Advanced Complex Analysis-Part 2
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Lecture 34
Proof of the other direction of Marty's Theorem

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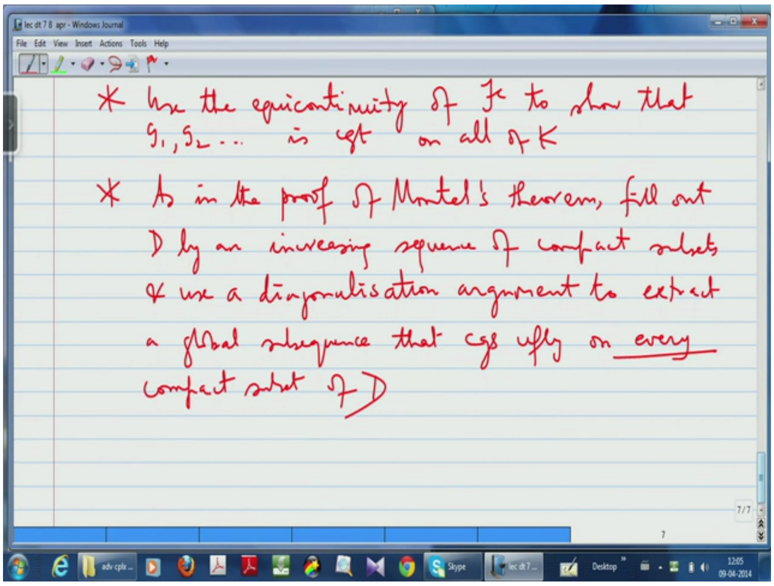

So let me just write down what I said last time. So we are trying to proof Marty's Theorem which is you know it is a analog of Montel's Theorem for you know meromorphic functions, okay and we have just so what I have written here is upto one way of the proof of the theorem, okay namely that if you have normal uniform boundedness of the spherical derivatives of a family of meromorphic functions defined on a domain in the complex plane then that family is compact in the sense that it is normally sequentially compact namely that given any sequence in that family you can find subsequence which converges uniformly on compact subsets of your domain, okay.

So and I just told in words the proof that the other way of the theorem is also true namely suppose you start with a family which is normal so in the sense that a family which this property that every sequence admits a normally convergent normally uniformly convergent subsequence, okay then ofcourse this is the family of meromorphic functions then the claim is that if you look at the family of spherical derivatives that family has to be normally uniformly bounded namely it should be uniformly bounded on every compact subset, okay and that is the way we have to write down the proof I think I told you in words how this can be done but now I will write it down I will write it down more accurately.

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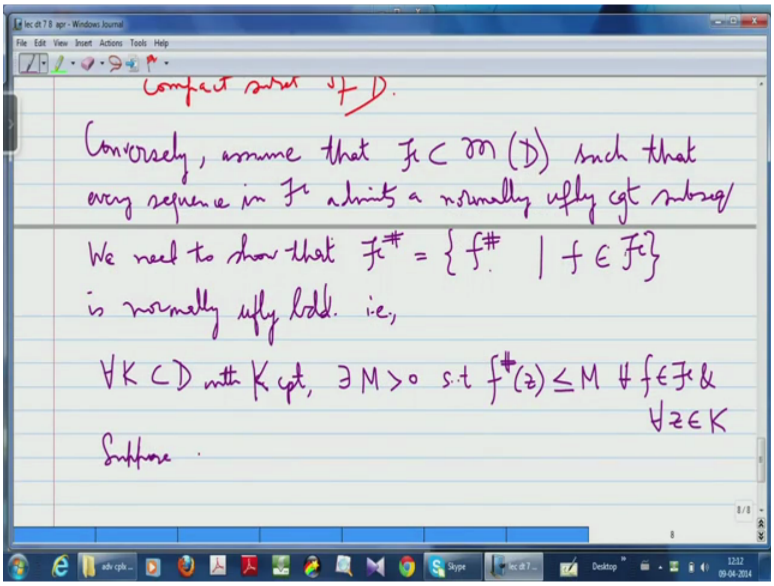

So let me write this here conversely so probably as I am going to the converse I will probably change colour something else, so conversely assume that script F is a family of meromorphic functions on D such that every sequence in script F admits a normally uniformly convergent subsequence so I am abbreviating uniformly to ufly cgt is abbreviation of convergent and subseq is for subsequence.

So basically this is the right notion of saying that a family is compact, okay what do I have to show? We need to show that the family of spherical derivatives in which this is the set of all you take each function small f in the family script F and take its spherical derivative, okay and then you get the family of spherical derivatives of this collection and we already have to show is that this family of spherical derivatives is normally uniformly bounded, okay is normally that means that you know you will have to show that it is that it is uniformly bounded on every compact subset of D okay.

