Advanced Complex Analysis-Part 2
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Lecture 35
Normal Convergence at Infinity and Hurwitz's Theorems for Normal Limits of Analytic and Meromorphic Functions at Infinity

Okay, so what we need to do now as I was telling you in the last lecture is to work with domains which probably include the finite infinity, okay so you know this is a important turning point little technical but it is very easily understandable. So all this time you know so let me say few things when you probably do a first course in complex analysis you are working with a domain you are working with a function which is defined on a domain in the complex plane, okay which means it is defined on an open connected set ofcourse non-empty, okay and the function is taking complex values okay.

But then we want to work not just with analytic functions we want to work with meromorphic functions and the problem with a meromorphic function is at a pole the function goes to infinity and therefore at a pole you do not get continuity, okay so you are forced to include the point at infinity in the co-domain of the function okay and that leads you to look at functions at values in the extended complex plane, okay so that is what we are being doing so far, what we are doing so far is we have been looking at functions which are defined on a domain in the complex plane, ok but taking values in the extended complex plane, alright and for such functions we have proved lot of theorems.

So for example we have proved that you know we have proved that if you have a family of functions which converges normally, okay then if all the functions are analytic that is holomorphic then the limit function is either analytic or it is identically infinity. Similarly if all the functions are meromorphic, okay then the limit function is either meromorphic or it is identically infinity and then we have proved ofcourse Montel's Theorem that if the functions are all analytic if you have a family of analytic functions okay then every sequence admits a normally convergent subsequence if and only if the family that family is normally uniformly bounded, okay that is uniformly bounded on compact subsets that was Montel's Theorem.

And then we also proved Marty's Theorem which is much more stronger namely you take a family of meromorphic functions and assume that then at the condition that any any sequence in that admits a normally convergent subsequence is equivalent to a spherical derivatives being normally bounded and we (())(3:25) the spherical derivatives because for meromorphic functions usual derivatives will not be defined at the poles, okay.

So in all these statements the domain of the functions was always a subset of the complex plane and but ofcourse the co-domain we took it to be the extended complex plane and mind you therefore what is happening is in the domain the metric you are looking at is the usual eucledian domain because after all it is a subset of the complex plane where you have the eucledian metric.

Whereas in the co-domain the metric you are looking at is the spherical metric because the co-domain in the extended complex plane is identified by the stereographic projection with the Riemann sphere and you actually take the spherical metric on the Riemann sphere, alright which is the distance between two points being given by the length of the minor arch of the greater circle on the Riemann sphere passing through those two points, okay.

So this is the setup of things now what we want to do is we want to extend all these theorems, okay to the domain not only a domain in the complex plane but you want to extend it to a domain in the extended complex plane, okay that is the next step that means you are now allowing infinity as a value of the variable you are allowing infinity in the domain, okay and you want to write out the same you want the same results again and the fact is it is true it will work all these results will be true for a domain even in the even in the extended complex plane but then we need to fix it, okay.

So I will tell you where the problem lies, the problem lies in let us try to be knife and usually see the advantage of being knife is that you will think naturally, okay and the disadvantage is that it may not work but the advantage is that you will know where you go wrong and then you can correct yourself, okay so there is always an advantage to being knife in the first place. So suppose D is a domain in the extended complex plane, okay and certainly I am looking at a domain which contains the point at infinity, okay because if it does not contain a point infinity then it is a usual domain I have all the theorems for usual domains in the complex plane I have already proved, okay. So let me look at the domain in the extended complex plane which contains a point at infinity, okay.

Now I can just say a sequence of functions converges normally on the domain if it converges on compact subsets this is usual definition, okay but the problem is that this definition will not work, why it will not work is because if you take a compact subset of a domain even if you take a domain in the extended complex plane that compact subset cannot contain the point at infinity because any (())(6:01) infinity is unbounded, do you understand?

So therefore the problem is that if you say a family of functions or suppose you are looking at a sequence of functions and if you say if you say it normally converges on a domain which contains the point at infinity actually you are not taking care of the normal convergence at infinity because when you say it normally converges what does it mean it means that it is converging on compact subsets uniformly convergent on compact subsets but what are compact subsets of a domain even in the extended complex plane?

See if you look at it with respect to the usual eucledian plane, okay no neighbourhood of infinity, okay can be compact subset of the usual plane, okay. So what is happening is that even if you naively define that a sequence of functions is converging normally on a domain containing the point at infinity what you are actually defining is only that it is that this sequence of functions is converging normally only on the domain the punctured domain with the point at infinity removed, okay.

So you are not able to take care of normal convergence at infinity that is the problem, okay. So how do you tackle this? So the way of tackling this is is again the same old philosophy of you know how to tackle the point at infinity, you tackle the point at infinity by you know inviting the variable and looking at the point at 0, okay. So you know whenever you want to look study f of z at z equal to infinity what was original idea we studied f of 1 by w at w equal to 0 and when the moment so the neighbourhood of infinity will translate to a neighbourhood of 0 and a neighbourhood of 0 is again now a good old neighbourhood in the good old complex plane and you can do work with it we have already proved theorems there, okay.

