take Epsilon Zeta it will become disk of radius Epsilon okay, so it will be smaller disk like this okay and this will have radius Epsilon okay and then if I had Z naught to it I am just translating it to give me a small disk of radius Epsilon centred at Z naught that is what this does okay.

So you are just multiplying Epsilon to Zeta then you are adding Z naught so that is a translation by Z naught okay. So basically what you are doing is that when you do this you know you are actually looking at this you are looking at this small disk centred at Z naught radius Epsilon and so you know in principle for what values of Zeta is g defined, so you will see that g is defined for…should be (())(13:29). See if mod Zeta is less than rho by Epsilon okay then mod Epsilon Zeta will be less than rho and mod Z naught plus Epsilon Zeta will be less than therefore also less than rho okay.

See mod Zeta is less than rho by Epsilon means that you know mod Epsilon Zeta is less than rho okay and what does this tell you? This tells you that the distance of Epsilon Zeta from the origin is at most rho and therefore if you add Z naught to Epsilon Zeta the distance of Z naught plus Epsilon Zeta to Z naught is at most rho, so you see this g is defined on this disk and you see this disk is actually…see this disk is larger see because you see rho the original disk centred at Z naught has radius rho alright. Now the radius has become rho by Epsilon and Epsilon being smaller rho by Epsilon should be larger if you think of Epsilon to be you know Epsilon is less than rho, so rho by Epsilon is greater than 1 okay.

So what you have done is by this transformation this disk centred at Z naught radius rho okay if you want to study the function value (())(14:45) the values of the function f there the behaviour of the function f there, what you have done is you have actually zoomed it by constructing this function G, so g is a kind of zooming function okay it is like you have a microscope or telescope and you have 10 X zoom 20 X zoom you see that also in camera these days okay you have 5X zoom and so on, so this is so many X zoom and the zooming factor is 1 by Epsilon okay. Original disk of radius rho has now become zoom to disk of radius rho by Epsilon okay and well the point is that… And of course rho by Epsilon will be slightly bigger okay you will get so this is Epsilon, this thing will be rho by Epsilon and it is in this bigger disk open disk that the function g is defined and studying this function g in this bigger disk is the same as studying the functions f in the original disk okay.

So you know I will write it like this I will write g of Zeta is equal to zoom I will think of it as zooming, zoom the function f centred at Z naught by a magnification factor 1 by Epsilon and use the variable Zeta okay I will use this notation okay, so when I say zoom f Z naught 1 by

Epsilon Zeta you are zooming the function f centred at Z naught, so your zoom is centred at Z naught okay and 1 by Epsilon is the magnification factor and Zeta is the variable of the zoom function okay. Now well… So this is how you zoom into the behaviour of function at a point okay, now I am given that this function, I am given that this family script F is normal suppose I am given it is normal at Z naught suppose I am given normal in the closed disk centred at Z naught radius rho which means that it is actually normal in an open set which contains that closed disk okay. Suppose that happens and let us see what it means for us.

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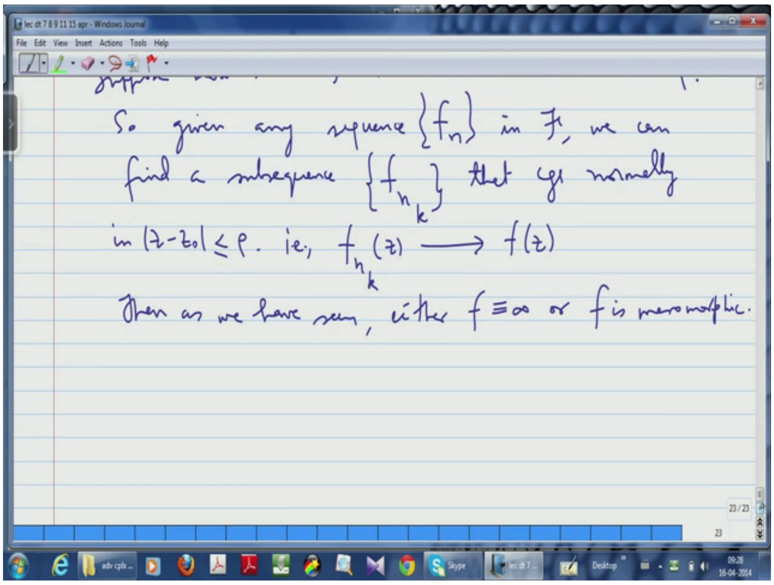