What does that condition mean that is for every compact subset K in D so I am using cpt to abbreviate the word compact so for every compact subset K in D okay so let me write it like a logical statement because I am going to negate because you are going to proof it by contradiction, for every K in D with K compact there exist M greater than 0 that is a uniform bound for all the spherical derivatives of the functions in your family when restricted to K that is so if I write it down it means that thee exist a uniform bound which I am calling as M such that the spherical derivative at each point is bounded above by M for all functions f small f in the family script F and for every point is that in the compact set, okay this is the this is exactly the condition that the family of spherical derivatives is normally uniformly bounded.

Mind you the spherical derivative is a non-negative real number, okay spherical derivative is a non-negative real number it is 2 times the modulus of the usual derivative divided by 1 plus modulus of the functions squared, okay and this is at all the points where the function is analytic and since we are considering meromorphic functions you could have points which are poles and that poles I have told you what the spherical derivative is, we have evaluated it by continuity if the pole is the simple pole spherical derivative is 2 divided by the modulus of the residue of the function at that simple pole otherwise it is 0 if it is a pole of higher order, okay.

And ofcourse for the exceptional function which is constantly infinity on the pole domain which is also possible in a limit of meromorphic functions or even for that matter even for analytic functions in a under normal limit. For this function which is constantly infinity equal to infinity spherical derivative is 0, okay you should remember that. So here is so what we have to show? We have to show is that we have to show this condition, okay and what we will do is we will proof by contradiction so what we will do is we will assume this condition is not going to hold and we will contradict the fact that you have normality of the family namely that every sequence in the family admits a subsequence which converges uniformly on compact subsets so we will do that we will show contour example of that, okay.

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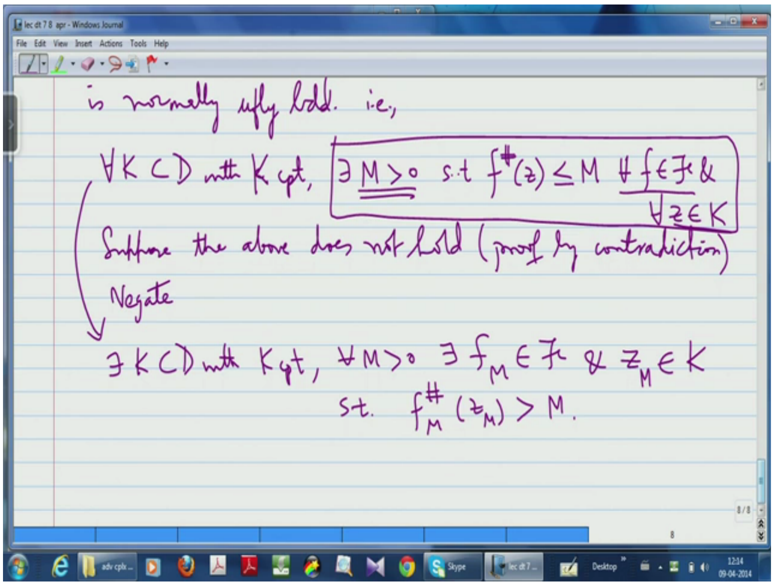

So suppose the above does not hold so this is proof by contradiction, suppose this does not hold then what you do? So you know what you will have to do is I will have to negate this, okay I have to negate this logical statement, okay and you know there is a very strict way of you know negating logical statements you normally replace for every with there exist and the other way round you replace there exist with for every okay that is the way you do it.

So you know if I negate this what I will get for every compact subset I want certain property to hold if I negate it it means there I can find a compact subset where this property will not hold. So this for every K in D with K compact will negate to there exist K in D with K compact so it will become there exist K subset of D with K compact, okay and what is the rest of it?

So look at the original statement for every K in D with K compact there is this this is the rest of the statement the statement is that for every for every f and for every z okay I can find an M such that the spherical derivatives of f at z are all bounded by M, okay. So this is there exist M greater than 0 on upon negation will become for every M greater than 0, okay and this for every small f will become a there exist small f and this for every small z will become there exist a small z.

So the way this part negates is you will get for every capital M greater than 0 there exist an f sub M in the family okay and a point z sub M in the compact set such that the spherical derivative of f sub M at z sub M is going to be greater than M this is how it negates, okay this is the negation of the statement. So in particular what I can do is that you know basically I want to show that if I assume if that this happens I want to show that I am going to get a contradiction in contradiction to what contradiction to what have assumed namely that the family whenever you have a sequence you can extract a normally convergent subsequence, okay.

