Advanced Complex Analysis-Part 2
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Lecture 32
Introduction to Marty's Theorem – the Meromorphic Avatar of the Montel and Arzela-Ascoli Theorems

So you see we are looking at compactness of families of functions, okay we want to look at Meromorphic functions alright and so you know what we did last time was Montel's Theorem, okay which was for analytic functions, alright. So let me recall this so that I will give you the background for the formulation of the version of Montel's Theorem for Meromorphic functions which goes by the name of Marty's Theorem, okay and once you have that then we can go ahead and try to we get closer to the proof of the Picard Theorems, alright.

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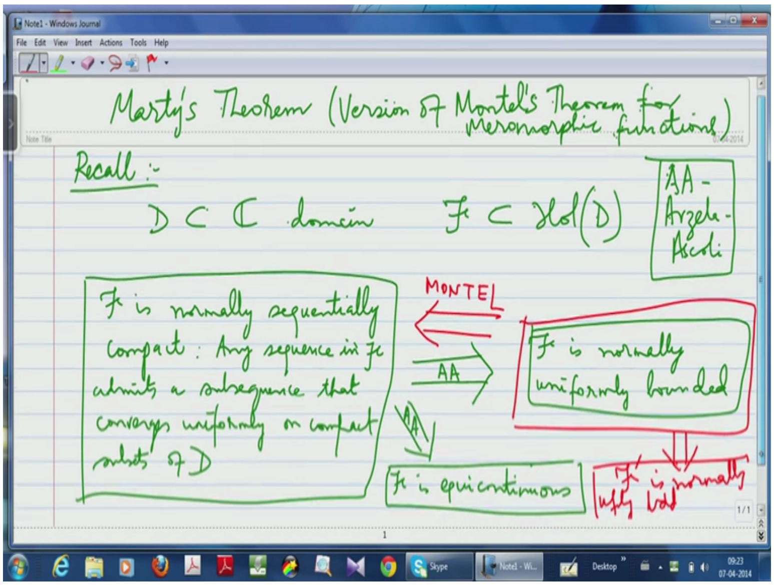

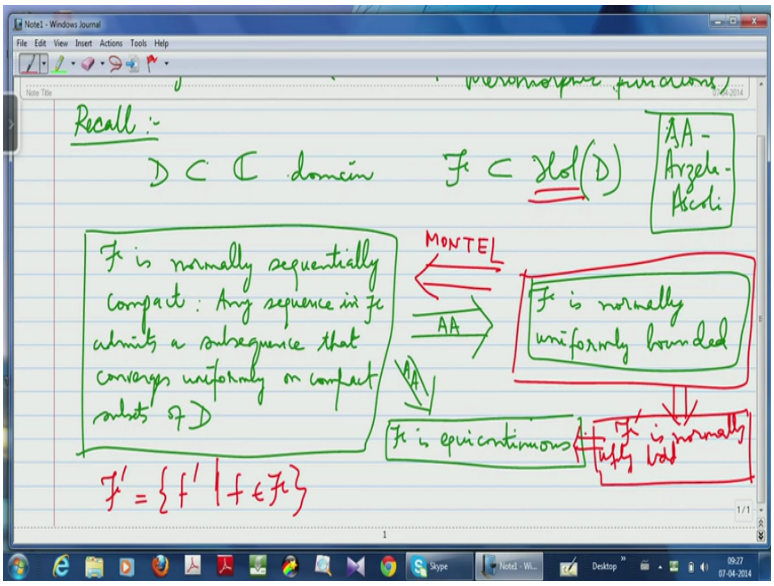

So you see so you can recall so maybe I will put the tittle as Marty's Theorem so let me put here version of Montel's Theorem for Meromorphic functions, okay. So let us recall see so the idea is the following so you have D inside the complex plane its domain it is an open connected set non-empty ofcourse, okay and you have script F is family of analytic functions on D so this is this is subset of holomorphic functions on D, okay and mind you these are ofcourse certainly continuous functions on D with complex values, alright.

So you have this family of holomorphic functions on D and then the whole idea is you want to worry about the compactness of these familiar functions, okay and compactness in what sense compactness can be you know in general compactness is the same as sequential compactness, okay as a general philosophy and so and you should expect therefore you want conditions for which given a sequence of functions in F in the family script F you want to pick out a convergent subsequence, okay.