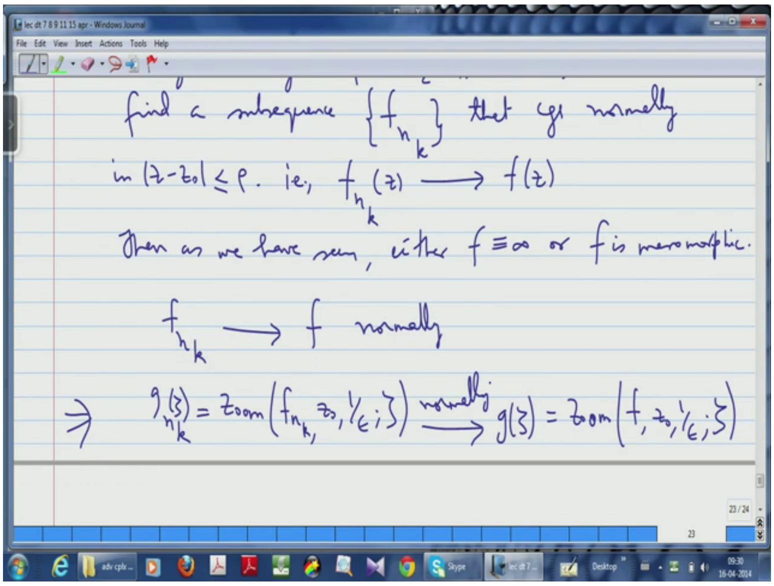

So let me continue like this suppose now that you know script F is normal in mod Z minus Z naught less than or equal to rho okay. By that I mean it is normal in a slightly bigger open disk okay. Then so what does normality means, it means actually normally sequentially compact is the correct notion of compactness for as when we do complex analysis okay, so what it means is that you give me any sequence of functions in script F I can always find a subsequence which converges normally okay it is normal sequential compactness, it is sequential compactness with respect to normal convergence. So given any sequence f n in script F we can find a subsequence let me call that as f n k that converges normally in mod Z minus Z naught less than rho okay I can put less than or equal to okay.

In fact mod Z minus Z naught less than or equal to rho is actually compact therefore it will converge even uniformly there okay. So it converges normally so what happens is that you will get this subsequence of f n k of Z it will go to some function f Z a and f is a normal limit of Meromorphic functions so you know what is going to happen you have body seen this a normal limit of Meromorphic function is either identically infinity or it is also a Meromorphic function it could even be analytic okay, so then of course then as we have seen either f is identically infinity or f is Meromorphic this is something that we have seen but the point I want to emphasise is that is the following issue, you see you have these functions f n k they are converging to f alright and what is happening is that now you know mind you I have the zoomed function okay I have these zoomed functions by zooming factor Epsilon is than rho.

So look at the zoom functions, so what happens is that I take g N k which is actually the zooming of…so g n k of Zeta is the zooming of f n k centred at Z naught with the magnification factor 1 by Epsilon and I am using the variable Zeta and similarly I get zoomed function for the limit function also, so I get g also is equal to zoomed function of f centred at Z naught magnification factor 1 by Epsilon my variable is Zeta okay, so this is g of Zeta so I have that and of course you know if f n k converges to f then the zoomed function of the f n k namely the g n k will converge to the zoomed function of f which is g and this is just because the zooming is just the scaling followed by a translation okay.

So after all what is the zooming? The zooming is you multiply, you multiply Zeta by Epsilon and then you add Z naught to it okay that is just a in fact it is actually a bilinear transformation it is a linear fractional transformation, it is a Mobius transformation okay so it will preserve all properties convergence everything okay. So you get this, now you see so let me write this, this is normally so here also I will get so this will imply that this will also converge normally okay and the point I want to make is that you see I have assumed that the original family is normal okay and we have seen namely Marty's theorem that normality is equivalent to spherical derivatives being normally bounded okay.