So I will have to cook up a sequence and the way I do it is since this this negative statement is true for K you start putting M equal to 1, 2, 3, 4 so you make M larger and larger so that M goes to infinity and then you know by going to infinity expect that the spherical what you will get is you will get a sequence of points and a sequence of functions which where the spherical derivatives are those corresponding points is going to infinity but then this cannot be this cannot come from the original family if it were normal that is because normal convergence of a family also implies normal convergence of the spherical derivatives and mind you the spherical derivatives is the finite quantity, it is not a infinity quantity, okay. So that is how we will get the contradiction.

So what I will do is so let me write it down these are (())(11:41) details but sometimes you should know how to write down things that is very important and sometimes you should also be able to just say it in words for example you would find that in several textbooks probably the textbook would become very voluminous if they write down every detail so they might just qualitatively say it in a few words, okay but then it is your duty to write it down you have to translate it, okay so this is part of this is part of the exercise whenever you read a book, okay.

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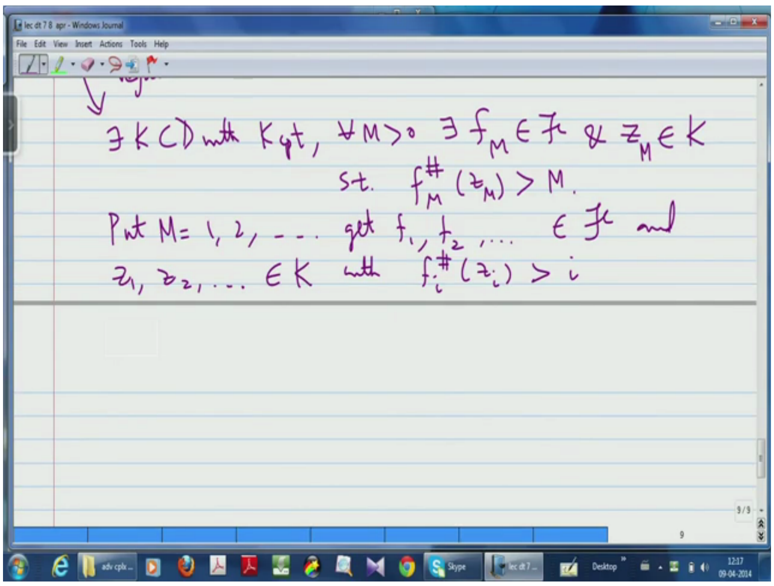

But since this is a lecture I am bound to explain as many details as I can so I will do it. So put M equal to 1, 2 and so on get f 1, f 2 and so on this is a these are all functions in the family script F and you get these points z 1, z 2 and so on which these are all points in K with the spherical derivative of f i at z i greater than i okay I get this this is when I put M equal to 1, 2 and so on, okay.

Now you know how the proof will go on the one hand I have this sequence of functions so I can always extract a normally convergent subsequence because as a part of my assumption, okay. On the other hand I also have the sequence of points, okay it is a sequence of points but where does it lie it lies in a compact set therefore if there is a convergent subsequence because you know any compact this compact set is a compact metric space and for a compact metric space you know that means compactness means is equivalent to sequential compactness which means that from every sequence you can extract a convergent subsequence.

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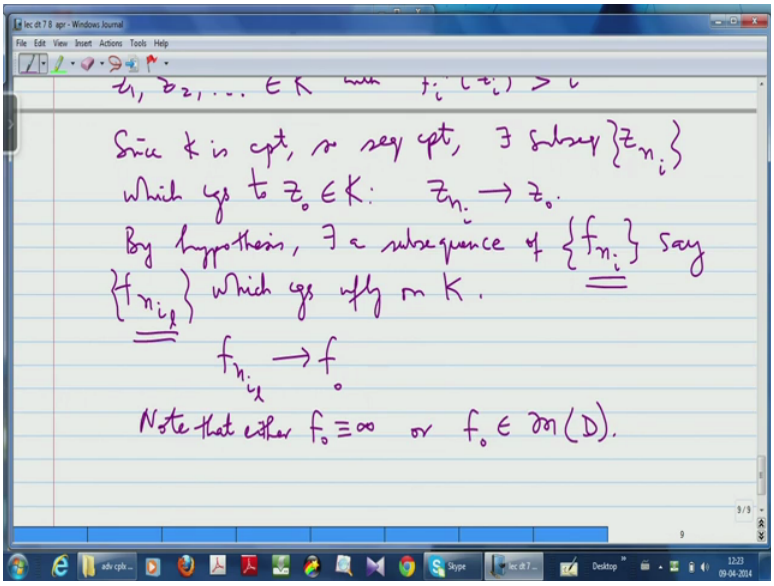

