Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

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Lecture No 36

Normal Sequential Compactness, Normal Uniform Boundedness and Montel's

Alright so what we did do now is worry about giving the statements of… and of course the proof of you know Montel's theorem and Marty's theorem in the case when the domain includes the point at infinity okay, so if you recall see all this time we have been worrying about domains in the complex plane but since the last lecture we also wanted to include the infinity as a point in the domain that means you are looking at a domain in the extended complex plane alright.

See the problem with including infinity is that you know compact neighbourhood of infinity make sense only in the extended plane okay and because it corresponds to a compact neighbourhood of the North pole on the Riemann sphere, the extended plane being identified with the Riemann sphere okay but if you take a compact neighbourhood of infinity and delete infinity okay then what you will get is you will get an unbounded domain in the usual complex plane, so basically it is an unbounded domain and along with of course the boundary, so on an unbounded domain we generally do not expect uniform versions okay. We expect uniform convergence only on bounded domain especially on compact subsets okay and of course you know compact implies closed end bounded, so since you are in Euclidean space is same as closed end bounded.

So you should not expect uniform convergence on an unbounded domain okay that is the rule you will get it only on compact subsets, so if you take a compact neighbourhood of infinity okay then you will see that it is too much in general are just too much to expect uniform convergence because it is unbounded as far as the if you look at it from the complex plane point of view any compact neighbourhood of infinity for that matter any neighbourhood of infinity will be an unbounded set in the usual complex plane okay. It will be bounded only with respect to the extended complex plane okay, so that is the reason why we defined what is meant by normal uniform convergences of a sequence or normal convergence of a sequence of functions defined on a domain which contains the point at infinity okay.

So now we will did it was if you remember if D is the domain in the extended complex plane of course and of course if infinity is the point of the domain of then what you do is that you consider 2 things you 1st of all remove infinity and you get D minus infinity that becomes a domain in the usual plane and for such a domain you know what uniform convergence of compact subset normal convergence means, so you make that definition and then to deal with you know the point at infinity to deal with normal convergence at infinity, what you do is at you invert the variable, so what you do is you take a neighbourhood of infinity, you take a neighbourhood of Z equal to infinity and treat it as a neighbourhood of w equal to 0 by putting Z equal to 1 by w okay and then you say now is neighbourhood of 0 is anyway it is anyway a neighbourhood in the complex plane.

So it makes sense to talk about normal convergence okay. So you define sequence of functions converging normally on a domain the extended complex plane containing the point at infinity if it individually it converges on D minus infinity and the sequence with the variable in from Z to 1 by w converges again converges normally in a neighbourhood of w equal to 0 which corresponds to a neighbourhood of Z equal to infinity and then you know with this modification you found that you know the limit function to which if the original functions are already continuous or example which is the case when you have an analytic function of Meromorphic functions than the limit function is also continuous and the limit function is of course unique and continuous because it is a two-piece definition.

There is a definition for the domain minus the point at infinity and there is another definition for a neighbourhood of infinity okay, so in principle you could have got 2 different functions but because of continuity you will get a unique function and therefore what we did we were able to extend these important results namely that you know if you have a sequence of analytic functions on a domain in the extended complex plane, if that sequence converges normally on the domain then the limit function is either analytic or it is identically equal to infinity and we also prove the same thing for Meromorphic functions if you have sequence of Meromorphic functions on a domain in the extended plane and if that sequence converges normally then the limit channel is either Meromorphic or it is identically equal to infinity.

Now what we need to do is, we need to worry about Montel's theorem and Marty's theorem which is a Meromorphic version of Montel's theorem do you know in the case when the domain of definition of the functions of family of functions is domain in the extended complex plane okay. So for that we will have to define what is meant by normally sequentially compact or family of functions defined on a domain in the extended plane and that involves a little bit of subtlety but anyway as we will see everything works out well.