So if you recall you know Marty's theorem says that a family of Meromorphic functions on a domain, the domain could even be a domain in the extended complex plane it could contain the point at infinity. Such a family is normally if and only if the spherical derivatives of those functions, the family of spherical derivatives is normally uniformly bounded so it should be on every compact subset of the domain the family of spherical derivatives should have uniform bound okay. So by Marty's theorem you know that the family script F itself has if you take the spherical derivatives than that has uniform bound.

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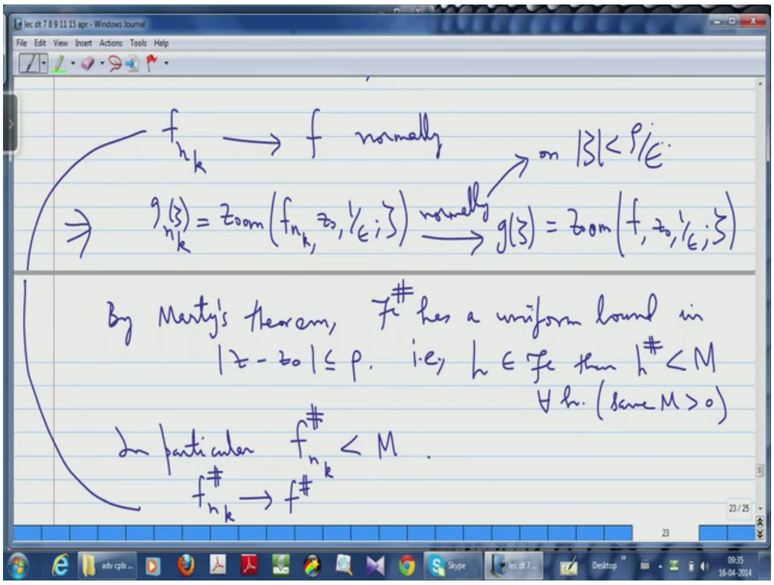

So let me write that down by Marty's theorem script F has uniform bound, script F hash, so when I put script F upper hash means the family of spherical derivatives of the functions in f okay. This has a uniform bound in mod Z minus Z naught less than or equal to rho I have this okay because this mod Z minus Z naught less than or equal to rho is a compact set alright and on a compact set I have a uniform bound for spherical derivatives and that is equivalent to normality okay this is something that I have this is Marty's theorem.

So what does it mean? It means that if you take any function f so let me not use f let me use h because I have already use f for the limit, if you take a h in f then h hash this is the spherical derivatives of h (())(23:45) than m for all h and same m right, so you have this you have uniform bound alright and this bound applies to all members of (())(24:01) spherical derivatives of all members of the family f script F so it applies also to the f n case okay, so in particular what you will have is you know, you will have that f n k hash the spherical derivatives you see these things are going to be bounded by m it is going to happen because all these f n case are anyway in the family okay but you know we have also seen this earlier if a family of Meromorphic functions converges normally to limit function then the spherical derivatives will also converge okay.

Taking the spherical derivative will respect convergence okay, so you also have from here if you want, so I need to go like this that f N k if I take the spherical derivatives that will go to f hash okay you have this okay and of course you know if I by the same token if I do it to the g n k the spherical derivatives of g n k will go to the spherical derivatives of g okay mind you g n k is just f n k translated and scaled, so when you translate and scale a function its nature does not change, so if you take a Meromorphic function and translate and scale it you will again get a Meromorphic function okay.

After all translating and scaling is not going to change the nature of singularities okay and if you translate and scale an analytic function you will again get an analytic function okay and so on and so forth, so g n k are also Meromorphic functions and mind you have to remember that I am considering the f n k in mod Z minus Z naught less than or equal to rho but I am actually considering the g n k in mod Zeta less than rho by Epsilon okay that is the zoomed disk centred at the origin where the zoom functions are being looked at, so let me write this here on mod Zeta less than rho by Epsilon okay this is the Epsilon less than rho alright so this is a quantity greater than 1, fine.