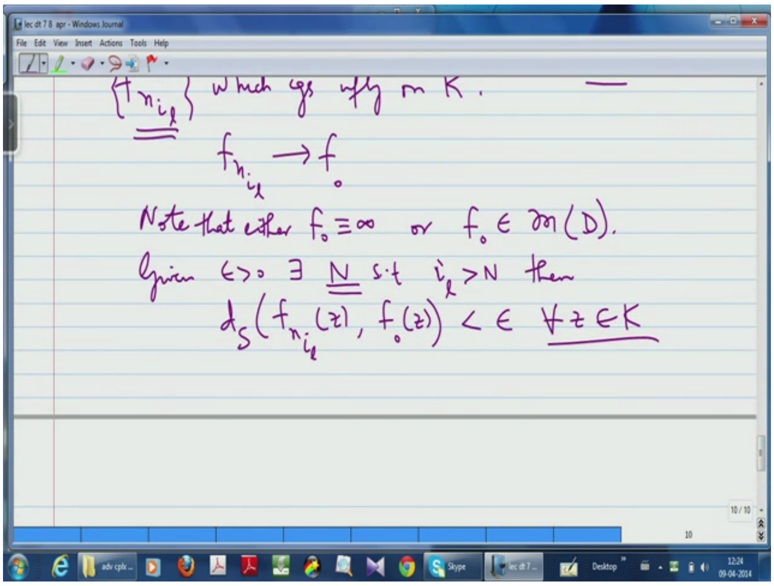

So what I can do is that I can do it either way so let me do it like this since k is compact, so sequentially compact because compactness in sequential compact is a equivalent from metric spaces there exist a subsequence z let me call this as z n z n i okay there is a subsequence which converges okay to it will converge to a point and that point has to be in K because the compact set is closed, okay because set is closed set is closed it will contain limits, okay so z sub n i is a subsequence and it is convergent subsequence of the limit should also be in K because K is closed, okay.

Fine so you have this so you have these z n i is going to z not on the one hand okay this is the compactness of K where that has being used and then I will also use the normal the fact that the family is normal that it is compact, okay. So well by hypothesis, there exist a subsequence so you know well there exist a subsequence of f sub n i okay so you see I already have the original sequence, okay and I have the original sequence of points when I take the subsequence of points which converges I have got a subsequence of the original sequence and I am considering the corresponding functions in the family of in the sequence of functions and I am that is already a subsequence and I am extracting a further subsequence from that because that is what the normal sequential compactness is all about from given give me any sequence I can always extract a normally convergent subsequence.

So I am not trying to I am not extracting a convergent a normally convergent subsequence from f 1, f 2 and so on but I am extracting a normally convergent subsequence from the subsequence given by f n 1, f n 2 and so on, okay. So there is a subsequence of f n i say let me call this as well the notation it is little bit bad but does not matter so I get this subsequence I get a subsequence of this which converges uniformly on K, okay in fact it will be a subsequence which converge uniformly on all compact subsets of D, okay that is what the hypothesis comes in but I am restricting I am just worried about K for the moment because K is the compact set when I am working.

Now but here is issue the issue is that these this f n i l, okay this will go to certain f not, okay because you see a normal limit if you take a normal limit of meromorphic functions that is also meromorphic that is something that we proved, okay. See a normal limit of meromorphic functions is either identically infinity, okay or it is a meromorphic it can even be holomorphic it can even be holomorphic okay this is what we have already seen, okay.

So this f not what can happen with f not is that f not can either be the function which is identically infinity or f not can be honest meromorphic function (())(17:09) be a holomorphic function, okay. So note that either f not is identically infinity, okay or well f not is meromorphic function on D okay so this is there, alright and what is it that I wanted to understand?

See this convergence is uniform on K this f i n f n i l converges to f not is uniform on K it is uniform convergence. So what it means is given a epsilon greater than 0, okay you can find an N such that if your i sub l is greater than N, alright then the distance between these two functions the distance between the function values at any point of K can be made less than epsilon and mind you know because you are working with meromorphic functions you could very well they could very well take the value infinity and it is not used the usual distance you should use the spherical distance.

So you will have so I have to write it like this d spherical of f of n i l f sub n i l of z, comma f not of z this can be made less than epsilon, okay the spherical distance between the values of these two functions in the sequence beyond the certain stage and the value at the limit that can be made as small as I want this is uniform convergence and the point is that this N has got nothing to do with z, so this is for all z in K this is the uniformity it works for all z, okay.

So in particular you know you can now see what I am getting at you see I have these z n i's the z n i's are going to z not, okay and therefore the f if you give me any function f in the family by continuity f of z n i will go to f of z not, alright and but the point is for f if I had taken f n i okay what will happen if you take the original sequence of functions f 1, f 2 and so if you take f i and evaluate it as z i not the function but I mean the spherical derivative, okay then it is greater than i, okay we have this.

