Advanced Complex Analysis-Part 2
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Lecture 33
Proof of one direction of Marty's Theorem

Alright so let me continue with this discussion about Marty's Theorem, okay which is basically an analog of Montel's Theorem except that your working with not with analytic functions but with meromorphic functions, okay.

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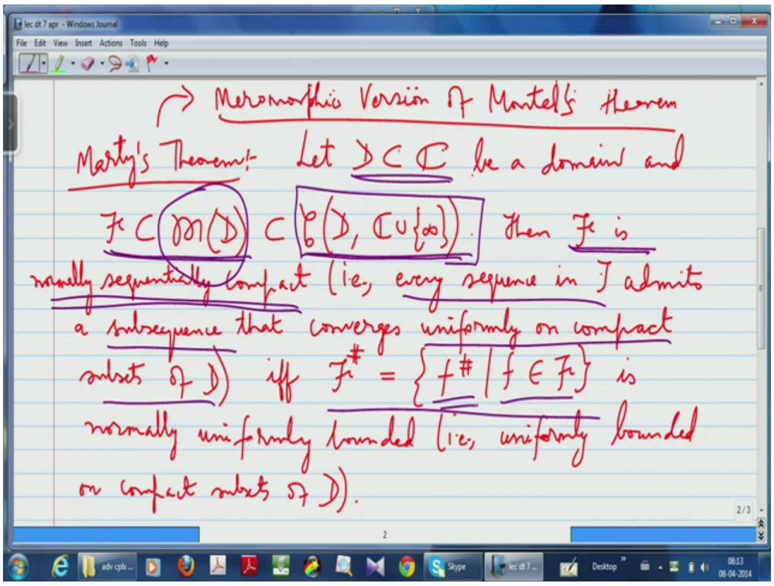

So let us look at let us again look at the statement so you have this domain D in the complex plane and you have this family script F of meromorphic functions on D so the script m D is the you know it is the set of meromorphic functions on D and mind you we are considering these as continuous functions from D into the extended complex plane, okay. So you are able to do that because meromorphic function normally at a point which is a pole which is it goes to infinity, okay but then you allow the value infinity and you declare the value at the pole to be infinity so it becomes continuous.

So the set of meromorphic functions is a subset of this the set of continuous functions from D to the extended complex plane C union infinity which as a metric space given the spherical metric we think of as the just the Riemann sphere, okay with infinity corresponding to the north pole, right. And so you have this family script F of meromorphic functions you can either use a word collection or family or subset whichever you prefer but the point is when is this family compact?

So in this case you know normally I should since the word normal is used technically I should say usually compactness is equivalent to sequential compactness and then that means that you know saying something is compact is same as saying that every sequence has a convergent subsequence. So if you want to say that the family script F is sequentially I mean it is compact you will like to say it is sequentially compact if you want to say it is compact.

And then you would like to which means that you know given any sequence of functions in this family you are able to extract a subsequence which converges. Now what kind of convergence? If we are usually the convergence that we worry about is the uniform convergence but ofcourse in the case of analytic functions, the meromorphic functions you will not get uniform convergence on the whole domain you will get only uniform convergence on compact you will get only uniform convergence on compact subsets and that is called normal convergence, okay.

So in other words the compactness of the family F is (())(4:09) of as normal sequential compactness which means that every sequence in F admits a subsequence that converges uniformly on compact subsets of the domain, okay so this is what compactness for us means and Marty's Theorem says that this is the same as the family of spherical derivatives of F namely you take for each small F in script F you take its spherical derivative F hash small f hash and you get this family script F hash this is the family of spherical derivatives and that should be normally uniformly bounded which means that it is uniformly bounded on compact subsets.

So in some sense boundedness of derivatives is equivalent to compactness I mean if you want to say it in a nutshell boundedness of derivatives is equivalent to compactness, okay and so there are a couple of aspects that I want to stress between this and the original Montel Theorem see the original Montel Theorem was for analytic functions, okay so you took instead of taking a family of meromorphic functions as we have done now, if you are taken analytic functions, okay then we would have put the condition that the family is uniformly bounded the family itself is uniformly bounded, okay.

And the and there the uniform boundedness of the family on compact subsets that would be equivalent to the family being normally sequentially compact that is the usual Montel

Theorem, okay and the way we work there is you have the uniform if you restrict to a compact set you have uniform boundedness of the family, okay then from the uniform boundedness of the family you derive equicontinuity because from the uniform boundedness of the family you get uniform boundedness of the derivatives and that is because of the fact that the derivatives are expressed in terms of the original functions using the Cauchy integral formula and you can make an estimation that are the Cauchy estimates.

So uniform boundedness of the family on a compact subset will give rise to uniform boundedness of the derivatives on the compact subsets and uniform boundedness of the derivatives always gives rise to equicontinuity. So you get along with uniform boundedness on a compact subset you get equicontinuity for free if you are looking at analytic functions, okay.

But you see Marty's Theorem is slightly different what is happening is whereas in Montel's Theorem uniform boundedness on compact subsets of the family is equivalent to the family being normally sequentially compact. In Marty's Theorem it is not uniform boundedness on compact normal uniform boundedness of the family but it is actually normal uniform boundedness of the spherical derivatives, okay.