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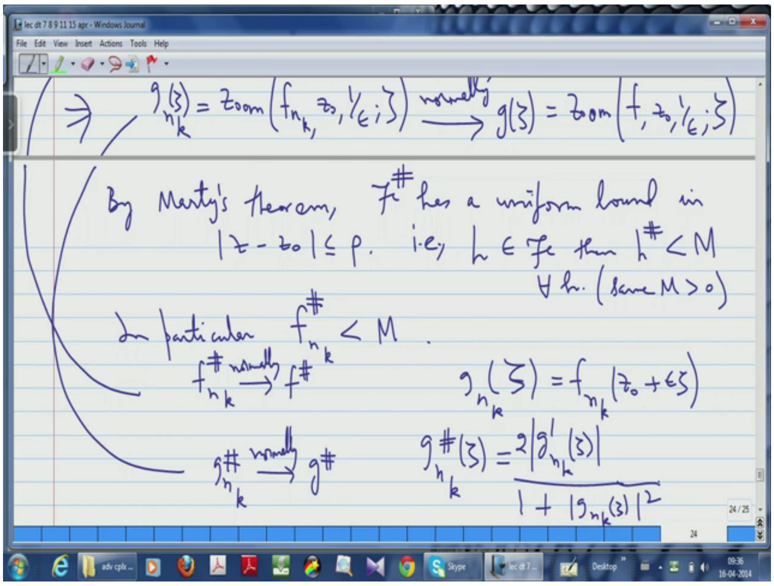

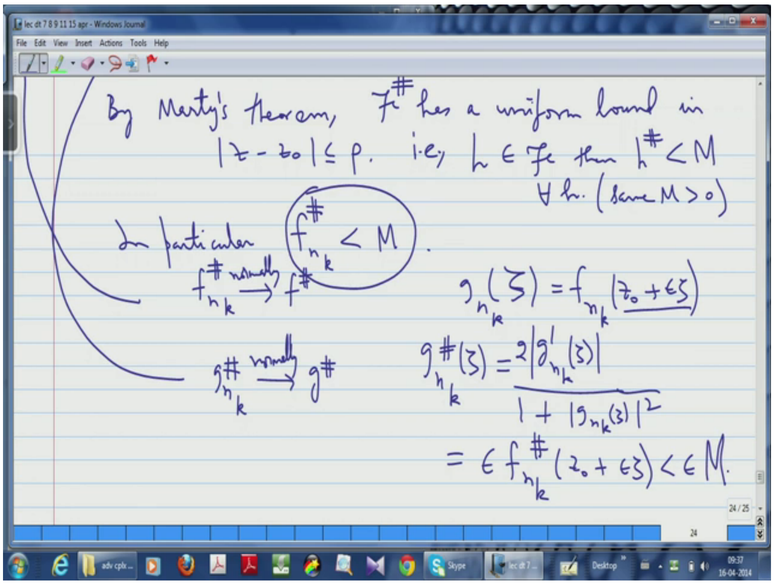

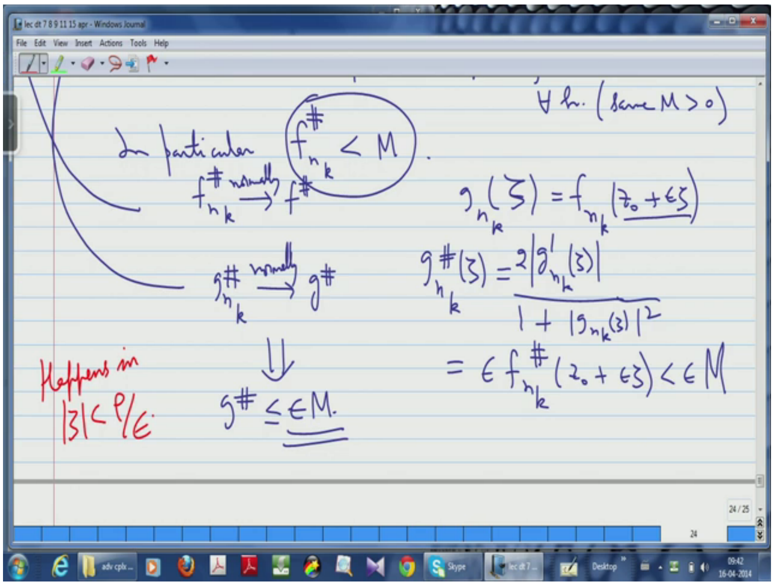

So I have this and I shall have the same thing for these guys I will also have g n k hash going to g hash alright and of course it is again normal this also normal okay and the point is that but there is an inequality coming out, see what is g n k? g n k of Zeta is by definition g n k of Z naught is Epsilon Zeta this is the zoomed function, so what is g n k hash of Zeta, what is it? Say it is by definition it is supposed to be 2 times the modulus of the derivative of g n k with respect to Zeta divided by 1 plus modulus of g n k of Zeta the whole square, this is the definition of spherical derivative okay and but then this is what is the g n k dash Zeta in the numerator?