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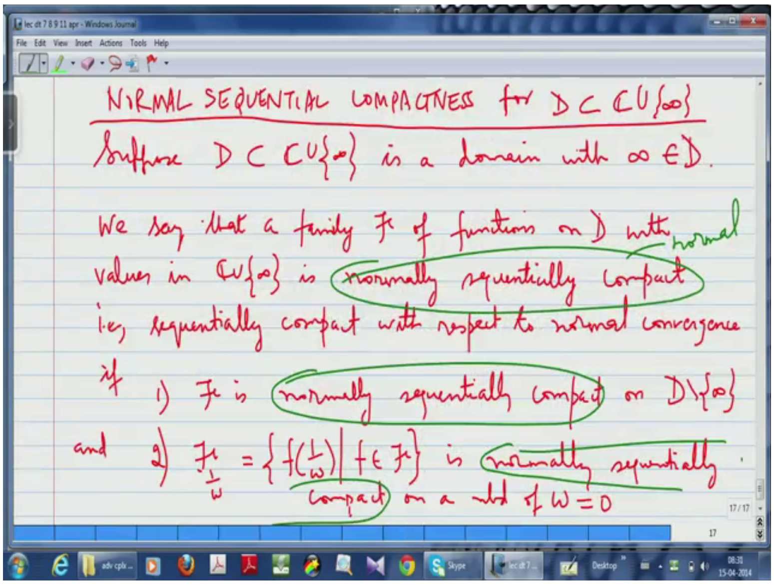

So let me write this down so suppose D in the extended complex plane is a domain of course nonempty it is an open connected set and let us assume that infinity is a point of D, so this means that if you remove infinity from D what you will get on the complex plane is an unbounded domain okay and what we want to do is suppose you have family of functions depend on this domain I would like to say what it means or the family to be normally sequentially compact okay because after all Montel's theorem, the usual Montel's theorem and Marty's theorem are just you know the correct generalisation of the Arzela-Ascoli Theorem okay which says that you know.

So Montel's theorem says that for analytic functions you know normally sequentially compact okay that is sequential compactness with respect to normal convergence that is the same as uniform boundedness on compact subset that is normal uniform boundedness for the family okay and Marty's theorem extends this from analytic functions to Meromorphic functions, so one can always be very (())(8:27) and say that well my definition of family f being sequentially compact with respect to normal convergence is just that it should be you know sequentially compact with respect to normal convergence then infinity is removed okay.

So on the domain punctured at infinity and to deal with the point at infinity you say that the same family you take the same family and change the variable to from Z to 1 by w and you say for this new family defined in a neighbourhood of 0 again you should have you know sequentially compact with respect to normal convergence and this is the definition that you will make in line with what we have been doing and you will see that it is the correct definition make okay, so let me write this down we say that our family script F of functions on D with values in this C union infinity that is extended complex plane is normally sequentially compact i.e. sequentially compact with respect to normal convergence if number 1 so f is normally sequentially compact on D the domain D punctured at infinity and now I will take care of the point at infinity and for that what I will do is that f let me put this as f sub 1 by w this is set of all f of 1 by w where f belong to script F.

So I put the subscript 1 by w to tell you that I have changed I have inverted the argument the independent variable in the function okay is normally sequentially compact on a neighbourhood of w equal to 0 because you know a neighbourhood of w equal to 0 will correspond to a neighbourhood of Z equal to infinity because w is 1 by Z alright. So this is the… So I should put the heading as normal sequential compactness for domain in the extended complex plane, so that is the heading, right? So well this seems to be the right definition to make, whenever you want to deal with a domain which contains a point at infinity you delete with it in 2 pieces one is you throw the point at infinity okay you throw it away namely you punctured the domain at infinity.

So you get a deleted neighbourhood of infinity and you give a definition for that and that is easy to give because deleted neighbourhood of infinity is also a domain in the usual plane okay and the other thing that you do is that to consider a neighbourhood of infinity, you consider a neighbourhood of 0 by inverting the variable okay, so well now so there are couple of things that I want to tell you with respect to the notation terminology and literature and also there is other subtlety that I want to point you about, so the 1^{st} thing is that you know in the literature there is normally there is normal sequential compactness is actually abbreviated to normal okay, so this is very important thing.

See people just use the word normal this is in fact Montel’s terminology that instead of every time saying normally sequentially compact you use the word normal and this is being thought of as property of the family, so when you say family is normal it means it is normally sequentially compact okay, so wherever normally sequentially compact comes you know then you can just replace it with the word normal okay and the whole point is that the family being normal is the correct notion of compactness of a family okay that is the whole point okay, so you know if you are working with general topology and if you are working with continuous functions say real valued or complex valued functions and you are working on a compact metric space or for that matter you are even working with continuous functions with taking values from one compact… I mean it takes values in another compact metric space okay and usually compactness because you are in the context of metric spaces, compactness is the same as sequential compactness okay.