So here is the definition so the definition is suppose D is a domain in the extended complex plane containing the point infinity, when do I see a sequence of functions converges normally on D? I say that I have to say it in two pieces, I have to first say that the convergence is normal on D minus infinity, okay that is throw away infinity you are throwing away one point, okay so it is still an open set, okay mind you infinity is a closed point if you look at in the extended complex plane which is identified by the Riemann Sphere the infinity is identified with the north pole on the Riemann Sphere, okay.

So you take D minus infinity that is an open that is a nice domain in the complex plane, okay it is continue it is going to just by removing the point you cannot make it disconnected, okay and because it is open okay so it is still going to be open connected. So it is a domain in the complex plane you require that on that domain punctured at infinity the given family converges normally, okay which means you are requiring that that is uniform convergence on compact subsets of on compact subsets of the plane which intersect D that is all, okay that is one condition.

The second condition is you take the same sequence of functions change the variable from z to 1 by w and say that that converges normally in a neighbourhood of the origin, okay. So you give a definition outside infinity and you give a definition in a neighbourhood of infinity by translating it to a neighbourhood of 0 and this is the definition that we will make and this is the definition that works, okay you will see that with this definition you can translate all the theorems that we have proved you can just use all the theorems that we have proved so far to get theorems for the case when the domain includes the includes the point at infinity, okay. So let me write this down.

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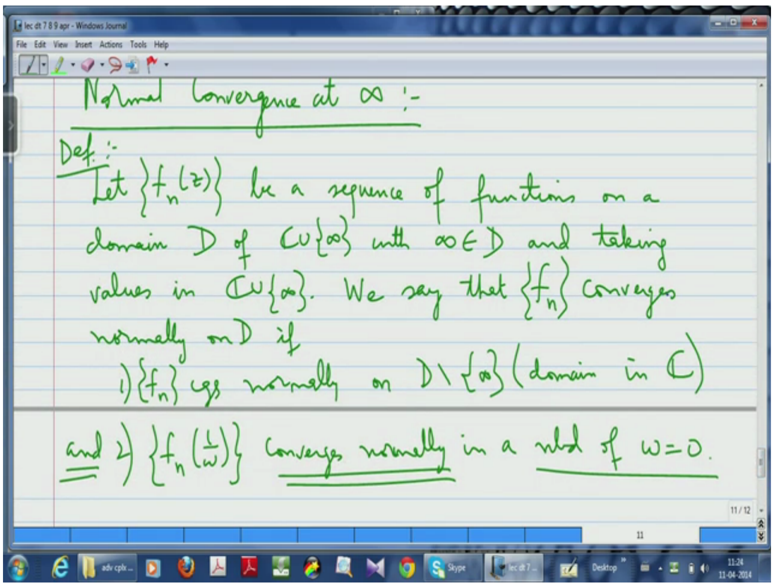

So the next thing is let me use a different colour normal convergence at infinity. So let f n of z be a sequence of continuous functions I do not even need continuous let me just say sequence of functions on a domain D of the extended complex plane with infinity in the domain, okay and taking values in the extended complex plane, okay. So basically you can think of f n as a sequence defined on and you know an open on a domain on the Riemann Sphere and taking values on the Riemann Sphere, if you think of the extended complex plane is Riemann

Sphere you can think that your store is a domain it is an open connected set on the Riemann Sphere, okay your D is on the Riemann Sphere it is an open connected set that contains a north pole which is supposed to correspond to the point at infinity and you have these functions each of these functions are defined on that D and they are talking values again in the Riemann Sphere which means they can take the value infinity, okay.

So you have to take a sequence of functions so when do we say that this sequence converges normally on D? So let me write that down we say that f n converges normally on D on D if number 1 f n so the sequence f n converges normally on D minus infinity which is D minus infinity is a domain in the (())(12:33) complex plane, okay and you know defining convergence on a domain in the complex plane is something that we have already done it is just uniform convergence on compact subsets, okay.

And and this and is very very important number 2, okay f n of 1 by w okay converges normally in a neighbourhood of w equal to 0. So here is the so the second statement is what actually takes care of normal convergence at infinity and you know it is very beautiful because normal convergence at infinity means you should ensure uniform convergence on compact subsets of infinity, okay but the problem is there is no compact subset of infinity that you can think off in the usual complex plane you can think of it only under Riemann Sphere.