So if I take this f n i and take the spherical derivative and then evaluate it as z n i I am going to get something that is greater than n i and that is supposed to go to f n i of z not and now if I let n i tend to infinity it will go to f not of z not. So essentially you know f not f z not is coming very close to a sequence of quantities which are becoming larger and larger then how can it be because I mean not f not of z not but I mean spherical derivative of f not z not.

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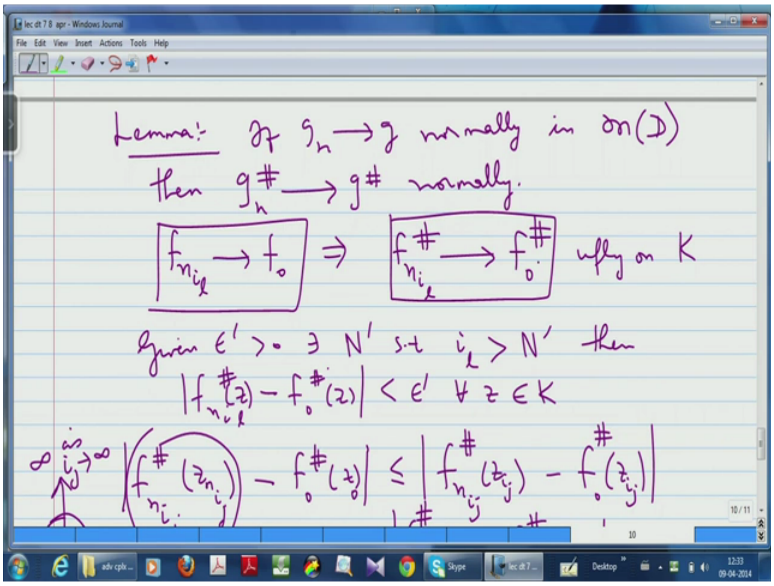

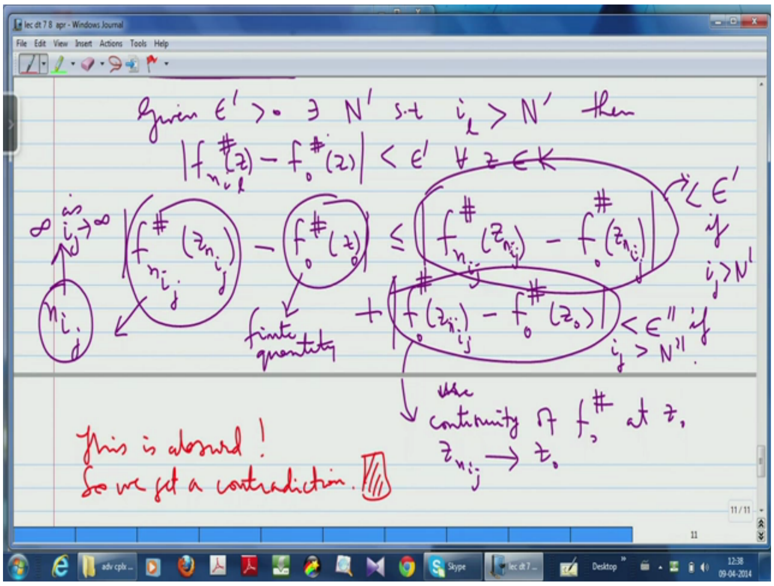

So let me write it out so the first statement I need to make is so this was the the lemma that I had rubbed out last time so this lemma if g n converges to g normally in suppose this is the normal convergence then the spherical derivatives also respect that so then the spherical derivatives of the g n's will go to the spherical derivatives of g normally, okay so this is the lemma that I rubbed out which we need to use at this point. If you have a sequence of meromorphic function which converges to a limit function and suppose all this is happening normally namely it is appearing uniformly on compact subsets of your domain D then ofcourse we have already seen that the limit function is going to be meromorphic or it is identically infinity, okay so this g can be either meromorphic or it is identically infinity and the point is no matter what it is if you take the corresponding sequence of spherical derivatives that will converge also to the spherical derivative of the limit and that will happen normally, okay.

And remember that if g is the function which is identically infinity there is no problem with the spherical derivative it is 0, it is not something that is straight, okay it is not identifying quantity of something like that, okay. So you have this so I am using this. So if I use this what will I get what I will get is that you see I have this f n i l going to f not. So this will tell me that f n i l be the spherical derivatives will go to f not spherical derivative of f not, okay and this is happening mind you this is going to happen normally so it is going to be all this is going to happen uniformly on K, okay so this is uniformly on K because K is compact, alright.

And what does this mean this means that you know the if you calculate mind you when I am comparing when I am comparing values of usual meromorphic functions I have to use the spherical metric okay but when I am comparing values of the spherical derivative, okay I have to use the usual distance function on R because spherical derivatives are non-negative real numbers.