So and there what happens is that you when you say sequential compactness the idea is that you are able to say that every sequence admits a convergence of sequence okay and this convergence is with respect to one example if you are working with complex valid function it is with respect to supremum norm okay, so there is a supremum norm which induces a metric and its convergence with respect to that metric okay but of course when we are considering the extended complex plane we are using the spherical metric that is one thing that you must always member okay, the extended complex plane is compact metric space and the metric you are using on that is the spherical metric which is actually the spherical metric on the Riemann sphere transported the extended complex plane okay by the identification of the Riemann sphere with an extended complex plane using the stereographic projection okay.

Now the point is that this is what you will get if you are looking at continuous functions but if you are looking at an analytic function the rule is that you cannot expect a uniform convergence on unbounded sets, you can expect uniform convergence only on compact subsets and this is called normal convergence. So when you are… In the context of analytic functions or in the context of Meromorphic functions okay then you have to worry only about uniform convergence on compact subset and that is called normal convergence okay and you have to do everything normally okay and the point is therefore in the context of analytic functions or Meromorphic functions the correct notion of compactness is not sequential compactness but it is sequential compactness restricted to compact subsets and that is called normal sequential compactness and Montel's terminology is that you do not say normally sequentially compact you simply say normal okay, so this is one this is something about terminology that you must know.

Then the other thing is of course about these subtlety in talking about normal sequential compactness, the subtlety is you see…when you say normally sequential compact means that it is sequentially compact when restricted to compact subsets and what does sequentially compact… I mean if you blindly read or any property P normal P means the property P is supposed to hold when restricted to compact subsets, so if you go by that normally sequentially compact will mean that you know it will mean that it is sequentially compact when restricted to compact subsets, so what does that mean?

It means at you give me a sequence okay and if you are restricted to a compact subset I will get a convergence of sequence okay but if I restrict to different compacts if I start with the given sequence okay functions and I if I restrict to different compact subsets I may get different convergence of sequences okay but the fact is that it is more than that you can get uniformly subsequence which will work on, which will converge on every compact subset and this is actually you know another diagonalization argument that we used if you check.

So actually you know normally sequentially compact is as strong as sequentially compact with respect to normal convergence okay when you say sequentially compact with respect to normal convergence what you mean is that given a sequence I can find a subsequence whose convergence is normal, normally convergence subsequence which means that same subsequence will converge when you restrict to any compact subset but when you say normally sequentially compact you might interpret it as you know give me a sequence for every compact subset I will get the convergence of sequence but it looks as if you change the compact subsets the convergence of sequence could change but the truth is that there is not much difference because you can use always a diagonalization argument okay and you can use a sequence of compact subsets that fill out and increasing sequence of compact subset that fill out your domain, that is the argument that we use.

So there is really no confusion in saying normally sequentially compact and sequentially compact with respect to normal convergence that is really no difference okay that is the cost of this diagonalization argument that you can apply on a sequence of compact, increasing sequence of compact subset that can cover your domain, so that is one subtlety then here comes the other technical issue, the technical issue is that you know again because you are dealing with a point at infinity we have defined normal normality in 2 pieces, we have defined normality outside infinity that is the 1st requirement and the 2nd requirement is normality at infinity okay. Now think of it for a moment, what does it mean? It means suppose I start with sequence my family, the normality outside infinity will give me a subsequence which will converge on compact subsets said infinity okay and what will happen is separately for the same sequence of functions I will get another subsequence which will converge normally at infinity okay.

Now I seem to be getting 2 different subsequence okay and I do not seem to be getting a single subsequence which will converge both outside normally outside infinity and also at infinity I do not seem to be getting that and the fact is that you can do this okay it is not much of discrepancy because you see suppose you have a family script F which is normal in the sense okay what you do is? Start with a sequence in the family okay first go to a neighbourhood of infinity okay go to the 2nd condition namely you go to a neighbourhood in infinity which is part of as a neighbourhood of 0 with the variable inverted, there you 1st pick a subsequence which is you know which converges normally at infinity okay.