So if you want it to translate it back to the usual complex plane you have to translate from neighbourhoods of infinity to neighbourhoods of 0 by making this inversion z going to 1 z going to 1 by z which is w, okay so you replace z by 1 by w, alright. So this is the definition, alright. Now comes now you see this definition is peace wise what you have done is you have got normal convergence outside infinity that is the first statement because outside infinity in the domain is again the domain in the usual complex plane you have no problems with that.

And the second part of the definition says you have convergence normal convergence at infinity, okay that is what the second one says because you know after all studying f n of 1 by w at w equal to 0 neighbourhood of w equal to 0 is a same of studying f of z in the neighbourhood of infinity, alright. But because it is a neighbourhood of 0 I know what converges normally means okay fine.

So this is the right definition and this definition works so I will put this as Def for definition so this definition works. But then there are certain remarks that need to be made so that you know you realize that you always make a definition if a definition has to suit a particular condition you make a twist in the definition when you have to check whether the definition is consistent.

So one of the thing that you can ask is the following. Since I have defined the normal convergence a sequence in two pieces can it happen that on each piece I get a different limit function? You can ask that and the answer is no, you cannot if you are working with continuous functions it cannot happen because of continuity, okay so this definition will work properly with continuous functions, okay so let me write this down.

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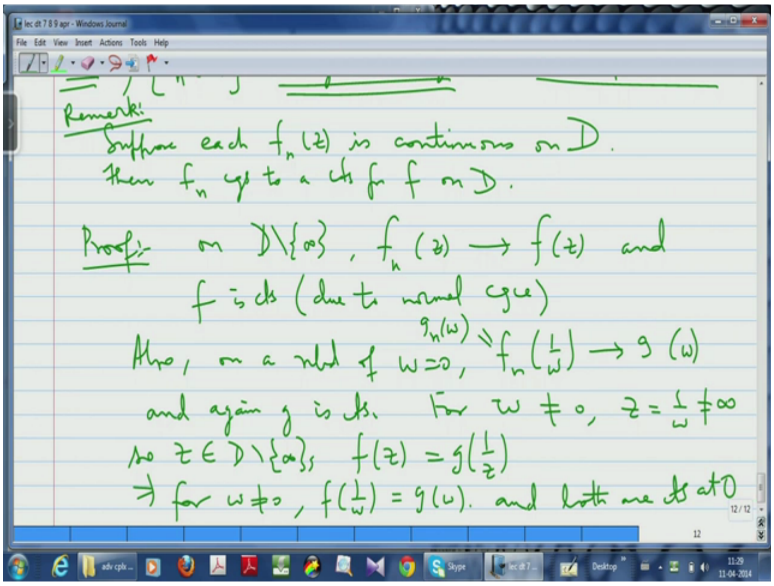

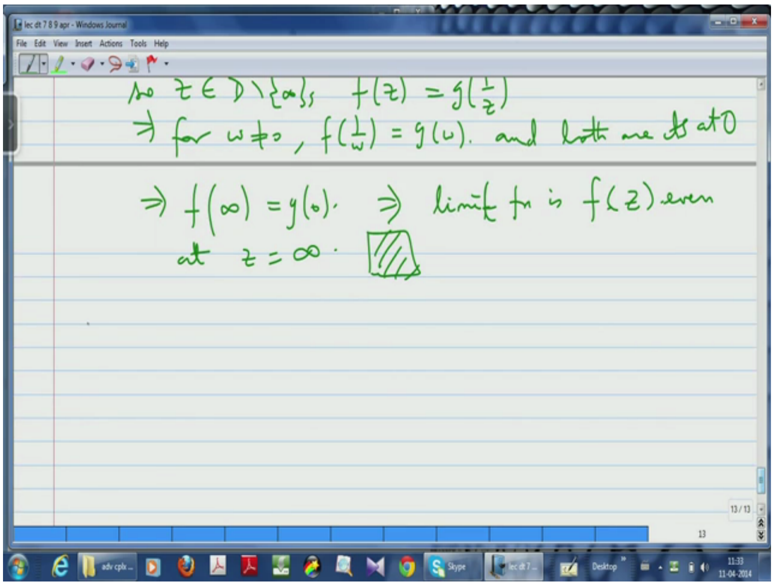

Suppose so here is a remark so the remark is suppose each f n of z is continuous on D, okay then f n converges to a continuous function f on D, okay so you get only one function, okay and what is the similar statements hold, similar statements hold okay let me say that because I need to expand on that, okay. So what is the proof? The proof is that well you see on D minus infinity, okay f n of z will converge to well let us say f of z alright and f z is continuous this is because of normal due to normal convergence you know a normal limit of continuous function is continuous and mind you should consider the limit function f also to be a function with values in the extended complex plane, okay.

And so this is because of part 1 of the definition, now part 2 of the definition will tell you that in the neighbourhood of the origin f n of 1 by w will also converge to something and I will call that as g of w, okay so also on a neighbourhood of w equal to 0 f n of 1 by w converges to g n of w sorry g of w so I in principle I should expect to get function g and again g is continuous at 0 and in fact g is continuous everywhere again because again I am using just the fact that normal limit of continuous function is continuous, okay.