So you move from the family in some sense you move from the boundedness of the family to the boundedness of the derivatives that is the switch, okay and the point is that in a sense this is stronger than the original Montel Theorem because in the original Montel Theorem if you know if you are looking at a family of analytic functions and suppose you know that their derivatives are normally uniformly bounded suppose I am not given that the family itself is uniformly bounded but suppose I am given just the derivatives are normally uniformly bounded, okay.

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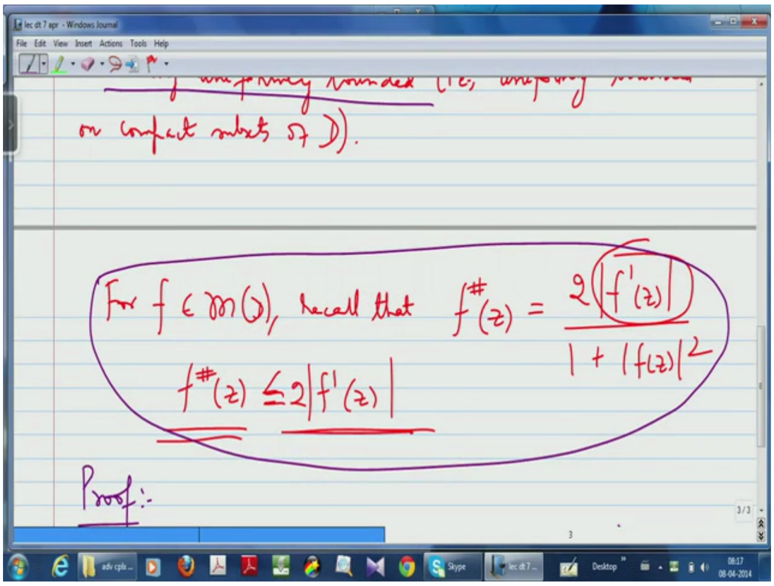

Then what happens is if the usual derivatives are normally uniformly bounded then it also happens that the spherical derivatives are normally uniformly bounded because of this reason, okay because you see if you the spherical derivatives are bounded by 2 times they bound for the normal derivatives the usual derivatives. So if the usual derivatives are bounded, okay then the spherical derivatives are bounded.

So if you take a family of analytic functions on a domain such that the usual derivatives are all uniformly bounded on compact subsets that is normally uniformly bounded then also you will get you know a normal sequential compactness, okay because of Marty's Theorem but the only thing is that now you could have you know because you are considering these as a meromorphic functions you could have the extreme case that all these analytic functions go to infinity, okay and by that I mean they go to the function which is infinity on all points of the domain which is also considered as a continuous functions, okay and mind you for such for that function the spherical derivative is 0 because it is a constant function, okay.

So now what I want to say is so this is one aspect that when you move from Montel's Theorem to Marty's Theorem you are actually moving from uniform boundedness on compact subsets of the family of functions to the uniform boundedness of the derivatives, okay and because you are worrying about meromorphic functions usual derivatives will not work. For example at poles so you will have to look at spherical derivatives, okay now that is one aspect.

Now here is another important aspect, see if you know that these Montel's Theorem for example is actually deeper version or it is an application of the Arzela-Ascoli Theorem, okay and what is the philosophy original what is the philosophy of the original Arzela-Ascoli Theorem? The philosophy is that if you want to say a family of functions is compact which is same as saying sequentially compact namely you want to extract a convergent subsequence from any given sequence.

See you will have to put the conditions of the family being equicontinuous and uniformly bounded that is why the Arzela-Ascoli Theorem is often referred at as uniform boundedness principle, okay. So you need uniform boundedness plus you need equicontinuity together to give you sequential compactness, alright. If you are working with analytic functions uniform boundedness is enough, okay because equicontinuity will come out as a immediately it will come out for free because you have the Cauchy integral formula, okay.

Now in the case of Marty’s Theorem there is a slight advantage the advantage is that if you see I have if I look at it in one direction that is why is it that the uniform boundedness normal uniform boundedness of derivatives should give me a normal sequential compactness, okay what you can guess immediately is that always boundedness of the derivatives gives rise to equicontinuity it always give rise to equicontinuity.

So even on a compact set if you want to extract a convergent subsequence from a given sequence, okay you would like to apply Arzela-Ascoli Theorem. So what is missing? What is missing is uniform boundedness because if you want to apply Arzela-Ascoli Theorem you need uniform boundedness together with equicontinuity so that you can extract from any given sequence as convergent subsequence.

So if I restrict to a compact set what I if I assume that the derivatives spherical derivatives are bounded, okay I can expect only equicontinuity, okay I will not get the I will get equicontinuity of the given family of functions but I cannot get I do not seem to be getting uniform boundedness of the family, but here is where the beautiful thing is you do not need any uniform boundedness, okay.