It is the derivative of g n k with respect to Zeta it is derivative with respect to Zeta okay and mind you this formula for spherical derivative also works when Zeta is a pole okay we have seen that you know by continuity if the pole is of higher-order then the spherical derivative at the pole is 0, if the pole is a simple pole then spherical derivatives at the pole is actually 2 divided by modulus of the residue of the pole which is something that we have seen already okay, so there is no problem if Zeta is a pole okay this formula works but then you know if I differentiate g n k with respect to Zeta it is the same as differentiating f n k and then you know by the chain rule differentiating the argument of the f n k with respect to Zeta and that will bring me a multiple of Epsilon.

So what I will get is, I will get Epsilon times f n k hash of Z this is what I will get okay not Z in fact I have to plug-in the right substitution scale to variable Z naught plus Epsilon so this is what I will get okay but the f n k are all bounded by what? The f n k are bounded by m alright so this is bounded by Epsilon m okay so the moral of the story is that because you because of your scaling okay. See what you must understand is that you zoomed the small disk centered at Z naught radius rho to a larger disk okay you scaled and you got the zoom function but what has happened is that the bound for the spherical derivative have become smaller because it has got multiplied by this the inverse of the zooming factor. Zooming factors 1 by Epsilon where Epsilon is very small okay then bound for the zoom function becomes much more smaller okay.

Now that is what it says, now this is true for every m K but you know g n k hash converges normally to g hash and each of the g n k is bounded by Epsilon m therefore g hash will also be bounded by Epsilon M, so these 2 put together will tell you that g hash will also be bounded by Epsilon m okay so this will happen alright, so that is because of normal convergence okay. So the moral of the story is that so this is what is happening, so what have we shown so far? You take a point Z naught a point where a family is normal okay and then you take any sequence in the family you will get a convergence of sequence, normally convergence of sequence you look at those functions in the subsequence okay and zoomed them.

Then the zoomed functions their spherical derivatives becomes very small and in fact the zoom functions if you take the limit of the zoom functions, the limit function will be a Meromorphic function whose spherical derivatives is very small that is what it says okay and now you know, so this is the first step of the argument. Now what I am going to do is I am going to introduce a level of complication by doing the following thing, what you do is instead of considering a single Epsilon you consider sequence of Epsilon going to 0. Imagine you are considering I am using the same Epsilon here for all the functions okay but suppose for g n I use an Epsilon n okay and such that the Epsilon n are going to go to 0 okay that means I am doing ultra-zooming, I am zooming into smaller and smaller and smaller and smaller neighbourhood of Z naught okay.

If I do that then what will happen is that you can guess what is going to happen? This g hash will be 0 because see g hash will the thing on the left side g n k hash that is going to be bounded by Epsilon n k m okay and if I take limit as N k tends to 0 Epsilon NK is going to go to 0, so Epsilon NK m is going to go to 0 because m is anyway finite quantity therefore g hash is going to go to 0 and what does it mean if g hash goes to 0? It means that g is a constant. A spherical derivative of function cannot be 0 unless it is a constant.

It can be even be constant function that is uniformly infinity mind you for the constant function which is uniformly infinity also the spherical derivatives is 0 okay in line with the fact that the philosophy that whenever you have a constant function the derivative of any type should be okay. So the moral of the story is that if you zoom in a family of convergent, family of functions at a point okay then the limit function is going to be constant that is the full point okay, so let me add that level of complication but now there is something very nice.

What is happening is that so let me put this in a different color here whatever is happening happens in mod Zeta less than rho by Epsilon okay this was the disk where everything is happening okay but now you know if I make these Epsilon goes to 0 by choosing Epsilon N which go to 0 something nice is going to happen as Epsilon N go to 0 rho by Epsilon N go to infinity, so your disks are becoming bigger and bigger and bigger and the beautiful thing is that any compact subset of the complex plane, the Zeta plane will be contained in a sufficiently large disk and therefore you get a family of functions for which you can talk about normal convergence on the whole plane okay, so this g will be defined on the whole plane and you will get a constant function okay that is the whole point, so let me write this down.