So there is only one thing that for w not equal to 0, okay z equal to 1 by w is not infinity, okay so z lies in D minus infinity, okay and therefore what will happen is by the uniqueness of limits of a sequence point wise you will get that f of z will be equal to g of what is this g of 1 by z okay so this implies for w not equal to 0, okay f of 1 by w is same a g of so maybe I should call this as you know here I have to I will have to say the following thing I should call this as g n of w, okay I should call g n of w as f n of w and g n of w tends to g of , okay and the point is that this g of w is f of 1 by f of 1 by w is g of w that is what I am getting, okay if I put z equal to 1 by w, alright.

So but then you see and both are continuous at 0, okay both are continuous at 0 mind you because of continuity therefore limit w tends to 0, okay. So what this will tell you is that you know the so this will tell you that f of infinity will be g of 0, okay so this implies that the limit function the limit function is f of z even at infinity so you get a unique limit, okay that is the whole point.

So this normal convergence defining it peace wise that is one outside infinity and the other one in the neighbourhood of infinity by translating to a neighbourhood of 0, okay though it is a two piece definition normal convergence will still give rise to only one function but mind you all functions are being taken with values in the extended complex plane and that in the target the metric you are using is always a spherical metric, okay.

So you know unless you give a argument like this, okay things could go round see after all I have defined normal normality separately in two pieces normality I mean normal convergence in two pieces, okay and then it could happen that on each piece I could get different limits atleast what this tells you that it will not happen if your functions are continuous, okay and that is going to be the case because we are going to deal only with analytic functions or meromorphic functions and you know analytic functions are ofcourse continuous and even if you recall even if you take an analytic function at infinity by definition it is a function which is continuous at infinity you know and by version of the Riemann's removable similarities theorem saying that function of analytic at infinity means that it should be bounded at infinity, okay.

So it means that the function value at infinity is a finite complex number, okay so analytic function at infinity make sense. So analytic functions on a domain in the extended complex plane containing the point at infinity make sense for us, okay. And similarly meromorphic functions also make sense because what is the meromorphic function on a domain which contains a point at infinity it is you see it is supposed to be a meromorphic function on the punctured domain with the punctured infinity and you invert the variable and that should give me a meromorphic function at the origin, okay always you go to saying that a function is meromorphic at infinity is a saying f of z is meromorphic at infinity, okay in the neighbourhood of infinity is same as f of 1 by z is meromorphic at z equal to 0.

So you always translate back to at infinity you always translate back to at neighbourhood at 0 so it make sense, okay. So meromorphic functions on an extend on a domain in the extended complex plane containing the point at infinity make sense and all these functions with values in the extended complex plane also make sense. So we are in a perfect situation and all these functions are all continuous, mind you meromorphic functions are continuous because you allow at a pole you define the function value to be infinity and you allow the infinity to be in the co-domain of the function, okay so we are in the right set up.

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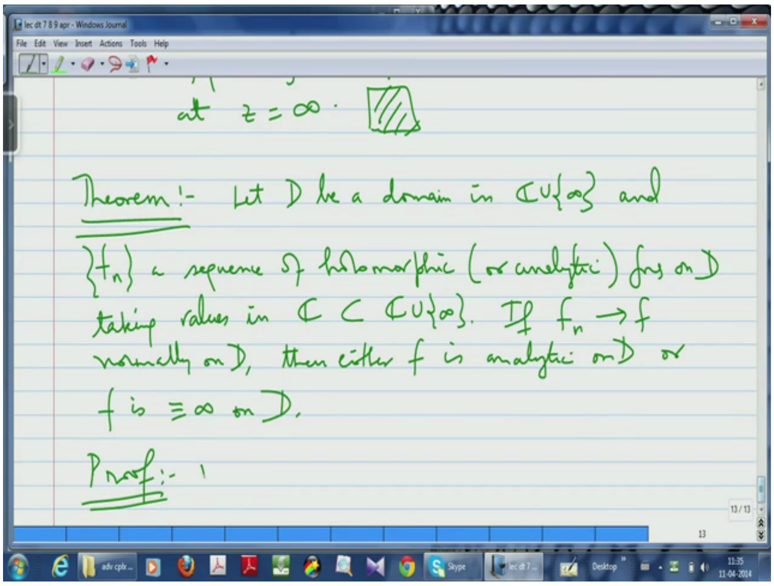

Now what we need to know is we need to check each of those theorems that we proved we need to deduce at those theorems also work for such a domain, okay so here is the first point. So theorem let D be a domain in C union infinity and f n a sequence so a sequence of holomorphic functions or analytic functions functions on D taking values in C considered as subsets of C union infinity, okay.