Now the point is that you know ofcourse since you are working with holomorphic functions, okay just convergence is not a very useful thing you would like to have a kind of convergence that will reserve properties of original functions which are converging to the limit function and therefore the best kind of convergence you can expect is uniform convergence but ofcourse uniform convergence is too much to expect what you will normally get is only normal convergence, okay that is you will get uniform convergence restricted only to compact subsets, alright.

So that is the background and so what is it that you have? See on the one hand you want F is normally sequentially compact and what do I mean by this? I mean that given any sequence in F, I can find subsequence which converges normally that means which converges uniformly on compact subsets, okay so let me write that any sequence in F admits a subsequence that converges uniformly on compact subsets on compact subsets of D, okay so this is this is compactness for us, okay that and that given any sequence you can extract a convergent subsequence but the convergence is only with respect to the normal convergence, okay so this is what you want.

Now how do you what what are the necessary and sufficient conditions for this to happen what are the necessary and sufficient conditions for this to happen? So you know if you want to go from here to here if you want to go from here to here alright so you will get immediately that F is normally uniformly bounded, okay so that means that you given this family of functions in scripts F and if you restrict it to any compact set then that family will have uniform bound.

So there is some positive number such that the modulus of the functions is bounded by that number and this will work for all functions and for all points on that compact set, okay. So this is something that you will get and the other thing you will get is you will also get that F is the family is equicontinuous, okay so plus so let me write this F is so for some reason let me write it at a distance so I will write here F is equicontinuous.

So how does this come? These two implications come because of the they basically come because of the Arzela-Ascoli Theorem, okay so this is so let me write this let me put AA here where you know I will put AA here also where let me write here as a legend AA stands for Arzela-Ascoli Theorem you know this is just the Arzela-Ascoli Theorem. See what Arzela-Ascoli Theorem says is that you know if you are looking at continuous complex or real valued functions on a compact metric space, okay.

Then the condition that such a family of functions is compact, okay is equivalent to that family of functions being uniformly bounded and equicontinuous that is Arzela-Ascoli Theorem. So if you now look at it if you now look at this thing that I have assumed on the this property on the left side which says that this family is sequentially compact if I restrict to any compact subset, okay. So if I take capital K a compact subset of D and I restrict this family to that, okay then I am looking at a family of continuous functions on a compact metric space, okay any compact subset of D is also a metric space it is a compact metric space metric is just metric on D restricted to that subset, okay.

And then by the Arzela-Ascoli Theorem that the family script F will become actually compact you know with respect to the topology given by the supremum norm, okay the supremum norm is defined and you have the metric induced by that norm and the topology induced by the metric so with respect to that this family script F actually becomes a compact family, okay it becomes a compact subset of points, okay and you are now considering this family inside C K, C namely the set of all continuous functions on the compact set K with values in C, okay because analytic functions are ofcourse continuous, okay.

So then by the Arzela-Ascoli Theorem what will happen is that on that compact set K what will happen is that this family will be uniformly bounded, okay that is bounded with respect to the sup norm which is uniform boundedness and you will get equicontinuity. So and since this and equicontinuity is something that is needs to be checked at every point so you will get equicontinuity for the family the whole family throughout all of D, okay to check equicontinuity at all of D at every point of D I just have to check equicontinuity at each point of D and to check at each point of D it is enough to check on each compact subset of D even a point is a compact subset if you want, okay.

So equicontinuity will fall out and restricted to that compact subset I also have uniform boundedness, okay therefore restricted to compact subsets I have uniform boundedness and that is normal uniform boundedness so that is how I get these two implications, okay but the serious thing is to go so that the serious thing is to go the other way round, okay so starting with starting with you know the Arzela-Ascoli Theorem in another direction tells you that is you are on a compact metric space and you are looking at continuous functions complex valued or real valued, if you know these collection of functions is uniformly bounded and it is equicontinuous then your family is compact, okay.