The reason is because the values are being taken in a compact metric space, okay see the values are being taken as far as meromorphic functions are concerned where are values being taken the values are being taken in the extended complex plane extended complex plane mind you is identified as a Riemann sphere and is a compact metric space, okay and you know a compact metric space is ofcourse bounded it is totally bounded, it is bounded, okay it is complete, okay.

So there is no unboundedness phenomena that is going to occur in a compact metric space, okay. So this uniform boundedness condition is not necessary that is the whole point, okay. So what I want to say is that your Arzela-Ascoli Theorem in the Arzela-Ascoli Theorem okay we were looking at functions continuous functions either real or complex valued on a compact metric space, okay.

Now I am saying and there for sequential compactness of a family of functions you needed both uniform boundedness and equicontinuity but if I instead of looking at real or complex valued functions suppose I was looking at functions with values in a compact metric space, okay that is the change I am making you try to look at functions defined on a compact metric space and taking values in another compact metric space the target is no real numbers or complex numbers but the target is another compact metric space.

Then because the target is already compact this uniform boundedness is not needed just equicontinuity is enough and it is equivalent to sequential compactness that is the whole point that is the whole point, okay so what I want to tell you is that when you go to Marty's Theorem, okay you switch to the uniform boundedness of the derivatives and you do not care about boundedness of the original family of functions locally that is because how the functions are already taking values in a compact metric space and you do not have to worry about it, okay.

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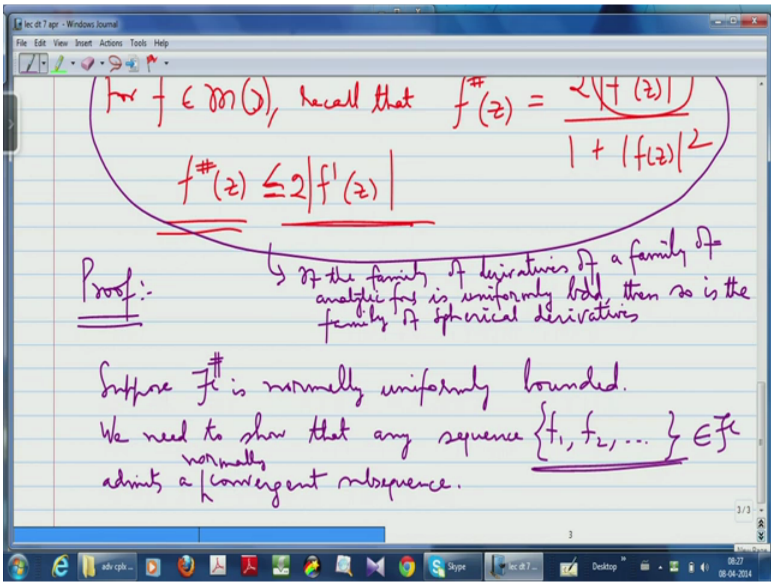

So let me explain the proof so whatever I have circle here is to tell you that what this tells you is that if the family of derivatives of a collection of a family of analytic functions functions is uniformly bounded then so is the family of spherical derivatives so the boundedness of the ordinary derivatives implies boundedness of spherical derivatives, okay so that is something that I am writing here I think I have cramped it a little bit so let me get rid of this lemma and rewrite it later, okay fine.

So what I will do is I will try to give you the proof of this so let us go in one direction so let me again rewrite the Arzela-Ascoli Theorem is valid, okay in the sense that sequential compactness is same as equicontinuity you do not worry about uniform boundedness, if you are looking at functions which are taking values in continuous functions values in a compact metric space, okay if that is if you replace real and complex numbers by a compact integrals that is the whole point so just equicontinuity is enough, right and I will try to instead of trying to prove a theorem in that generality I will even explain to you how you can get sequential compactness so what you do is.

So let us start this way suppose so maybe I will use so suppose F is suppose F hash is normally uniformly bounded, okay suppose it is normally uniformly bounded, what do I have to show? I have to show that it is normally sequentially compact that means you will have to pick up given any sequence in the family script F you have to show that there is a convergent subsequence, right convergence in the sense of normal convergence that is convergence on compact subsets so that is what I have to do, we need to so let me write that down.

We need to show show that any sequence f 1, f 2 and so on admits this sequence in ofcourse in I should not say well when I put subset this is I am not writing this sequence as set, okay because there could be repetitions in the sequence, okay so this is by this notation let me put let me put belongs to okay so this I mean that f 1, f 2, etc is a sequence in F you have to show that any sequence admits a convergent subsequence, subsequence and ofcourse it should be a normally convergent subsequence that is something that converges on compact subsets, okay uniformly on compact and ofcourse on compact subsets the convergence is uniform, alright so uniform convergence.

So now so how do I go about this? So as usual the moment usually if you have boundedness of the derivatives the first thing that you do is you get equicontinuity of the family, okay that is always always you should remember as a philosophy boundedness of the derivatives is a strong condition that will imply equicontinuity of the original family. So what you do is that so that is what I am going to demonstrate we will demonstrate that this family script F is equicontinuous, okay.