If f n converges to f normally on D then either f is analytic on D or f is identically infinity on D, okay. so you see the point is that we have already proved this theorem when D does not contain the point infinity when D is a subset of D usual complex plane we have already proved this theorem that is you take a sequence of analytic functions and you assume that it converges normally then the limit function has to be either analytic or it will be identically infinity, okay and you can get anything in between.

For example you cannot get a strictly meromorphic function in between that, okay and the reason is a pole cannot pop up at infinity I mean in the limit and if you remember this was because if you invert the variable 0 cannot pop up at the limit because of (())(24:50) so (()) (24:51) is working in the background, okay so we will have to only worry about the case when D contains infinity that is the extension we are particular interested in so let me write that down.

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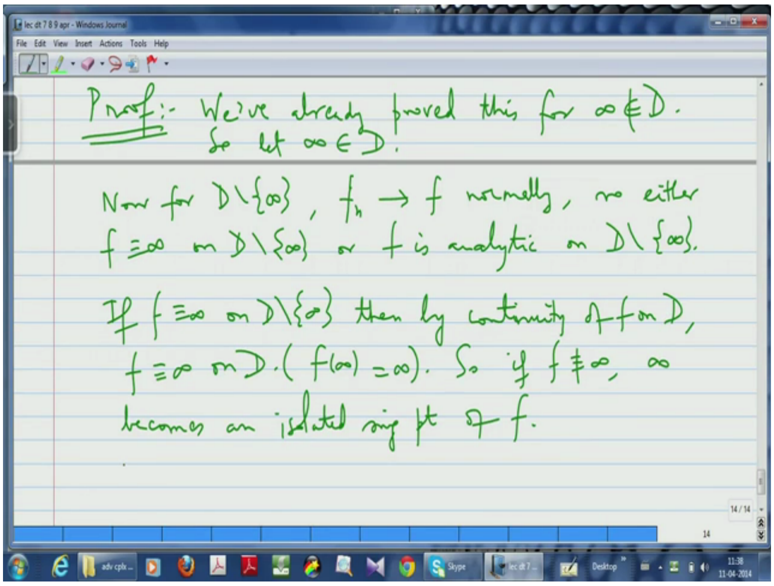

We have already proved this for infinity not belonging to D so we are the essential thing that so let the essential thing is we allow infinity to belong to D, okay. Now go back to the definition of normal convergence for a domain containing the point infinity what is the definition? First thing is you throw out infinity and on the remaining thing which is a domain in the complex plane there is normal convergence, okay.

And the other thing is that you take a neighbourhood of 0 and look at the functions of the variable invertible in the neighbourhood of 0, okay. So now so let me write that down now for D minus infinity f n will converge to f normally so either f is identically infinity on D minus infinity or f is analytic on D minus infinity, okay this is because of the fact that D minus infinity is the usual domain the usual complex plane and for the usual complex plane we have proved such a theorem, okay whenever you have normal convergence of analytic functions the limit is either identically infinity or it is identically or it is uniformly an analytic function, okay.

But you know if f is identically infinity on D minus infinity it will also be infinity at infinity because f is continuous, okay. So we have to only take care of the situation when f is not identically infinity and proof that f is analytic on even at infinity so infinity is the only problem, okay. So let me write that down if f is identically infinity on D minus infinity then by continuity of f on D, f is identically infinity on all of D because f of infinity will become infinity, okay.

So if f is not identically infinity, infinity becomes an isolated singular point of f okay because mind you f is analytic on D minus infinity D minus infinity is the neighbourhood of infinity it is a deleted neighbourhood of infinity D minus infinity is deleted neighbourhood of infinity, okay and f is analytic on that that means f is analytic in neighbourhood of infinity that means infinity is an isolated singular point for f and to check that f is analytic at infinity I have to only check f is bounded at infinity but why is that true that is because I have normal convergence at infinity, okay which is normal convergence if you change the variable to 1 by w and look at w equal to 0, okay.

Now we will use the second part of the definition of normal convergence which is compact convergence I mean all convergence at infinity we use that and then you are done, okay.

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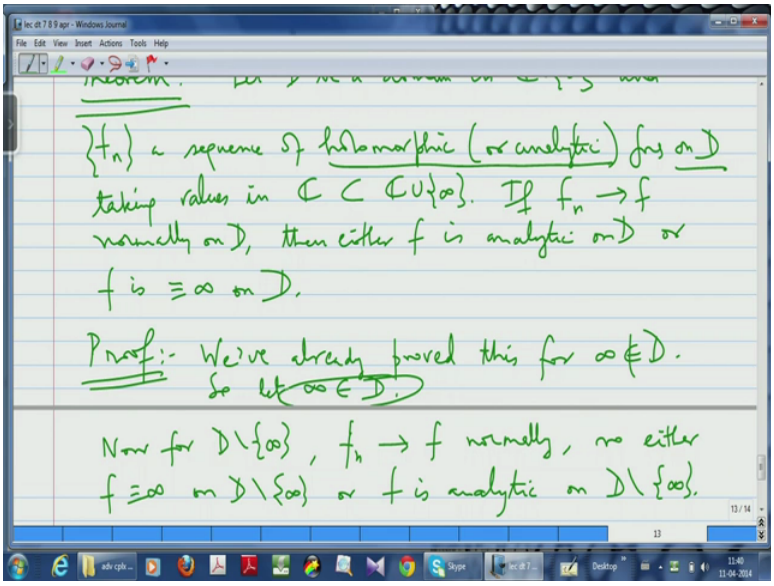