Now so if I start go from this direction suppose I assume F is normally uniformly bounded script F is normally uniformly bounded namely this condition so I will purposely change colour because I want to emphasize something else. So you know I take this I take this condition script F is normally uniformly bounded alright then the beautiful thing is we do not have to add equicontinuity that is the big deal, the big deal is you can go from here to here directly as a theorem and this is the this is Montel's Theorem so this is Montel's Theorem that is you start with a normal uniformly bounded family of analytic functions, okay then it is normally sequentially compact that means you can given given a sequence you can extract a subsequence which converges normally, alright.

And mind you so I have I have to tell you few things here this is this Montel Theorem is stronger than Arzela-Ascoli, okay in the following sense. See what Arzela-Ascoli Theorem will say is that you know if you restrict yourself to a compact subset of D if you take a particular compact subset of D, okay and suppose I have this condition that this family script F is normally uniformly bounded then family script F will become uniformly bounded on that compact subset and again I will get equicontinuity and I will apply the Arzela-Ascoli Theorem and I will be able to pick a subsequence which converges uniformly on that compact subset, okay.

And the ofcourse equicontinuity comes because of the derivatives being bounded which comes as a result of the Cauchy integral formulas and some estimations, okay and the Cauchy estimates of the first derivative, okay but let us forget that for the moment basically what happens is you are able to for every compact subset if I start with the sequence in the family F for every compact subset I am able to get a convergent subsequence I am able to pick a subsequence such that on that compact subset the convergence is uniform.

But the point is if I change the compact subset the subsequence can change, okay if I apply only the Arzela-Ascoli Theorem if I change the compact subset then the subsequence can change but the Montel Theorem is very strong what it says is that I can uniformly find a single subsequence of the original sequence which will converge uniformly on every compact subset it will work for every compact subset, okay that is the power in the Montel Theorem and if you remember this we got this by diagonalization argument, okay we covered the domain D by an increasing sequence of compact sets, okay which fill out the domain and on each member of this sequence of compact sets we picked out a convergent subsequence using Arzela-Ascoli Theorem, okay and then we wrote down this metric of convergent subsequences by you know for the first compact set in the sequence we picked out a subsequence in the sequence of sets covering the space D we picked out one sequence from the original sequence we from Arzela-Ascoli, okay.

Then from this sequence we picked out another subsequence which will work on the next bigger compact subsets and then we went on like this and all these compact subsets eventually filled their union filled the whole of D, okay and the diagonal sequence gave us a sequence of the original sequence which will converge uniformly on every compact subset of D because every compact subset of D is contained in one of the members of this sequence of compact sets that we increasing sequence of compact sets that we constructed to cover D, okay.

So you see we have got this very strong statement from Montel's Theorem, okay so that is one point you have to remember, okay and the other important point about Montel's Theorem is that you do not worry about you really do not worry about equicontinuity, okay and this is basically because you are working with analytic functions. So what is happening is that this condition that the family is normally uniformly bounded tells you that if you take the family of derivatives of these functions then that family of derivatives is also uniform normally uniformly bounded, okay.

And so let me write this so I will write this as F prime is normally uniformly bounded so there is this thing here in between, okay so when I write ofcourse when I write F prime I mean the set of all I mean all those derivatives of functions f which are in small f which are in script F, okay so so script F prime is just the derivatives of the functions in script F, okay and mind you the functions in script F are all analytic therefore the derivatives you know if you take a function which is analytic then all orders of derivatives of that function exist and they are also analytic, okay.

So script F prime is also a bonafied family of holomorphic functions on the same domain, okay and the point is that that is normally uniformly bounded, okay and that is because that is simply because of the fact that the derivative of a function can be expressed in terms of the function using the Cauchy integral formula, okay and therefore if the original functions are uniformly bounded then the derivatives are also uniformly bounded on closed on sufficiently small closed disks, okay.

So the moral of the story is that the derivatives are normally uniformly bounded and because the derivatives are normally uniformly bounded you know this is the philosophy that I told you last time whenever the derivatives are bounded uniformly then the original family is equicontinuous, okay because you just have to integrate, okay so this so there is another implication that is going like this whenever the derivatives whenever the family of derivatives is uniformly bounded then the original family is equicontinuous.

So what happens is that because I assumed that the family is normally uniformly bounded I am also getting equicontinuity the way I am getting equicontinuity is because I am getting actually normal uniform boundedness of the family of derivatives that is the whole point and the reason I am able to get this is because of the Cauchy integral formula because of the Cauchy estimates, okay. So this is how this is how everything works.