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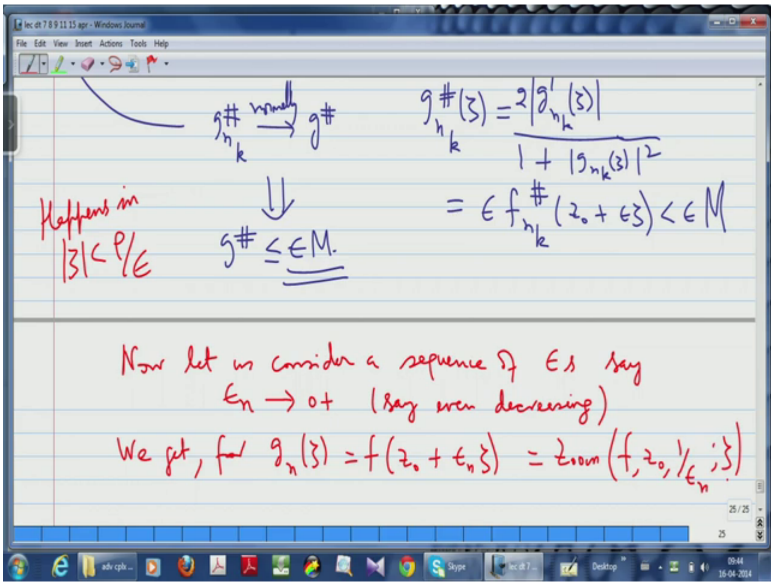