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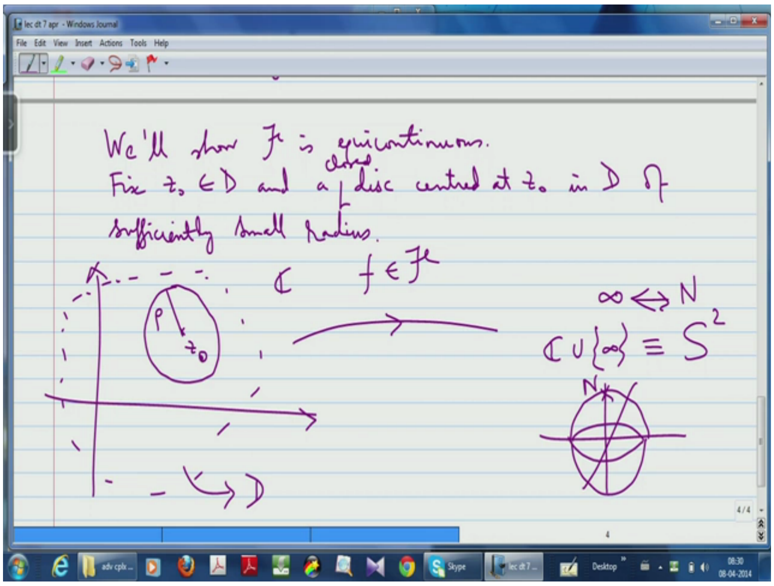

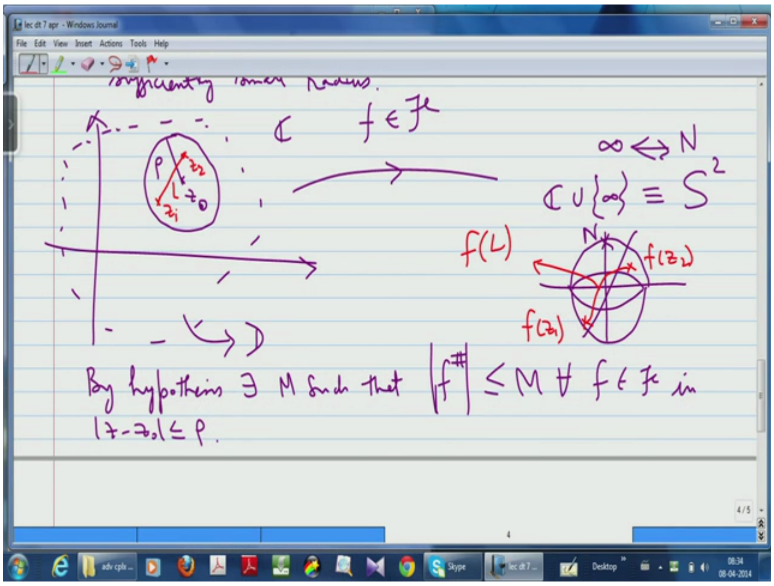

So we will show script F is equicontinuous, how do I do that you check equicontinuity at every point so what you do is fix z not at D and disk a closed disk centred at z not at z not in D of sufficiently small radius, okay so now you know so the situation is like this you have the you have the complex plane and you have this you have some you have this domain D okay and so this is D, I am always trying to draw a bounded domain but it will not be a bounded domain, okay because it is an unbounded domain which I cannot show on a picture.

So here is the domain D it is the boundary is this dotted line and what I am having is a point z not in D and I am choosing a sufficiently small disk such say of radius rho, okay rho sufficiently small so that the whole closed disk is inside D okay the open disk with z not centre z not, radius rho along with the boundary circle that is also in D, okay and what do I do I just so I remember that you know my if you take a function f small f in script F mind you the function is being now (())(21:05) as going into the Riemann sphere, okay it is going into C union infinity and the C union infinity is identified so I put a triple line, okay this is identified with Riemann sphere so what is it? It is just so this is just S 2 the real two sphere in three space real three space radius 1 centred at the origin.

So it is this you know it is this thing so this is the Riemann sphere and this points corresponds to the north pole which corresponds to so this infinity corresponds to the north pole, okay. So here function is taking values on the Riemann sphere that is how you think about it, right and now what is it that I am given? I am given that I am given that the family I am given that the family of spherical derivatives is normally uniformly bounded so that means it is uniformly bounded on compact subsets of D and this closed disk centred at z not, radius rho is a compact subset of D so it is uniformly bounded on that, okay.

By hypothesis of normal uniform boundedness of the family script F there exist an M such that the spherical derivatives of all the spherical derivatives in the family are bounded by M so let me just put in mod z minus z not less than or equal to so I have this, okay this is just the uniform boundedness of the spherical derivatives restricted to this compact subset given by this (())(23:02) right.

Now what you do mind you that in this situation since the functions are taking values in the extended complex plane, okay on the target the target metric space is extended complex plane and the target metric is the spherical metric that is what you have to remember, okay the target metric space is the extended complex plane and on the extended complex plane the metric is the spherical metric it is actually the spherical distance on the Riemann sphere transported by the (())(23:31) of the Riemann sphere with the extended plane, okay.