Now what is that so what is that we want to do with meromorphic functions okay so if you now if you are see so far we are working here in the set of all holomorphic functions I mean analytic functions, alright but you know you want to work with meromorphic functions the problem with that is that if you are working with meromorphic functions then you are going to allow the value infinity, okay.

So you are going to take values not if you are going to take a meromorphic function you cannot just considered it as a function into complex numbers because then at a pole you cannot define it. Whereas if you considered it as a function into the extended complex plane, okay then at a pole you can define the function value to be infinity and still keep the function continuous even at a pole, okay.

So if you are working with meromorphic functions you want to do a same you want to have the same kind of theorem, okay then you know it is little it is little troublesome somehow you can see that in this whole game you have to pass through this red box that I have put here which is that the derivatives are normally uniformly bounded, okay and that see will work for analytic functions as it is not work for meromorphic functions because the problem will be at the poles at a pole I cannot apply any kind of Cauchy integral formula I cannot express in fact even derivative at a pole is not defined it is a singular point, okay so I am in trouble.

And you know in order to overcome this we had introduced this concept of spherical derivatives, okay so that is what we are going to use. So in fact we will get this theorem that now you again take a domain in the complex plane, you take family of (holo) not holomorphic but meromorphic functions on the domain but mind you you are now considering this as functions not into the complex plane but functions into the extended complex plane, okay.

And when you consider it as functions into the extended complex plane mind you the target plane is not complex plane it is the extended complex plane and the extended complex plane has been made into a metric space by putting the spherical metric and with respect to the spherical metric it is a compact metric space, it is a beautiful metric space, it is just the Riemann sphere with the spherical metric on that, okay alright.

And now what you do is you get this version of the Montel the correct version of the Montel's Theorem for meromorphic functions will tell you that now you again take a family of meromorphic functions the condition that it is normally sequentially compact is equivalent to saying that the spherical derivatives are bounded that is it, okay. So what I want you to understand is that it is rather funny when you move from the holomorphic version of the Montel Theorem to the meromorphic version of the Montel's Theorem which is called Marty's Theorem, okay your condition changes from the normal boundedness of the of the family of functions to the normal boundedness of the derivatives but what derivatives spherical derivatives that is the big change, okay and with that everything works, okay so that is what I am going to state next.

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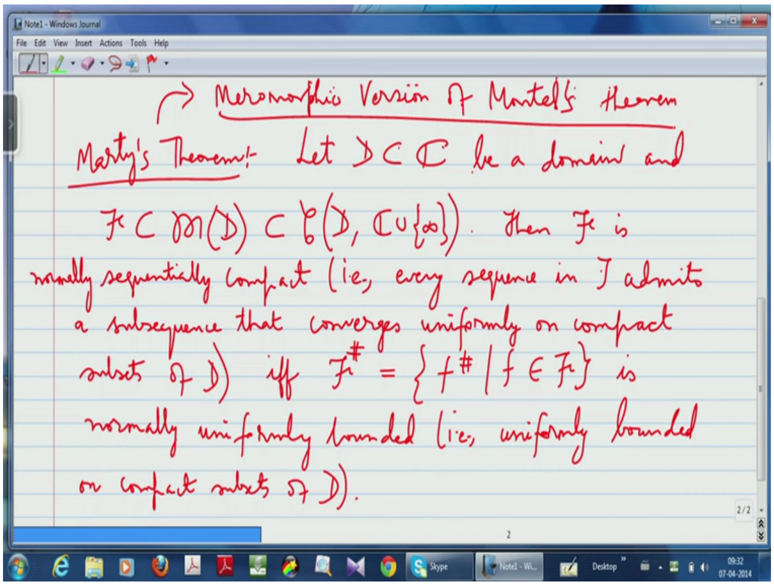