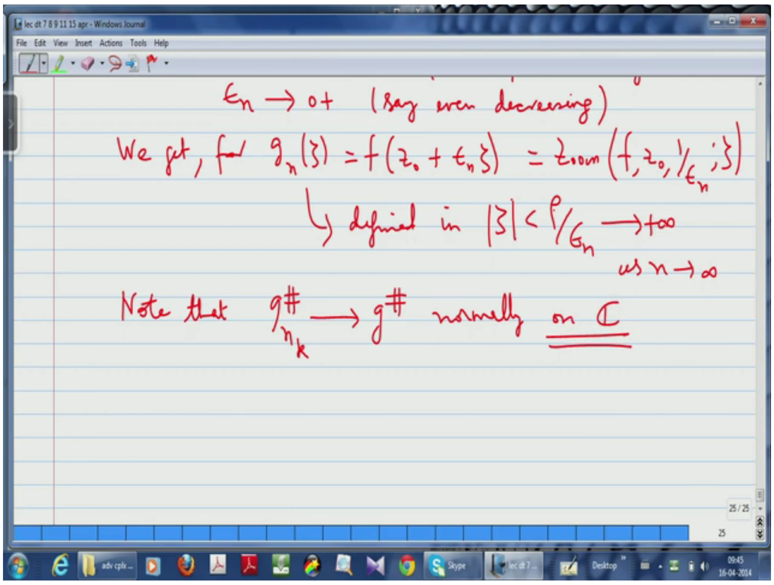

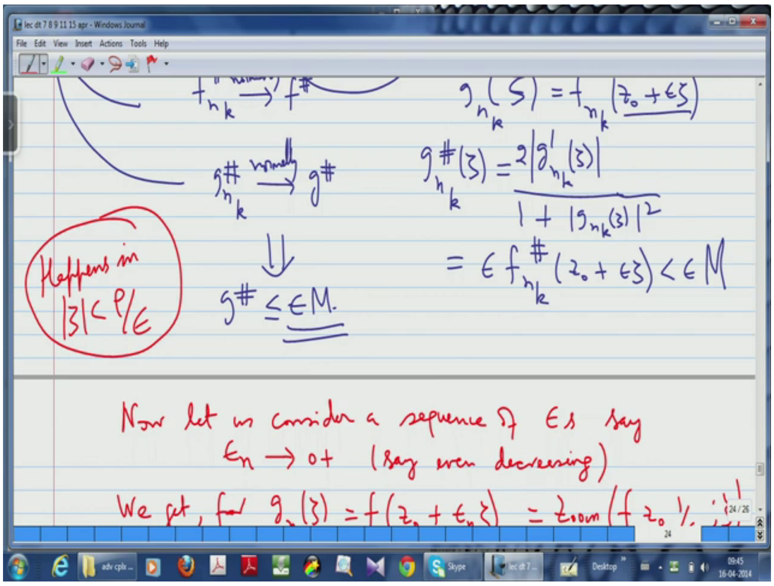

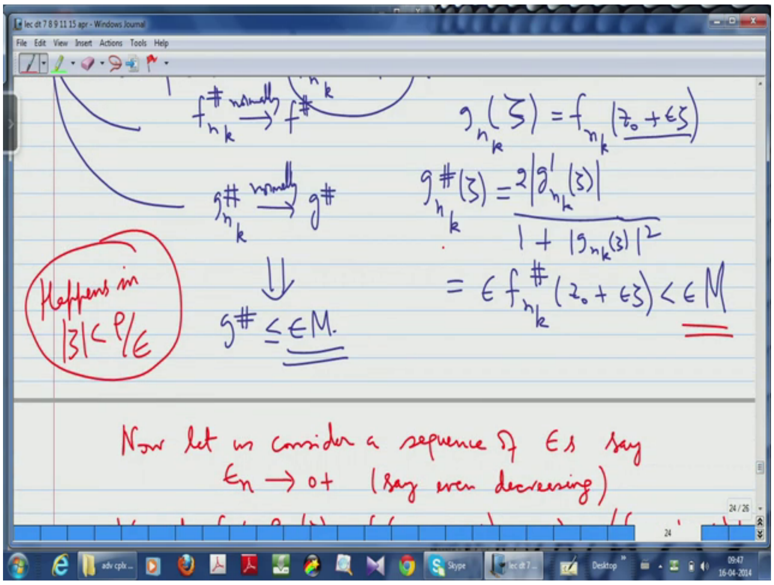

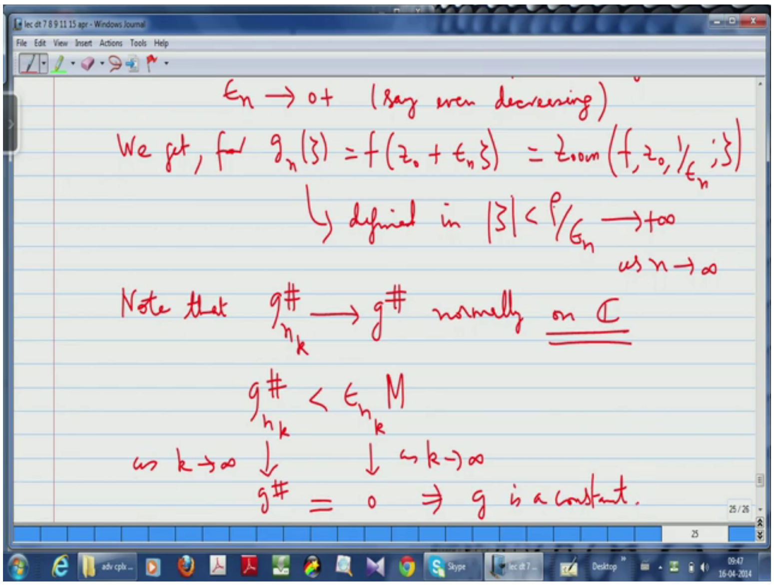

Now let us consider a sequence of Epsilon say Epsilon n going to 0, so Epsilon going to 0 plus so they are all positive (())(34:29) and you know let me say even I mean say even decreasing okay anyway sequence going to 0 eventually it is going to be a decreasing sequence alright strictly decreasing sequence you can think of it as smaller and smaller neighbourhoods then what happens is that you we get for you know so here is the funny thing you define g m all Zeta to be you zoom f so you take f you write Z naught plus Epsilon n Zeta, this is actually the zooming of the function f centred at Z naught scaling factor is now 1 by Epsilon n and the variable I am using a Zeta okay and mind you this 1 by Epsilon n is going to go to infinity because Epsilon n is going to 0 plus 1 by Epsilon n is going to infinity that means I am actually I am doing ultra-zooming you know at the point Z naught and I am looking at the zoom functions alright.