So what this means is that given an epsilon prime greater than 0 there exist an N prime such that if the index is greater than N prime then the distance between f n i l hash and f not hash for any z is can be made less than epsilon prime, okay I can do this for all z in K this is what it means and I am using a modulus function here because these values are all real values, okay. you see I will now interpolate so what I will do is that I will use a triangle inequality and write this is f n i j of z i j and then I will add a f not of z i j okay I can do this plus then I will write this f not of z i j minus f not and ofcourse everything is a spherical derivatives not just the original functions so and then I will have f hash of z not so this is just by (())(24:09) triangle inequality on the real line, alright I have just added and subtract the spherical derivative z i j, okay and of f not okay.

So now you know now you know what I am going to do so this will be n i j that is what this quantity is alright and now you know by now you know what is going what is happening.

You see this quantity is n i j and as i j tends to infinity this quantity this is going to go infinity okay this is going to go infinity as i j tends to infinity, okay you know that alright. So what you are saying is what is there on the left is the distance from f not hash of z not, okay to a pint to a value which is going bigger and bigger and you are saying that can be bounded by the sum of two quantities okay see the first quantity see the first quantity can be made less than epsilon that is because of the uniform convergence of f n i j to f not, okay.

See I have just written it down above the f n i j's they converge uniformly to f not on K so the lemma says that therefore the spherical derivatives also converge. So the distance between f n i j hash and f not hash at any point can be made as small as I want. So the moral of the story is that this quantity here can be made less than epsilon, okay and this is okay this is okay for any z okay so I plug in z i j so I can make this less than epsilon, okay.

And look at this quantity here what is it? So this is less than epsilon ofcourse I will have to put in some condition and the condition is that if this n i j is chosen greater than N I think it was so it is less than epsilon prime if you choose the i j greater than N prime so let me write that so it is this can be made less than epsilon prime if so let me write it correctly i j can be made greater than N prime, okay I can do this, right.

And again I think I missed the subscript needs to be corrected here so let me do it this is an issue with double subscripts or triple subscripts, okay so this quantity this can be made less than epsilon, okay so I have this and then what about the second quantity the second quantity is spherical derivative at so there is a 0 missing here the spherical derivative of f not at z i j minus spherical derivative of f not at z not, okay.

But mind you this also I can make sufficiently small that is because of the continuity of the spherical derivative spherical derivative is anyway continuous, okay. So the moral of the story is that I can make this also this also can be made less than if you want epsilon double prime if you choose some ya again I have this problem copying up that I will have to worry about this double subscript let me write this n i j if n sub i j if i j is sufficiently large is greater than say N double prime I can do this and this is just what am I using here I am just using you have space to write but may be I will continue here continuity use continuity of f not hash at z not and remember that the z n i j's also tend to z not the z not was a limit point, okay.

So that is it you have a quantity on the left okay which is very close to very large number okay and this large number is becoming larger that it is very close to a large number is because it is bounded by a quantity on the right which is very small okay the sum of epsilon and epsilon prime I can make it as small as I want. So it means that the quantity on the left which is the spherical derivative at z not of f not that comes arbitrarily close to any large number okay and that is impossible because that is a this is a contradiction because this quantity is a finite quantity the spherical derivative at any point is a finite quantity so it cannot go it cannot be within an epsilon distance of an increasing sequence of numbers just cannot happen so that is a contradiction.

So let me put this here so this is finite quantity and the totality of all is that well that the whole thing that you have written down manage to get is observed, okay. So let me write that down so this is observed, okay all this is observed cannot happen so we get a contradiction, okay and that finishes the proof, okay so spherical derivatives need to be bounded okay and the point I wanted to remember is that ofcourse you know sometimes writing down these (()) (30:00) details is a pain you will have to worry about subscripts and things like that but then once in a while you should do this because then you get a you get a (())(30:12) of having written down something very accurate, okay.

And ofcourse you also should try to be very elegant and say without any notations you must be able to say in words. So if you want to say all this this crazy things in words you know what you will say is that well I have a sequence of points and I have a sequence of functions which that is if I want to proof by contradiction I will find a compact set on which the uniform boundedness of the derivatives will not be true on that compact subset I can get a sequence of functions and a sequence of points such that the corresponding spherical derivatives are going to be unbounded, okay and but then the points have to converge because it is a compact subsets atleast subsequence of points have to converge because it is a compact subset, okay.

So and also the functions must allow a convergent subsequence if you put these two together what will happen is that you will get a sequence of spherical derivatives becoming unbounded, okay but coming arbitrarily close to the spherical derivative of the limit function at that given point and that is not possible because the spherical derivative of a function at a given point is a finite quantity, okay.