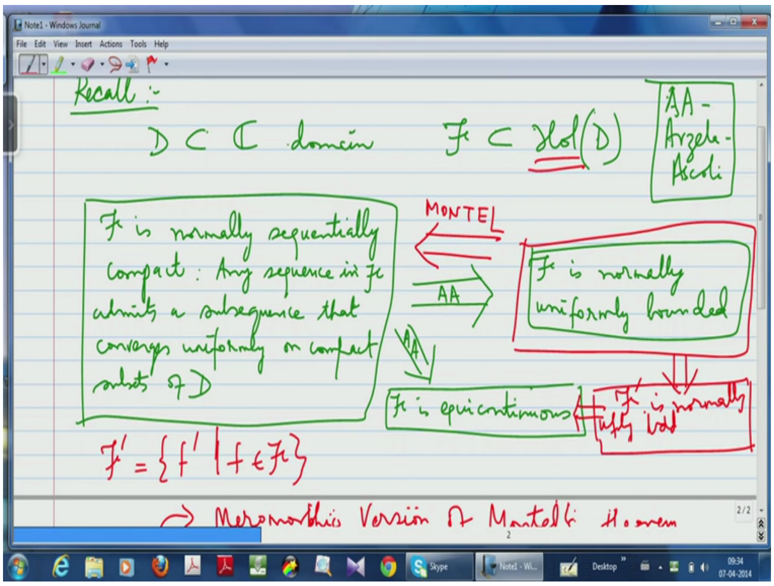

So here is Marty's Theorem. Let D in C be a domain ofcourse non-empty as usual and script F is a family of meromorphic functions on D, okay mind you this is a subspace of the continuous functions on D with values in the extended complex plane, okay you have to remember this this is very very important we are considering meromorphic when you say meromorphic function you are allowing the value infinity, okay otherwise you will not continuity at the pole at poles, okay that is very very important, okay.

Then F is sequentially is normally sequentially compact compact i.e, every sequence in script F in script F admits a subsequence that converges uniformly normally that is uniformly on compact subsets of D if and only if so I will write F hash, what is F hash? This is the collection of spherical derivatives of the functions in F so we use prime for derivative when it is an analytic function when it is not an analytic function but it is a meromorphic function we use hash, okay which is which is the notation we introduced earlier so this is set of f hash such that f belongs to script F is normally uniformly bounded that is uniformly bounded on on compact subsets, okay so this is Marty's Theorem so this is meromorphic version of Montel's Theorem meromorphic version of Montel's Theorem.

So let me write that here and the big deal in this statement is essentially to say that instead of requiring that the original family of functions is normally uniformly bounded which is what the original Montel Theorem want you know needed you shift to the spherical derivatives of these functions that is the difference, okay.

Now you see what does this say in retrospect I mean what it says in retrospect is that in principle it says that if you take a family of even if you take a family of analytic functions, okay even if you take a family of analytic functions the condition that the the usual derivatives are normally uniformly bounded is also equivalent to the normal sequential compactness of the family that is the big deal, the big deal is you know if you go back to this diagram, okay we had this we had this red box here which said that the derivatives the usual derivatives which in this case are they make sense because functions are analytic the usual derivatives are normally uniformly bounded they are uniformly bounded on compact subsets, okay.

Now this itself this itself is good enough to give you normal sequential compactness, okay but there is only there is only one small issue since the compactness is I mean since the convergence is with respect to functions which can take the value infinity the convergence point wise convergence is with respect to the spherical metric that is the difference, okay. So what it means is it means the following suppose I have a family of analytic functions on a domain how do I decide that this family is compact, okay that is it is normally sequentially compact one direct way is use the usual Montel Theorem for which I need all the functions in the family to be normally uniformly bounded to be I must be able to find uniform bound for this family on every compact subset.

There is another way the other way is verify that the derivatives the family of derivatives of these functions that is normally uniformly bounded if you verify that okay then what happens because of this meromorphic version of Montel's Theorem that family of see if the usual derivatives are uniformly bounded on a set then the spherical derivatives are also be uniformly bounded on the set, okay that is because of the way in which the spherical derivatives are defined.