So you should remember this is the big point to remember you have to remember that whenever you are working with values in the extended complex plane, okay the in the target space the extended complex plane the metric involved is spherical metric. So if you keep that in mind this is what is going to happen. See if I take two points suppose I take so let me use the different colour.

Suppose I take two points say z 1 and z 2 okay inside this closed disk and I take the straight line segment from z 1 to z 2, okay then and suppose I call this segment as L, okay then under if I take the image of the segment straight line segment under this map f, where f is any function any meromorphic function and the collection script F, okay what I am going to get is I am going to get something on the on the Riemann sphere I am going to get something, okay.

So it is going to be again it is going to be a contour with starting point f of z 1 and ending point f of z 2 mind you now f of z 1 and f of z 2 are being thought of as points in the extended plane, okay and the image contour is going to be just f of L, okay and what is the if you now you know you can you know that from f z 1 to f z 2 on the Riemann sphere that is in the extended complex plane the spherical distance is actually the shortest distance on the sphere, it is just the is it the minor arch of the greater circle passing through f of z 1 and f of z 2 on the sphere, okay.

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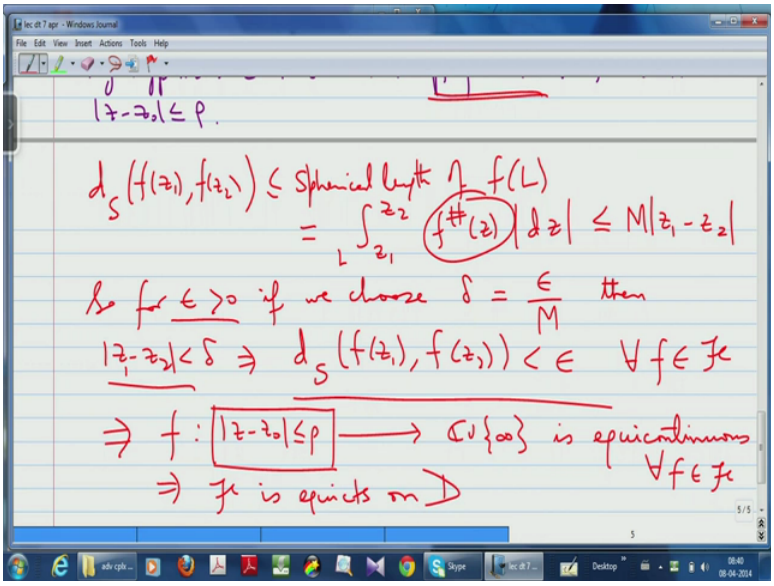

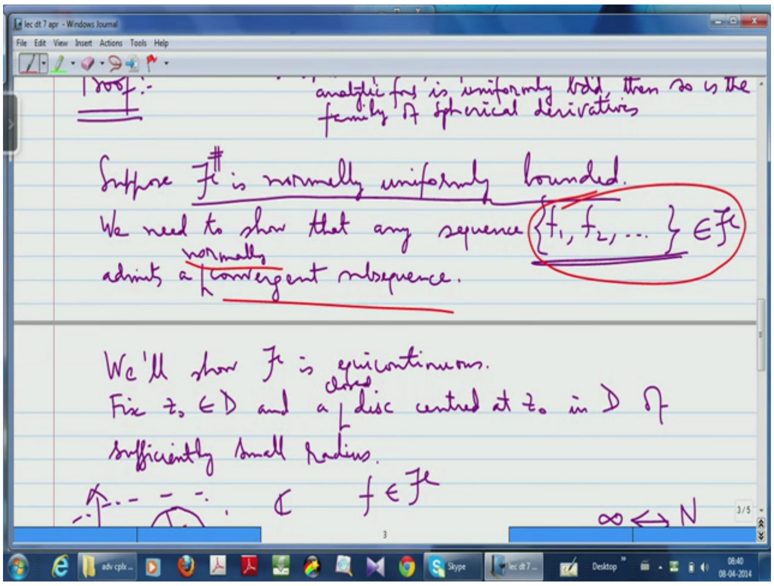

And so what you can write is the distance the spherical distance between f z 1 and f z 2 this is certainly is the shortest distance because it is the geodesic okay curves of shortest length on a surface occur geodesic okay in general if you have a space with a metric then the if you give me two points in the space it is not necessary that the straight line distance if it makes sense is the shortest, okay there could be some other curves depending on the metric especially you could have you could find the distance along the curve to be smaller than the straight line distance in some cases.

For example for spaces for negative curvature, okay but in any case if you take a space where metric is defined on if you take two points in the space then the shortest distance the curve of shortest distance from this point to that point on the space is called geodesic and that is geodesic distance on the sphere the geodesics are all given by the minor archs of the major surface, okay so that is the spherical distance and this is certainly this is the smallest and so this is certainly less than the length the spherical length so let me now abbreviate it spherical length of f of L, okay said to be and well what is the spherical length of f of L you know that how to get the formula for the spherical lengths the formula for the if you give me a curve on the plane that is a contour on the plane then the length of the contour is just given by integrating mod d z okay where z is very low you integrate mod d z from the initial point of the contour to the final point of the contour you get the length of the arch or contour on the plane.