Then what you do this subsequence is also anyway a subsequence of the original family which is normal outside infinity okay, so did the same subsequent and now apply it to the domain outside infinity and you get a further subsequent which will converge normally outside infinity, so this new subsequent that the picked up that will be one which will converge both outside infinity and at infinity okay so even though you are doing it piecewise everything works out fine okay, so you do get a global give me a sequence you do get global subsequence okay which will converge normally both outside infinity and at infinity okay there is no confusion alright. Only thing is you have to do this twice alright.

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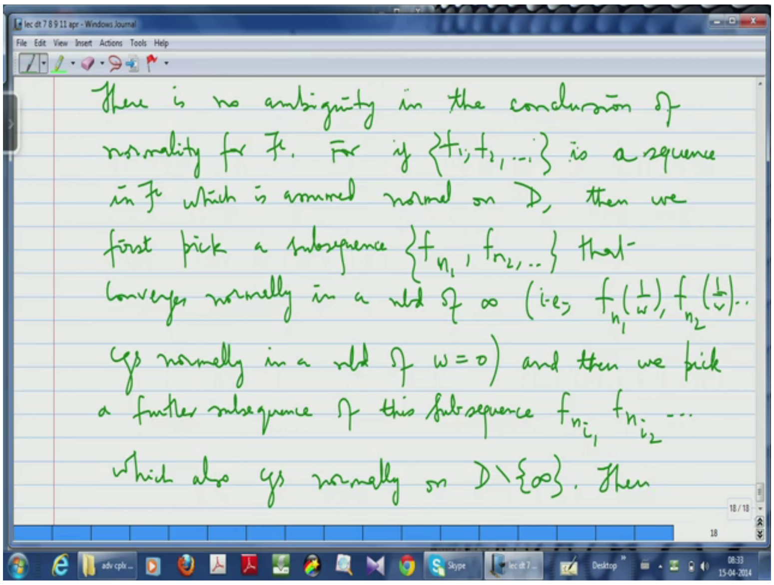

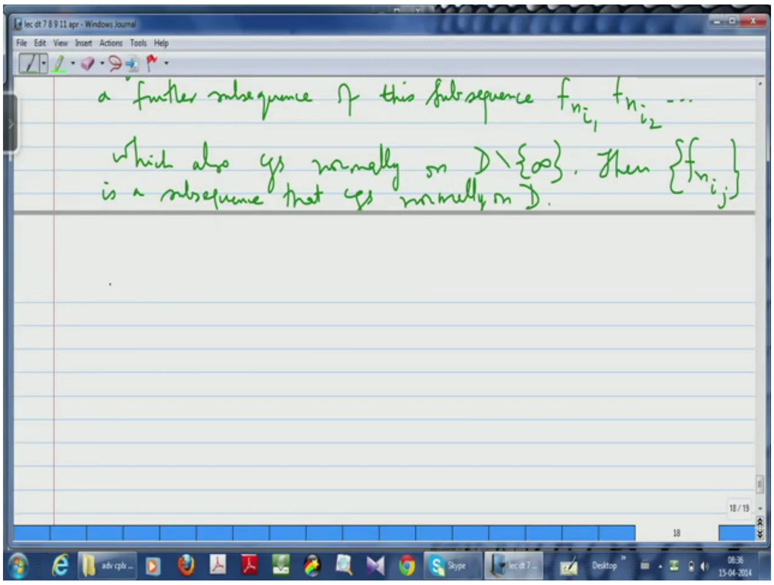

So let me write that down there is no ambiguity in the conclusion of normality for f for if f 1, f 2 and so on is a sequence in f which is assumed normal on D then we 1st pick a subsequence f n1, f n2 and so on that converges normally in our neighbourhood of infinity which means that f n1 of 1 by W, f n2 of 1 by w and so on converges normally in our neighbourhood of w equal is to 0 this is what it means okay and then we pick a further subsequence of this subsequence, so f n i 1, f n i 2 and so on and which also converges normally on D minus infinity then the subsequence f n i 1, f n i 2, so let me use is a subsequence that converges normally on D okay, so there is no problem alright you are able to get one subsequence that will work both outside infinity and at infinity okay, so this is a little fact that you need to know, so you know see this is just a part of usual philosophy in mathematics.