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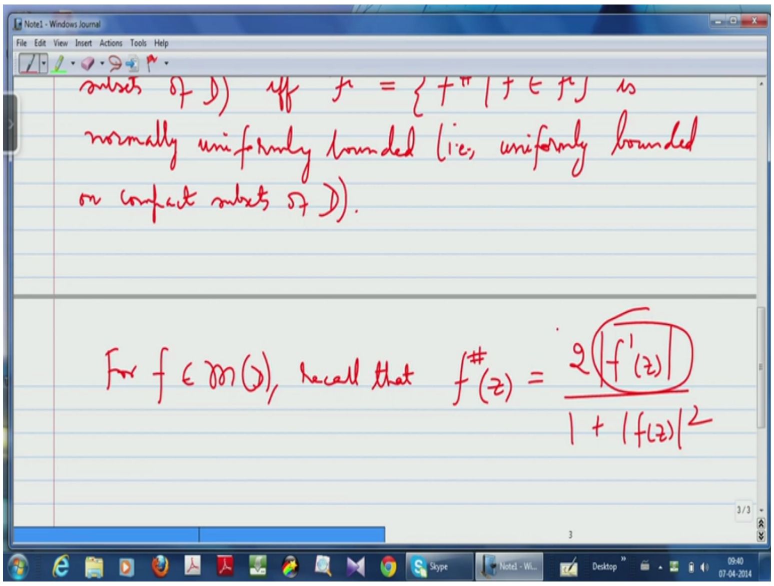

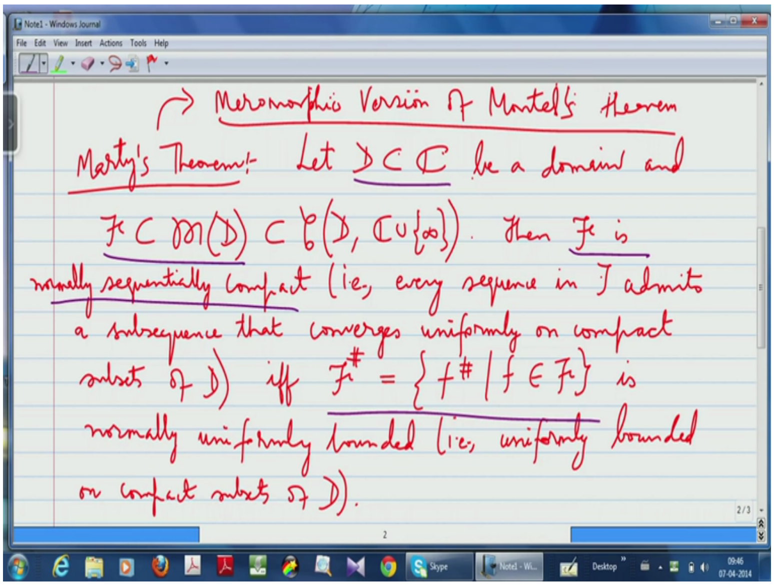

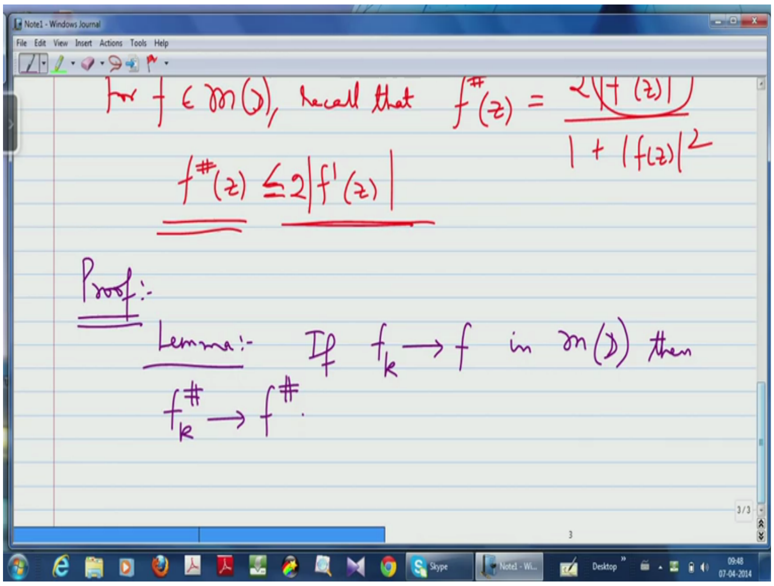

See how is the spherical derivative defined see it is defined like this for for f meromorphic function, recall that the spherical derivative of f at z you know it is defined as 2 times mod f dash of z divided by 1 plus mod f z the whole square this is the definition of spherical derivative this is how spherical derivative is defined, okay this is how we define spherical derivative.