But if you want to get the length of the image of the arch what you will have to do is you have to multiply by the factor which is given by the spherical derivative, okay if you multiply the ordinary derivative and if it is a analytic function you will get the length of the image arch in the complex plane itself, okay that is if you use modulus of the derivative of the analytic function as a scaling factor but if you use the spherical derivative or scaling factor and you will take the spherical derivative corresponding to meromorphic function then you will get the spherical length of the image of this arch on the Riemann sphere, okay.

So what is this? This is going to be just integral from z 1 to z 2 of f hash of z mod d z this is the spherical derivative, alright and what will happen is that you see now since you know now the point is that this integration is being carried out from z 1 to z 2 and ofcourse this integration is over let me put L here because this integration is along the straight line path from z 1 to z 2, okay and that path lies inside this closed disk, okay and on this closed disk all the spherical derivatives are all bounded by M.

So you know mind you spherical length is always a non-negative quantity, okay it is a non-negative real number, okay so what I will get is that this is this is certainly less than or equal to M times mod z 1 minus z 2 this is what I will get because I can replace this f hash of z by M because M is upper bound and the integral from z 1 to z 2 mod d z is just the is just along the straight line segment is just the length of that segment mod z 1 minus z 2, okay so I get this.

But now what is the advantage what is the advantage of this now it tells me I have got equicontinuity. See so for epsilon greater than 0, okay if we choose for epsilon greater than 0, if we choose delta to be you know epsilon by M okay then mod z 1 minus z 2 less than delta will imply that the spherical distance between f z 1 and f z 2 is going to be less than epsilon I will get this inequality, given epsilon greater than 0 whenever the distance between z 1 and z 2 is less than delta I can find a delta such that whenever distance between z 1, z 2 is less than delta this is the spherical distance between f z 1 and f z 2 is less than epsilon and this works for all f in the family script F so long as z 1 and z 2 lie in that closed disk.

So what have I got? I have got equicontinuity, I have got a kind of uniform equicontinuity you can think of this as either equicontinuity at z 1 or thinking at z 2 as a variable or you can think of equicontinuity at z 2 thinking of z 1 as a variable in any case it is a uniform equicontinuity, okay. So what I have got is that f from D f from this disk mod z minus z not less than or equal to rho to the extended complex plane is equicontinuous and this but then ofcourse I can cover the source domain D I can cover every point by such a closed disk lying in the domain therefore I have got equicontinuity at every point so this implies that so and this is equicontinuity f in script F. So basically what I am saying is that F is this family script F is equicontinuous on D so I get equicontinuity, okay.

So basically what I have done is I have just shown that boundedness of the spherical derivatives gives me equicontinuity and that is a very general philosophy whenever you have boundedness of the derivative you integrate and you get equicontinuity that is a very general thing, alright.

Now what I have to show what do I have to show? I have this I started with this I have this sequence here in script F okay and I will have to extract a subsequence which converges uniformly on compact subsets that is what I have to do, what I want to indicate is that you can now do it exactly in the way you proved equicontinuity and uniform boundedness implies sequential compactness in one way of the proof of the Arzela-Ascoli Theorem, okay so what you do is you do so these are the steps, okay.

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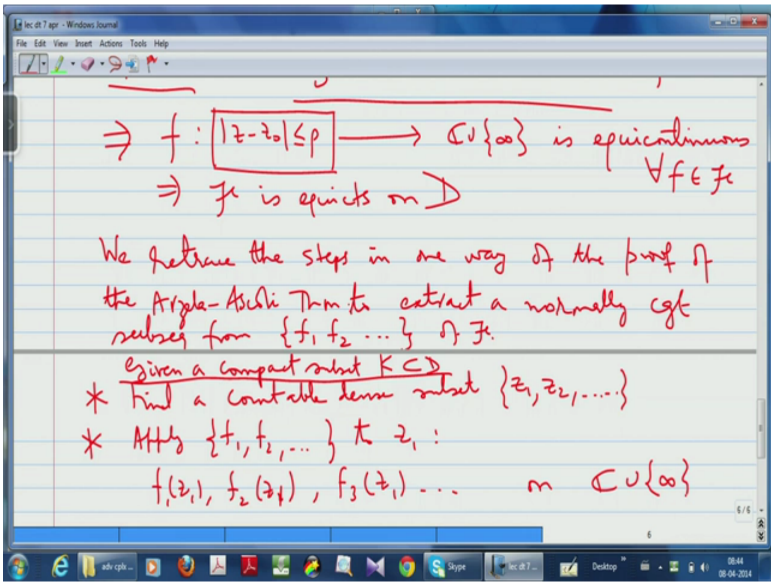