So there are 2 things that I want to say usually what happens is (())(24:47) properties of function is hold an open set, the other thing is to verify that a good property is true you verify it only locally that means you can verify it at each point in a neighbourhood on an open cover okay. For example this is the case with continuity okay or analyticity and so on, good properties if you check a function is analytic you check at each point or in a neighbourhood of each point. If you want to check a function is continuous it is enough you check at each point okay, so the same way you see this normal the idea of a normal family is also local okay if you say that a family is you know normal in pieces okay that is it is normal on an open cover alright then it continues to be normal.

So for example what we are saying is that if a family is normal outside infinity and if a family is normal in the neighbourhood of infinity okay this outside infinity and neighbourhood of infinity together constitute a cover okay and for the whole domain and when you say the function is normal outside infinity and at infinity okay then you are getting it is normal on the whole domain okay, so you see normality is a local property that is what is happening alright and it is a good property and all good properties are usually local properties you can verify them locally and they are valid on open sets okay, so this is something that you should remember as a philosophy. So this should have been normally alright okay so now what I am going to do is now let us go on with now you know with this background it is very easy to write out the analog of Montel's theorem and Marty's theorem for domain in the extended complex plane.

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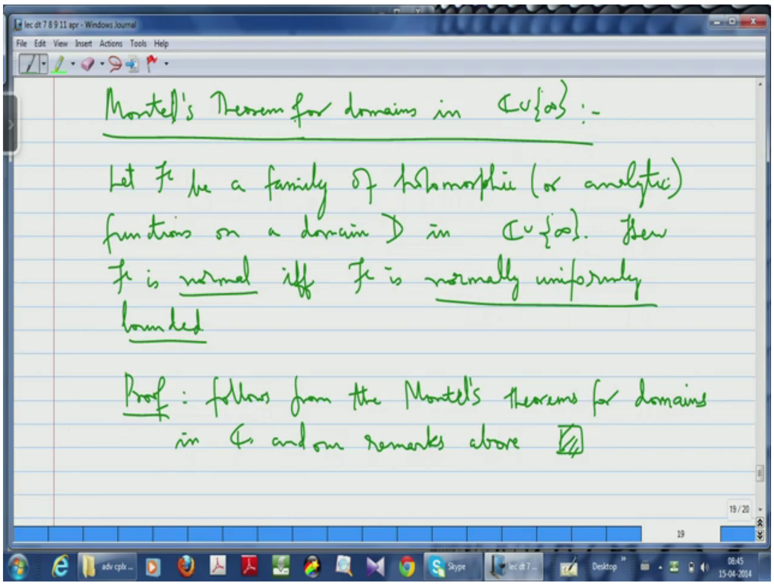

So here is Montel's theorem, so here is Montel's theorem so let me put it for as domains in the extended complex plane, what is the theorem? The theorem is let script F be a family of holomorphic or analytic functions on a domain D in the extended complex plane. Then script F is normal if and only if script F is normally uniformly bounded okay, so this is Montel's theorem okay that you say that, so where you say that normality of the family is the same as normal uniform boundedness of the functions in the family alright and mind you normality in the family means that it is normally sequentially compact that is it is sequentially compact with respect to normal convergence alright every sequence admits a subsequence which converges normally that is a subsequence which converges uniformly on compact subsets alright and normally uniformly bounded is uniformly bounded on compact subsets alright and this normally uniformly bounded on a domain in the extended plane again how do you define that? You define it piecewise.

So what you say is that if you want to say a family of functions is normally uniformly bounded on a domain in the extended plane, what you do is that you 1st say that it is normally uniformly bounded in you throughout infinity, so it should be normally uniformly bounded on D minus infinity and then you say that the corresponding family with the variables inverted okay is normally uniformly bounded in a neighbourhood of 0 for the inverted variable which corresponds to a neighbourhood of infinity for the original variable okay.