So that is how you can say it elegantly inverse, okay fine so that finishes Marty’s Theorem. So mind you Marty’s Theorem is very very powerful because you see it is more powerful than it is more powerful than the usual Montel Theorem because you see Montel’s Theorem the original Montel's Theorem says that you know it is only for analytic functions and what you had required there was well you needed normal uniform boundedness of the family of analytic functions, alright.

Whereas in Marty's Theorem you generalize from analytic functions to meromorphic functions, okay and you do not require normal uniform boundedness of the functions but you require normal uniform boundedness of the spherical derivatives of the functions, okay so this is the difference.

So if you so suppose I have a family of analytic functions, okay and suppose I know that their derivatives are uniformly bounded normally uniformly bounded I can still conclude something from I cannot apply the original Montel Theorem, okay but I can apply Marty's Theorem and say that I can still conclude that this family of analytic functions will converge to a limit and since it is normal convergence, okay I mean if you take a sequence of functions on this family I can extract a subsequence that will converge to a limit and that limit will either be holomorphic or it will be identically infinity. So you see Marty's Theorem is more powerful that is what I want to tell you, okay.

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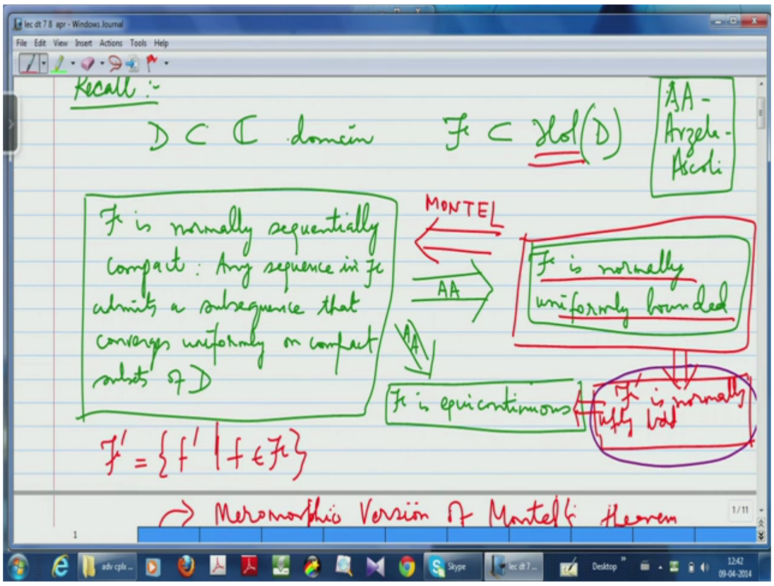

So let me again go back to that slide that I wrote out here it is so here is so this is let me go back to this slide you see there was this thing that I have circled in violet in surrounded by a small I have written it in red the family of derivatives is normally uniformly bounded is an intermediate step, okay so if I start with a family which is normally uniformly bounded and suppose it is a family of analytic functions because of the Cauchy integral formula okay I get the family of derivatives is normally uniformly bounded, okay and from that I get equicontinuity and once I had equicontinuity and uniform boundedness I can apply Arzela-Aascoli Theorem, okay and then whatever I want I get by doing a diagonalization argument.

But now see suppose I am given a family of analytic functions, okay on a domain. Suppose I am not given that the family is normally uniformly bounded suppose I am not given that, okay I cannot apply the original Montel Theorem. Suppose I am given instead of given that the original family of analytic functions is bounded suppose I am given that the family of derivatives usual derivatives which makes sense for analytic functions suppose I am given that family is uniformly normally uniformly bounded, okay I cannot apply Montel's Theorem because I do not have the uniform boundedness normal uniform boundedness of that family itself I have only the normal uniform boundedness of the derivatives of the family, I cannot apply the usual Montel Theorem.

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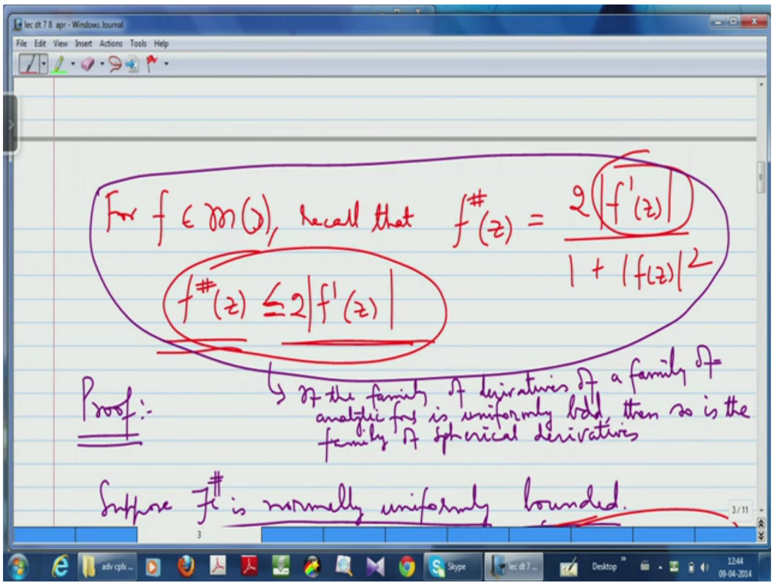