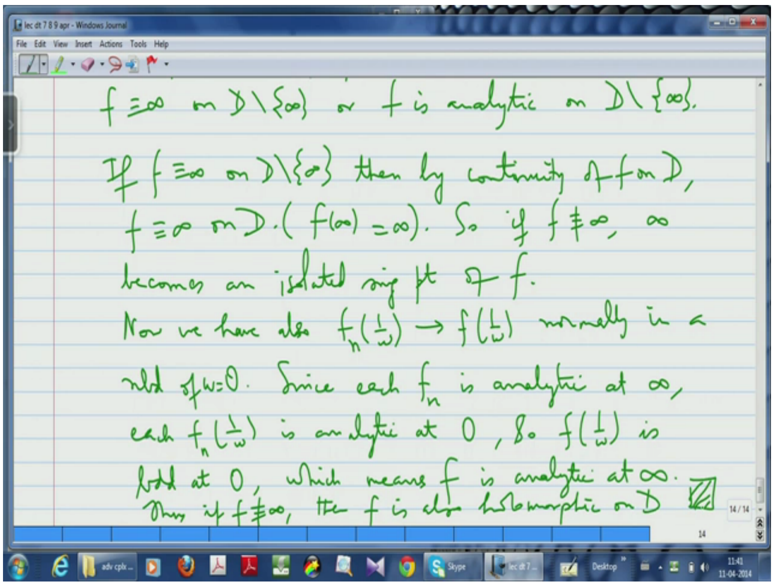

Now we also have we have also f n of 1 by w converges to f of 1 by w in normally normally in a neighbourhood of 0 of w equal to 0 this is the second part of the definition, okay and mind you that is a that is you could have taken a you could have taken a small enough open disk at the origin which will be a domain it will be connected, okay in fact it could (())(29:09) be simply connected, okay.

And now on that on that domain okay you look at the f n's okay the point is that each of the f n's is also analytic at infinity you see what look at what was given to me what was given to us, we have started with sequence of holomorphic functions on D and we are looking at infinity and we have assumed infinity belongs to D it means that each f n is already analytic at infinity each f n is already analytic at infinity, okay that means f n of z is analytic at z equal to infinity that means f n of z is bounded at z equal to infinity that means f n of 1 by w is bounded at w equal to 0.

And because the f is the normal limit of the f n's f of 1 by w will also be bounded at w equal to 0 and that is the same as saying that f is analytic at infinity and you are done, okay so that is it. So let me write this down. Now since since each f n is analytic at infinity, each f n of w is analytic at 0, so f of 1 by w is bounded at 0, which means f is analytic at infinity. Thus is f is not identically infinity then f is also holomorphic on D and that is the proof, okay.

So you see you are able to extend the theorem that we already proved that a normal limit of analytic functions can either be analytic or it will be identically infinity even if your domain contains infinity you are able to do that, okay and the technical point was to deal with normal convergence at infinity and that is cleverly done by translating a neighbourhood of infinity into a neighbourhood of 0 by inverting the variable which is the usual philosophy that we have always been using to study the point at infinity, okay.

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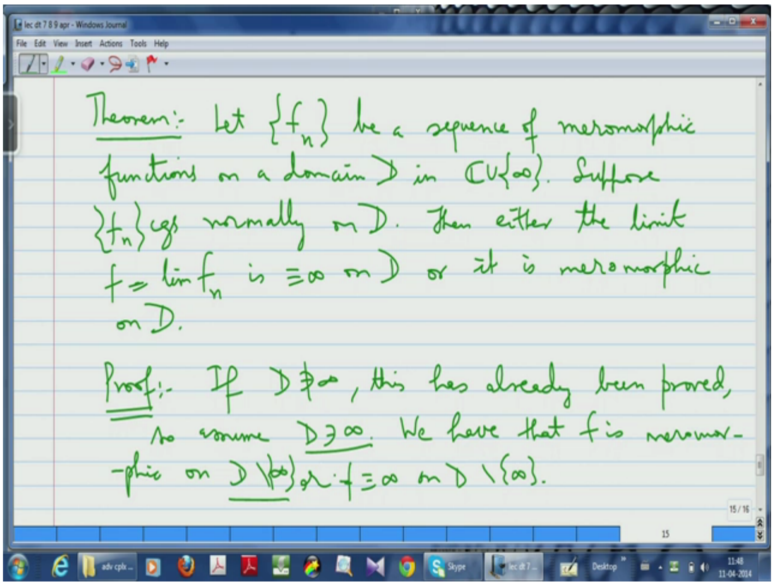