So this normally uniformly bounded also needs to be defined when you consider the point domains which contains the point at infinity but again you define it in 2 pieces you make one definition outside infinity and then at infinity the definition you make is by inverting the variable so that you change Z equal to infinity, neighbourhood of Z equal to infinity is a neighbourhood of infinity you change that to a neighbourhood of w equal to 0 where w equal to 1 by Z or Z equal to 1 by w okay, so this is Montel's theorem and you can see that you know the proof of this theorem just comes in the usual Montel's theorem because finally what you have done is to deal with the point at infinity you are actually going back to 0 by inverting the variable.

So the moral of the story is that by this device you are able to deal with the point at infinity, so the proof of this Montel's theorem will follow from the usual Montel's theorem okay and of course you have to worry about one thing namely that when you come from normal uniform boundedness to normality okay what you will get is that you will get separately normality outside infinity and you will get separately normality at infinity but we just discussed at normality is the local property, so you will get normality on the whole domain okay so everything works out fine alright.

The only thing that I want you to remember is that at the back of all this how does this differ from the usual Arzela-Ascoli Theorem. The Arzela-Ascoli Theorem says that you know if you want compactness then it is the same as sequential compactness and that is equivalent to you know uniform boundedness an equicontinuity okay, so you need equicontinuity alright and of course the usual Arzela-Ascoli Theorem is for functions defined on compact metric space but then here you are working with functions on a domain certainly not a closed set okay it is an open connected set and therefore what you will have to do is that you will have to go from usual convergence you have to go to normal convergence, so you must…instead of expecting uniform convergence on the whole domain you should except only uniform convergence on compact subsets of the domain and then the big deal is as usual you know the you do not have to worry about equicontinuity because equicontinuity is automatic.

In the case of analytic functions you know if the function are original functions of bounded then the derivatives also become bounded and that is because of the Cauchy integral formula and the Cauchy estimates for the 1st derivatives which can be expressed as an integral of the original function okay therefore if you have a normal uniform boundedness of the functions then you have normal uniform boundless of the derivative and the normal uniform boundedness of the derivatives always implies a equicontinuity okay so you get equicontinuity for free okay and therefore the only important condition is the uniform boundedness, normal uniform boundedness of the family and that is the whole point about Montel's theorem and then the extension of this theorem to Meromorphic functions is of course Marty's theorem.

So that also works okay but the point is that when you go to Marty's theorem you will have to worry about not using usual derivatives because what you are working with are Meromorphic functions and at a pole they are not differentiable and then you know the trick is to not use the usual derivatives but use the spherical derivatives okay and then you get the analog of Montel's theorem for Meromorphic functions okay and that is Marty's theorem, so let me write this down. So here let me just mention that proof follows from the usual Montel's theorem for domains in the complex plane and our remarks about okay and then now let me go onto Marty's theorem.

In Montel's theorem you know you can allow the functions you take only complex values because they are analytic functions okay there is no question about taking the value infinity alright, so whereas if you go to Marty's theorem you are considering Meromorphic functions and to make such a function continues at a pole you define its value at the pole to be infinity, so you will have to necessary look at functions with you know values in the extended complex plane okay, so here is the statement similar to the statement of Montel's theorem, so in Montel's theorems we say f is a family of holomorphic function on a domain in the in the extended complex plane, now you will say f is a family of Meromorphic functions on a domain D in the extended complex plane okay but taking values in the extended complex plan you will have to say that okay and then you also say that then of course the statement is that the family is normal if and only if the family of spherical derivatives is normally uniformly bounded that is the theorem okay so let me write that out.

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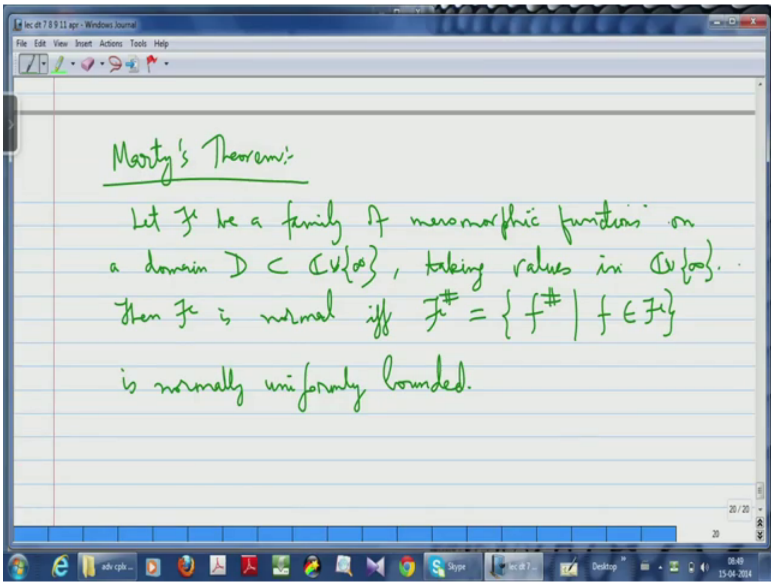