But then the uniform boundedness of the ordinary the boundedness of the ordinary derivative will give rise will imply the boundedness of the spherical derivatives because of this inequality you see if you go down because that is the so this is what I have circled in violet you see the spherical derivatives is bounded by 2 times the bound for the normal derivative the usual derivative.

So if I am given a family of analytic functions such that the usual derivatives are normally uniformly bounded it follows that the family of spherical derivatives is normally uniformly bounded, okay so I can apply Marty's Theorem Marty's Theorem will tell me now therefore that any sequence in that family admits a convergent subsequence normally convergent subsequence, okay and that normally convergent subsequence is what it is a convergent sequence of holomorphic functions but the only thing is that the convergence is with respect to the spherical metric and then you know what happens my limit function will either be holomorphic that is analytic or it will be identically infinity, okay.

So what is upshot of all this? The upshot of all this is let me state it if you have a family of analytic functions on a domain and suppose you know that usual derivatives of the analytic functions are normally uniformly bounded, okay then that family is compact in the sense that give me any sequence in that family I can find a subsequence of convergence normally convergent subsequence and the limit function will either be again analytic or it will be identically infinity this is what I will get because of Marty’s Theorem.

And mind you I cannot apply the original Marty’s Theorem because for the original Marty’s Theorem I need boundedness uniform boundedness on compact subsets of the original family that is not given to me, what is given to me only uniform boundedness on compact subsets of the derivatives. So in that sense Marty’s Theorem is very very strong is stronger than Montel’s Theorem, okay.

So with that you know I have more or less we are more or less come to one point here in our discussion I will tell you what we need to do next, okay. So if you see in all these things that we have proved we have been looking in the complex plane, okay. So what are the things we proved first we proved that so for a domain in the complex plane we defined normal convergence, okay that is uniform convergence on compact subsets and then we proved that if you take a normal limit of analytic functions, okay then the limit is either identically infinity, okay if you use the spherical metric or it will be again an analytic function, okay.

And if you take same thing happens for meromorphic functions if you take a family of meromorphic functions okay if you if it is if the family for example if you take a sequence of meromorphic functions which is normally uniformly convergent then the limit function is again going to be either meromorphic or it will be identically infinity we do not get any strange situations, okay.

The moral of the story is when you take a normal limit of analytic functions you will get a analytic functions and the extreme case is you will get the function is identically infinity, same for meromorphic functions, okay. And we have seen that for example if you take a sequence of meromorphic functions is not going to go to sequence of analytic functions is not going to go to a strictly meromorphic functions okay a pole cannot pop up at the limit such things cannot happen so it is all very well behave and then we have proved the Montel's Theorem and we have proved Marty's Theorem, okay.

Now all this is for usual domains what about a domain which can tends the point at infinity that is the next we want to include all the extend all these results to a domain in the extended plane, what is the problem with the domain in the extended plane? The problem with the domain in the extended plane is the point at infinity for the point at infinity, okay you cannot find a compact neighbourhood atleast some usual plane, okay.

So if you take suppose D so what I am trying to say is that you know if you say that if I have a if I am taking a domain in the extended plane that means it is actually a domain in the usual plane it is in the exterior of the large enough disk and it introduce the point at infinity, okay and whenever I take compact subset of that it will not include the point at infinity the point at infinity cannot come into any compact subsets, okay.

So there is a problem if you take a domain which contains the point at infinity, okay you are never able to get a compact neighbourhood of infinity in the usual sense, okay. So what is the meaning of normal convergence at infinity that has to be made (())(40:11) you have to understand what normal convergence at infinity means you will have to understand that in that context you have to extend all these theorems, okay.

So what I will do in the coming lecture is coming lectures is you know try to look at normal convergence at infinity and extend all these results to the case when your domain is a domain in the extended plane which can possibly contain even the point at infinity, okay. So this is a very very important step, okay and then we have enough tools to go ahead with the with another theorem called Zalcman's lemma and that will lead directly to a stronger version of Montel's Theorem on omission of values and that will lead to finally to the Picard Theorems, okay. So we will see that in the fourth coming lecture.