And ofcourse you know there is an issue I have used a mod f dash in the numerator that makes sense only if f dash exist and therefore I can write this only at point z which are not poles but at poles what happens at poles you know we did this in an earlier lecture at poles you extend the spherical derivative by continuity and what happens is that the spherical derivative will become 0 at a pole of higher order, okay and at a pole of order 1 namely at a simple pole the spherical derivative is 2 divided by the modulus of the residue of the function at that pole, okay.

So therefore this so in particular you know if you look at it in a very logical kind of way even if you take the function which is identically infinity, okay mind you that function is also there in (())(31:01) because we are considering functions with you know values possibly being infinity also. If you take the function which is always infinity that function is also by definition one for which you have to define the spherical derivative and the spherical derivative will be 0, okay so this is something that we make as a default definition, okay.

And this so I am just trying to say if you take the function which is uniformly infinity whose infinity at every point on your domain that function is also included and that function also spherical derivative is also included spherical derivative is 0, the way you think about it is a that you know usually the derivative should be 0 if the function is constant after all the function which is infinity at every point is just the constant function infinity so you should expect the derivative to be 0 that is one way of looking at it.

The other way of looking at it is what is the spherical derivative? If you take a meromorphic function you treat it as a map into the Riemann sphere, okay and what it does is it is the magnification factor of the length of the image I mean it is a magnification factor that you will have to put in to calculate the length of the image of an arch under this mapping. So suppose you have an arch on the suppose you have a suppose you have a an arch or contour on the complex plain in your domain where your meromorphic function is defined, okay and you take its image under this meromorphic function it will land on the Riemann sphere where I am thinking of the extended complex plane as the Riemann sphere.

So I am going to get an arch on the Riemann sphere, ofcourse this arch can pass through infinity, it will pass through infinity if the original arch in the plane pass through some poles of your meromorphic function wherever original arch in your complex plane hit a pole the image will hit on the north pole on the Riemann sphere which corresponds to the point at infinity, okay.

And if you take the image arch how do you get the length of the image arch, what you will do is you will integrate the meromorphic function not the meromorphic function in fact you will integrate spherical derivative along the original arch and you will get the length of the image arch. So the length of the spherical derivative is a magnification factor, okay and if this if your original function is just the function which is constant function infinity then it is going to map your whole domain onto a point, okay if you take the function which is constant function infinity then your whole domain is going to be collapsed to the point which corresponds to the north pole.

So any arch is going to be collapsed to a single point, okay so what is the magnification factor 0 and that should be the spherical derivative. So this is another way of saying that you know if you take if you take the constant function infinity you must think of the spherical derivative of that to be 0, okay that is another point that you will have to remember in mind. So but in any case the spherical derivative as it is is always a continuous function and that is the reason where we are able to integrate integrate it always even if your path of integration passes through some poles of f that is very very serious, okay.

But anyway what you see from here is that because that is this mod f dash term here, okay what it will tell you is that if you are looking at a family of analytic functions whose derivatives are normally uniformly bounded, okay then these numerators are normally uniformly bounded, okay but then you know I can forget the factor 1 plus mod f z squared denominator, okay because that is a factor greater than or equal to 1 and its reciprocal is less than or equal to 1.

So actually I can write f hash of z is actually is less than or equal to 2 mod f dash of z I can write this this make sense, okay because the denominator I can forget the denominator, okay and what does this tell you? This tells you that whenever f dash is defined okay in particular if you are looking at a family of analytic functions okay and the derivatives are make sense then if you know the derivatives are bounded then it means that the spherical derivatives are bounded because the spherical derivative is bounded by 2 times bound for the usual derivatives.

So if you start with a family of analytic functions such that the derivatives are bounded then the spherical derivatives are bounded and Marty's Theorem will tell you that these family of analytic functions considered as a family of meromorphic functions mind you analytic functions are also meromorphic functions but when you considered it as a family of meromorphic functions you are allowing the value infinity with that consideration this family becomes sequentially normally sequentially compact that means given any sequence you can get hold of a subsequence which converges normally.