Let script F be a family of Meromorphic functions on a domain D in the extended complex plane taking values in the extended complex plane this. Then this family is normal if and only if the corresponding family of spherical derivatives namely you take the spherical derivatives of the functions, the original family is normally uniformly bounded okay so this is Marty's theorem, so again let me pinpoint a couple of things, the 1st thing is that the reason why I am repeating this several times is I want you to note that there is a I want you to notice these important differences, see when you looking at Meromorphic functions necessarily you will have to use values in the extended plane or 2 reasons the 1st reason is that you want the Meromorphic function to be continuous at a pole so you define the value at the pole to be infinity okay that is one thing, the 2nd thing is that in the target space you want to have a metric.

So you know I want to a function if you are looking at a Meromorphic function at a pole it can take the value infinity okay, so I will have to compared infinity, the value infinity with finite values, with finite complex values and the only way I can do that is by using the spherical metric which is available on the extended plane master is very important that you have to take values in the extended plane and you have to use the spherical metric this has to be done if you are working with Meromorphic functions okay and then the other important thing is that you need to worry about not the usual derivatives at the spherical derivatives, usual derivatives of course will not work because usual derivatives will not… they will not even be defined at a pole okay.

So you will have to work with the spherical metric which is define even at a pole and I told you there is no problem with the spherical derivative it is always continuous and even at a pole is defined by continuity, the pole is a simple pole then the spherical derivative is 2 divided by the modulus of the residue at that simple pole for that function and if the pole is a pole of higher-order the spherical derivative is 0 at that pole, so and this is done in a very continuous fashion alright, so the point is spherical derivative is a continuous function, continues nonnegative real valued function and that is these are the functions that we have to ensure the uniform bounded of the spherical derivatives okay when restricted to compact subset okay.

Now that is normal uniform boundedness of spherical derivatives is what we want and that is equivalent condition to the original family being normal namely that the original family you know is normally sequentially compact okay, so this is Marty's theorem alright. Now so you know this brings us to a very important point namely at this point you are able to get all the theorem is that you want okay by you know including functions which can take the value infinity okay namely Meromorphic functions and you can also have functions defined at infinity okay.

So what you must understand is that all these lectures all the rules that we built were to tackle 2 things first of all you wanted to tackle the function in a neighbourhood of infinity okay that means you wanted to study functions in the neighbourhood of the point which is infinity for example that is why we initially were worried about trying to define when infinity is an isolated singularity and if it is an isolated singularity, what kind of singularity it is okay and the track in all these cases was replace (())(40:19) Z equal to infinity by the neighbourhood of w equal to 0 where w equal to 1 by Z okay and you know you must understand that this Z going to w that is kind of it is a homeomorphism okay of the Riemann sphere onto itself.

If you want it is a self-homeomorphism of the extended plane onto itself which you can think of it also as a isomorphism of self-homeomorphism Riemann sphere onto itself and what happens is that infinity is identified the North pole origin 0 is identified to the South pole okay and it is just this the inversion is just switching North and South poles of the Riemann sphere and this inversion is a very nice thing, it does not change the spherical distance because essentially inversion corresponds to a rotation of the Riemann sphere by 180 degrees with respect to the real axis the x-axis okay.

So the point is that you can deal with the point at infinity okay that is one aspect then the other aspect is that you can deal with these functions which have poles namely Meromorphic functions by allowing the value infinity okay, so on the whole we have built up all these goals to deal with Meromorphic functions even so you can look at a family of Meromorphic functions even defined in a neighbourhood of infinity and work with that okay that is the generality to which we have defined things now we will use all of this in the next few lectures prove the Picard theorems okay. So I will stop here.