Now see this is defined as I told you this is defined in mod Zeta less than rho by Epsilon n and the point is that this goes to infinity this goes to plus infinity as N tends to infinity because Epsilon N goes to 0 rho is a fixed quantity alright and note that what will happen is that g n k hash will go to g hash as before okay this will be normal but the beautiful thing is this will be normally on C on the whole of C that is the beautiful part. Earlier this g n k hash going to g hash was on only on this bounded domain it was on mod Zeta less than rho by Epsilon, it was normal on the convergence g n k hash going to g hash was…it was normal convergence on this bounded domain but now since this Epsilons are becoming smaller the g n are you being defined on bigger and bigger domains okay and that is an increasing sequence of open disk centred at the origin that will eventually cover the whole plane.

So it will eventually cover any compact subsets of the plane and therefore on any compact subset of the plane I can say that this sequence is going to converge normally because I will have to consider the sequence only beyond a certain stage okay. See whenever you are looking at convergence the 1^{st} finitely no matter how large, the 1^{st} finitely many terms do not matter okay, so the moral of the story is that the advantage of taking smaller and smaller Epsilons is that you are zooming into the point, you are zooming into the behaviour of the functions at that point but then you get a limit function which is defined on the whole plane, so you get a Meromorphic function on the plane, so this g it will be a Meromorphic function on the whole plane okay g will be a Meromorphic function on the whole plane and the nice thing is that as I was telling you the bound for the spherical derivatives of g n will be what?

It will be Epsilon n k times m, so you will get g n k hash is bounded by Epsilon n k times m okay because you know earlier we got the bound as Epsilon m for g n k okay but now the Epsilon is Epsilon n k in this case so I will get this okay and this goes to g hash and as k tends to infinity and well this was to 0 as k tends to infinity. So the moral of the story is g hash 0 and this implies that g is a constant and when I say g is a constant mind you g could be the constant function which is identically infinity that is also allowed okay. So you know how do you argue that g is a constant mind you g becomes G… anyway the limit function g is Meromorphic on the whole plane okay.

So it has only poles on the plane okay and outside the poles it is analytic function okay but outside the poles the spherical derivatives is 0 means ordinary derivatives is 0 okay and if ordinary derivative is 0 the function is constant, so except for these isolated set of points which are poles everywhere else the function is constant and those points cannot be there you cannot have (())(39:45) on a single pole because if you have a pole as the points approach the pole the function goes, the modulus of the function value goes to infinity. How can it be constant in a deleted neighbourhood of a pole? So that forces the function has to be identically constant that is how you get g is identically constant g is only a Meromorphic function okay. So this is the important thing, the important thing is that if you take a normal family, so what have we prove?

You take a point where a family is normal okay and you take any sequence in the normal family you can find a normally convergence subsequence. If you take those sequence of functions and do zoom in at the point you are able to construct…see the zoom functions will converge to a constant function that is what it says okay. If you look at normally convergent family of Meromorphic function zoomed at a point you will get only a constant function limit that is what you are saying okay this is the behaviour of a normal family at a point okay and what I want to do next is introduce another level of complication okay and this next level of complication is to vary Z naught also okay.

So instead of considering a point Z naught all this time I had fixed Z naught but now what you do is you take a sequence point Z N which goes to Z naught okay and at each Z n you zoom to Epsilon N or the function f n and do this process okay and you will still get the same result okay and the beautiful thing is that whatever you get there okay that behaviour is good enough to characterise normal families okay and the point about Zalcman's Lemma is the negativity of that statement okay, so roughly Zalcman's Lemma is a Lemma that is actually a theorem which will explain when family is not normal it will give you a (())(41:52) sufficient conditions for a family to be not normal and guess what the condition will be, the condition will be at you can do all this but the limit function that you get g it will be non-constant that will be the only difference.

You will get a limit function which will be a non-constant Meromorphic function, so its spherical derivatives will not be 0 okay and that is the characterisation of non-normality okay. See this is the motivation for Zalcman's Lemma okay, so in my next lecture I will you how to instead of considering Z naught consider a sequence Z n which goes to Z naught and have the same argument okay and continue with Zalcman's Lemma alright, so I will stop here.