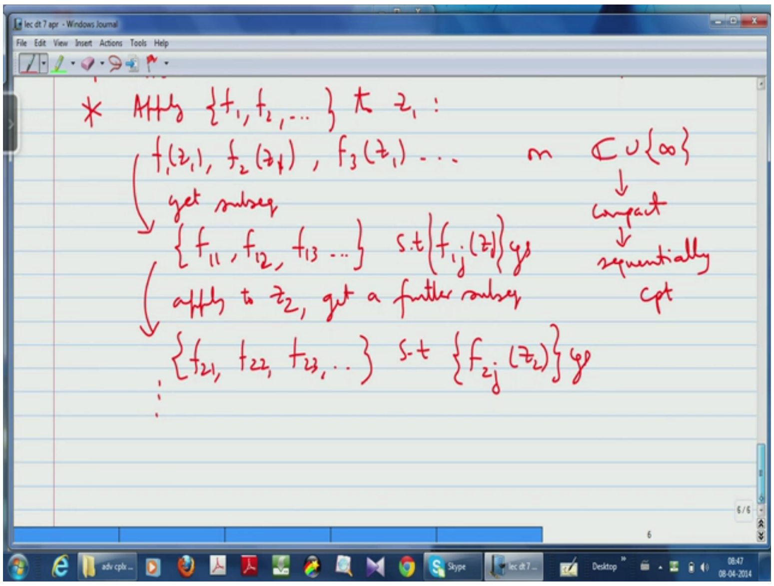

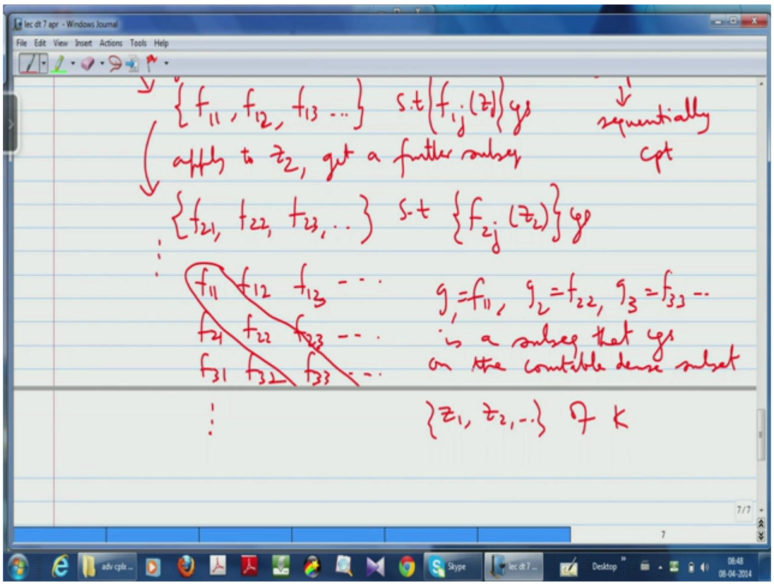

So what you do is we retrace we retrace the steps in one way of the proof of the Arzela-Ascoli Theorem to extract a normally convergent so I am using cgt for convergent as a an abbreviation subsequence from the given sequence, okay. So what you do so I will put it as a star list so the first thing is find a countable dense subset x 1, x 2, etc so you given given start with a compact subset K of D given a compact subset K of D first find a countable dense subset, okay here it is just the general statement that a compact metric space is separable, okay.

Then what you do is now you have now go back and think about the the proof of the Arzela-Ascoli Theorem what you do is that you take the original sequence you will apply it to x 1, okay and you apply it to x 1 you get all these real or complex numbers, okay and now you will use the fact that the original sequence is uniformly bounded to say that you have a sequence of bounded sequence and you will extract a subsequence, okay any bounded sequence of real numbers or complex numbers admits a subsequence convergent subsequence that is how you use it.

But now you see look at the present situation if I apply f 1, f 2 if I apply this sequence to x 1 mind you let me change just change the notation to from x 1, x 2 if you want to z 1, z 2 because all my points are actually my compact subset K is actually a point is a subset of D and all my points are complex numbers so let me change it to z 1, z 2 and so on, okay. Now what I will do is I will apply to z 1 I will apply the sequence, okay and I will get a convergent subsequence I will get a convergent subsequence, why is that? That is because if I apply these functions I am going to get a sequence of points on the Riemann sphere which is compact therefore it is sequentially compact therefore every sequence gives me a convergent subsequence you see so it works that is the whole point.

So apply apply the sequence f 1, f 2 to z 1 what will you get? You get f 1 of z 1, f 2 of z 1, f 3 of z 1 on the Riemann sphere, okay but but this is compact it is a compact metric space so it is sequentially compact and because of that what I will get I get a subsequence f 11, f 12, f 13 such that f 1j of z 1 converges, okay so this is the key step okay this is the key point of difference. When we were looking at real numbers or complex when we are looking at real or complex valued functions, okay when you apply the sequence to a point you got a sequence real sequence or a complex sequence but then you extracted a subsequence because you know it is bounded and where from did the boundedness come, it came from the uniform boundedness of the original family, okay.

But now you do not need any uniform boundedness in this case to extract a subsequence because the values are already being taken in the extended complex plane which is compact and is already sequentially compact I do not need anything more to extract a convergent subsequence that is the big difference, okay. Now what you do is now you iterate, what you do is apply to z 2, okay and you get a subsequence further subsequence which is f 21, f 22, f 23 and so on such that if you take f 2j z 2 this converges, okay and you do this ad infinitum what you will end up with is that you will end up with this matrix as usual so you know you will get this you will get this f matrix of functions 13 and so on f 21, f 22, f 23 and so on f 31, f 32, f 33 and so on so it goes on like this and you know it is the diagonalization trick that we used what we do is that we extract this diagonal subsequence, okay.