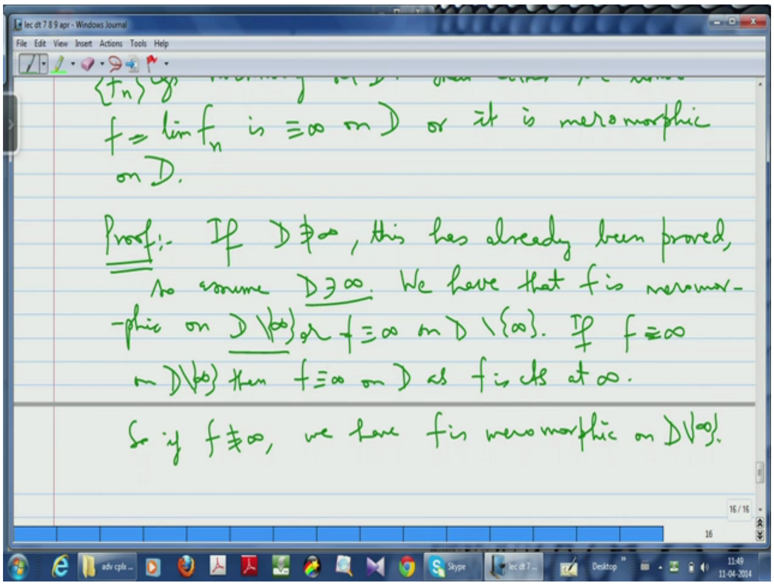

Fine so the next see the same kind of argument will give the corresponding theorem for you know it will give you the corresponding theorem for meromorphic functions, okay. So let me write that down theorem. Let f n be a sequence of meromorphic functions on the domain D in the extended complex plane. Suppose f n converges normally on D then either the limit f equal to limit f n is identically infinity on D or it is meromorphic on D, okay.

So so this is just extending the previous theorem to the meromorphic but now the point is that essentially you want to do deal with the domain which contains the point at infinity and what is the proof? Proof is exactly the same, okay the proof is exactly the same let me write it out. If D does not contain infinity, this has already been proved, so assume D contains infinity, okay. We have that f is meromorphic on D minus infinity, okay.

So again let me stop and say couple of things please remember that when you say normal convergence now, okay it is uniform convergence on compact subsets, alright but either you must look at compact subsets of D minus infinity or you should look at compact subsets of a neighbourhood of the origin with the variable inverted that is the point, okay and in both cases for the variable you are using only the eucledian metric but for the you values you are using the spherical metric, okay you have to remember that that is a big difference.

And the other important thing is that you know you are trying to make use of the theorems that you have already proved for a domain in the usual complex plane and trying to reduce the corresponding theorems when the domain contains the point at infinity, okay so these are things that you should highlight in your mind, okay. So you see so let me write repeat what I said if infinity is not in D then D is the usual domain and you know for a limit of meromorphic functions the limit can either be identically infinity or it can be meromorphic, okay.

And mind you this is a very important thing, you know it tells you the normal limits are good because after all what is meromorphic function? A meromorphic function is a function is analytic except for poles, okay but when a function goes to a limit the limit function could be horrible, see the limit function could have been an analytic function with non-pole singularities it could have been an analytic function with essential singularities or even worse the limit could have been an analytic function with non-isolated singularities such horrible things could happen but the fact is normal convergence prevents that.

See normal convergence you know is always a it is locally uniform convergence, okay because every point has a neighbourhood which is compact, okay so at every point you can find a neighbourhood compact neighbourhood where you will have uniform convergence and therefore on that neighbourhood also you will have uniform convergence should it is locally uniform convergence normal convergence and therefore because of the local uniformness everything nice happens you know the moment you have uniform convergence limits of continuous functions are continuous, limits of analytic functions are analytic and so on.

But so this is happening globally, alright and there is one more thing that I have to tell you that here the moment I assume f n is sequence of meromorphic functions on the domain which contains a point at infinity and I assume that it converges but mind you unique limit function is already defined that is because you know the f n's are meromorphic and therefore are continuous and I told you that whenever you take a continuous limit of continuous functions if you take a normal limit then the limit function is continuous even though your definition of normal convergence has been split into two pieces, one for the domain infinity punctured and the other for the neighbourhood of infinity part of the neighbourhood of the origin, okay.