So finally what happens is that this box that I wrote down here is the crucial condition that is crucial for both the original Montel Theorem and also the meromorphic version of Montel's Theorem which is Marty's Theorem so this is the crucial thing the boundedness of the derivatives, okay. So but the only thing that can happen is that your sequence of analytic functions may go to infinity because that is (())(36:50), okay.

See if you go back and think we proved the following thing you take a sequence of analytic functions on a domain, okay if you take convergence with respect to the spherical metric either they will converge to an analytic function or they will converge to a function which is identically infinity, okay and the same kind the of thing happens in meromorphic functions you take a sequence of meromorphic functions which converges normally on a domain, okay then either the limit is a meromorphic function or it is the function which is you know identically infinity you do not get bad behaviour you do not get a sequence of holomorphic functions or analytic functions going to a function which is meromorphic strictly meromorphic or you do not get a sequence of meromorphic functions which goes to a function which has funny singularities namely it may have non isolated singularities or it may have isolated essential singularities such these kind of horrible things do not happen, okay.

So if you take this thing that I have put down which I have now rounded in a long ellipse as the important condition, okay then that is the condition for sequential compactness that is what I want to say and see these conditions work if the functions are analytic, okay and the analogous condition namely the derivatives replaced by the spherical derivatives that works if the functions are meromorphic, okay so this is the very very important point in our theory that we are finally managed to translate compactness of a family of meromorphic functions or analytic functions to just uniform boundedness of derivatives that is all, okay.

And if they are usual analytic functions use the usual derivatives, if they are meromorphic functions use spherical derivatives that is all, okay. So bringing in the derivatives is the big deal here, okay. So we will now need to see a look at a proof of this theorem and the proof is pretty except that you will have to worry about all these issues there are little little things that need to be checked, okay.

So I will try to write down the proof pretty short steps and I will ask you to do some small verifications so let me say the following thing so you see let me write the so here is what I have to proof I have domain in the complex plane, I have a family of meromorphic functions I assume if I have to assume first that the family is normally sequentially compact and I have to show that the spherical derivatives are bounded, okay normally uniformly bounded and I have to do the other way round.

And what does the proof actually involve it involves the few simple results so let me write this so here the few lemmas that I want to worry about or rather let me say lemma if a sequence f k of meromorphic functions converges to f, okay in m D okay then the same thing happens to the sequence of spherical derivatives, okay ofcourse here again I must so I have written in very very simple words but I must again insist when I say converges I mean converges normally, okay it means it is uniform of compact subsets.

So if f k is a sequence of meromorphic functions on D it converges uniformly on compact subsets to a function f we have already seen that this f can either be meromorphic or it can be the function which is identically infinity that we have already seen then this normal convergence preserves derivatives, okay. So this is something that we have seen with analytic functions if a sequence of analytic functions converges normally to a given function then the limit function is also analytic and you know you can the nth order derivatives of the original sequence of functions will converge normally to the nth order derivative of the limit function this is all just because of normal convergence uniform convergence of compact subsets, okay.

So the same thing happens with spherical derivatives this is one fact that we will have to use, okay. So you can check this and well if you want to go back to the let me tell you about the let me tell you atleast in words about the proof of this theorem ya atleast one way is very clear, okay. Suppose your family is normally sequentially compact, okay and suppose contrary to what is required the family of derivatives spherical derivatives is not normally uniformly bounded then you know there is a compact subset on which these derivatives will go to infinity, okay.

And so there is a compact subset and a sequence of functions where the spherical derivatives will go to infinity, okay that is what you get if you contradict a normal uniform boundedness, okay of the spherical derivatives but if that happens then the original family could not have be normal because if the original family were normal then what would happen is that the original from every sequence you can pick out a normally convergent subsequence if the subsequence is normally convergent then the spherical derivative is also normally convergent, okay but then we have already obtained a sequence of spherical derivatives which does not converge, okay.

So the point you will have to remember here is that when you are considering spherical derivatives the convergence is with respect to the usual distance function on the real line, okay mind you that is another important point the spherical derivative is a positive non-negative real valued function, okay and whenever you talk about convergence of the spherical derivatives you are working with convergence on the real line that is something that you should not forget, okay and therefore you get a contradiction. So this is one way of the theorem the other way I will prove in the next lecture, okay.