Then what is the advantage of this diagonal subsequence? This diagonal subsequence will give you a sequence which will converge at all points of this dense subset this countable dense subset of K, okay so f 11 so g 1 is equal to f 11, g 2 equal to f 22, g 3 equal to f 33 and so on is a subsequence that converges on the countable dense subset z 1, z 2 of K, okay alright.

And now what you do is I will not repeat those steps now you use we have just now proved that all the functions in this family are equicontinuous, okay we have just now proved that. So just use equicontinuity and on this sequence of functions to hook up to show that this sequence is actually Cauchy on the whole space, okay and therefore it is convergent, okay. So the moral of the story is that at this point you use the equicontinuity of the family and mind you that equicontinuity came from the boundedness of the spherical derivatives that is what you have to remember, okay.

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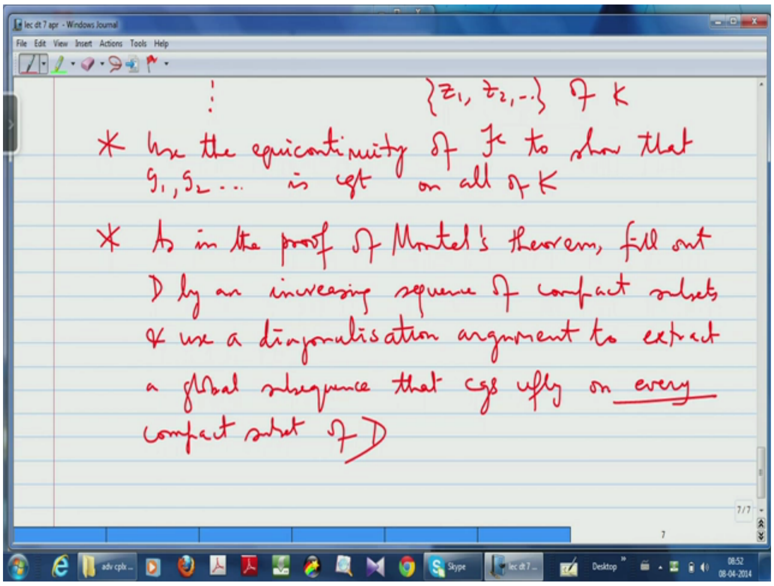

So use the equicontinuity of use the equicontinuity of this family script F to get to show that to show that g 1, g 2, etc is convergent on all of K, okay. So this is exactly as we did in the Arzela-Ascoli Theorem I am not going to repeat it, okay. Now so what have we succeeded using what we have succeeded is given any compact subset of D given a sequence, given any compact subset I am able to extract a uniformly convergent subsequence, okay.

But then what do I need? I need one if I change the compact subset, okay my subsequence could change but I want one global subsequence which works on every compact subset and how do you get that you again get by another diagonalization argument, what you do is you fill up D by a sequence of increasing compact sets, okay with the property that any compact subset is contained in one of one set of this sequence, okay and then use again a diagonalization argument as we used in the proof of Montel's Theorem to extract from this we have global subsequence which is going to be convergent uniformly on every compact subset and that finishes the proof one way of proof of Marty's Theorem that boundedness of spherical derivatives implies if family is normally sequentially compact, okay boundedness of the derivatives on compact subsets, okay so normal boundedness of derivatives implies normal sequentially compactness.

So let me write that (())(42:00) as in the proof proof of Montel's Theorem fill out D by an increasing sequence sequence of compact subsets and use a diagonalization argument to extract a global subsequence that converges uniformly on every compact subset of D so this proves one way, what is the other way you have to show that if you have a normal family you will have to show that it is the spherical derivatives are bounded and the other way is proved by contradiction, if the spherical derivatives are not bounded, okay then I can extract a sequence I can find a compact set and a sequence of functions and a sequence of points at which the spherical derivatives are becoming bigger and bigger and bigger, okay.

Now from this sequence of functions because I assume normal sequential compactness I can also get a subsequence which converges, okay if the functions you know if a family of functions converge meromorphic function converges to a limit function then the family of spherical derivatives will also converge, okay but mind you the spherical derivative of any function is always a finite quantity spherical derivative of any function meromorphic function is only a finite quantity even if you take the function which is uniformly infinity (())(44:00) derivative is 0, okay you will only get a finite quantity.

So if this sequence of functions converges to a function then the sequence of spherical derivatives converges to the spherical derivative of the limit function and that is a finite quantity but on the other hand the original sequence had points where the values were becoming larger and larger so that is a contradiction so that contradiction will proof that you know if you assume that the family is normally sequentially compact spherical derivatives have to be normally uniformly bounded that is the other way for the proof of Marty's Theorem, okay and with that we we are through with the proof of Marty's Theorem, alright.