So even this existence of f a uniform function f a single function f is because of the continuity of all these functions because meromorphic functions are continuous functions considered as functions into the extended complex plane that is something that you should not forget, okay so alright so if you take the domain which contains the point at infinity then D minus infinity is a usual domain in the usual complex plane and you have already proved the theorem for that so the function f is meromorphic on D union infinity or f is identically infinity on D minus infinity, okay so this is something that we have already proved we have that f is meromorphic on D minus infinity or f is identically infinity on D minus infinity, alright.

And again the same old argument if f is identically infinity on D minus infinity then it has to be infinity at infinity because of continuity of f, okay so we have to only deal with the condition when f is not identically infinity, right. So let me write that down. So if ya so if f is identically infinity on D minus infinity then f is identically infinity on D as f is continuous at continuous on D in fact continuous at infinity, okay.

So if f is not identically infinity we have f is meromorphic on D minus infinity, okay and therefore infinity becomes singular point, okay infinity is certainly a singular point because the problem now is slightly more complicated as it tends it looks you can say infinity is a singular point but you cannot immediately say that infinity is an isolated singular point that is the point that is the issue.

See infinity is a singular point because in if you take the if you take D minus infinity there are only poles on D minus infinity f is you know it is meromorphic. So on D minus infinity it is there are poles and poles are ofcourse isolated. So as far as D minus infinity is concerned all the singular points are isolated but infinity itself may be a non-isolated singular point it could happen poles could accumulate at infinity. If you have a sequence of singular points going to a point then that point is not an cannot be an isolated singular point so you have this problem.

But then that will not happen because of the normal convergence at infinity because what is happening at infinity is being controlled by what is happening when the variable is inverted in a neighbourhood of 0, okay.

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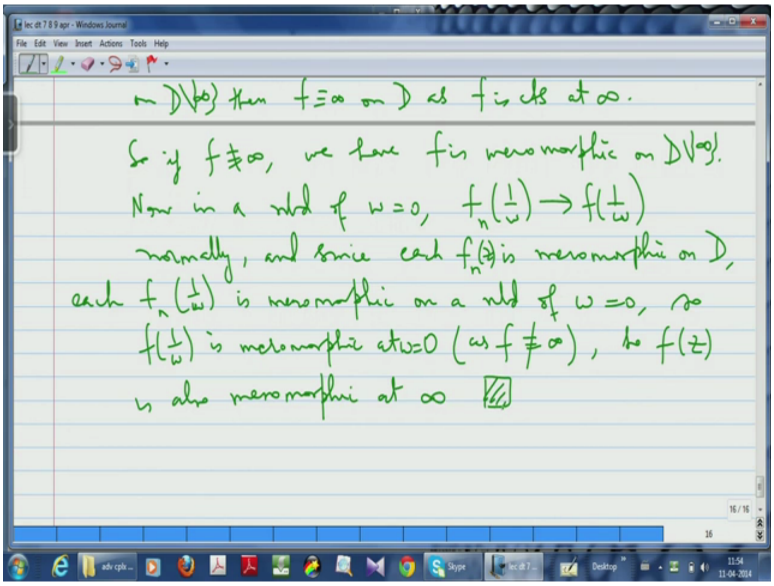

So let me say this now in neighbourhood of w equal to 0, f n of 1 by w converges to f of 1 by w normally and what I want you to understand is that you have changed the variable from z to 1 by w alright and if you are looking at z not equal to infinity or looking at w not equal to 0, okay and the point I want to make is that even at w equal to 0 I want to say that the f n's okay they are continue to be meromorphic that is because that is already given to you.

See what is given to you is that each of these functions is meromorphic on D, so infinity the point infinity which is in D can either be a pole by itself for each of the f n's or it can be a point of analyticity, infinity cannot be any verse, okay. So that means each of this f n's of 1 by w are meromorphic in a neighbourhood of w equal to 0, okay but this neighbourhood of w equal to 0 on that neighbourhood that is the usual neighbourhood in the complex plane and you have a normal convergence of the sequence of meromorphic functions therefore the limit functions can be either identically infinity or it can be meromorphic but the limit function is not identically infinity because f is not identically infinity, okay.

So f of 1 by w has to be you know a meromorphic function at w equal to 0 that means you are saying f of z is meromorphic at infinity and you are done, okay. So you escape and you get the proof of the statement that you want. So let me write that down so let me write this and since each f n is each f n z is meromorphic on D, f n of 1 by w is meromorphic on neighbourhood of w equal to 0, so f of 1 by w is meromorphic at 0 at w equal to 0 as f is not identically infinity, so f is so f of z is also meromorphic at infinity and we are done, okay and so you this horrible thing of infinity being a non-isolated singular point for f does not happen. So you see we are still in very nice in a very nice situation, okay.

So what I need to do is what we need to do next is try to generalize a Montel Theorem and Marty's Theorem for the case of meromorphic functions and we will do that in the next lecture.