Advanced Complex Analysis-Part 2

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Lecture 31
Completion of Proof of the Montel Theorem – the Holomorphic Avatar of the Arzela-Ascoli Theorem

So let us get along with the proof of Montel's Theorem. See so the key point that you have to remember is that you know the difference between Montel's Theorem and the Arzela-Ascoli Theorem is that the Arzela-Ascoli Theorem is for continuous functions, okay defined on a compact metric space alright continuous functions ofcourse real valued or complex valued in fact you can even have continuous functions with values in a compact metric space, okay.

Whereas the Montel's Theorem is for ofcourse it is for analytic functions and then we are going to extend it for meromorphic functions and these are going to be defined on domains, okay which are certainly not compact, alright. So one thing that you have to contrast and compare is that the Arzela-Ascoli Theorem as it is for continuous functions defined on compact metric space whereas the Montel Theorem is for analytic functions and later on for meromorphic functions defined on a domain, okay so it is not a compact set, right it is an open connected set.

Then that is so that is one difference that is actually two differences the domains are different in the Arzela-Ascoli Theorem the domain is a compact metric space, in Montel Theorem the domain is the domain of the function is actually the domain in the complex plane alright. Then the second thing in the Arzela-Ascoli Theorem we are worried about real or complex valued continuous functions, whereas in the Montel Theorem you are worried about analytic functions and later on meromorphic functions, okay.

Then the other important thing is that what is the similarity? The similarity is that both of these theorems tell you when a family of functions is compact, okay. Now if you look at it from the view point of the Arzela-Ascoli Theorem compactness is it corresponds to sequential compactness, okay and sequential compactness is just the condition that given a sequence of functions in the family you are able to extract a convergent subsequence, okay.

And whereas in the Montel Theorem what will happen is that you will have a normal version of this. So the Montel Theorem will tell you that you are given family of functions is compact in the following sense that given a sequence in the family you can extract a subsequence but now this subsequent is not convergent in the sense of functions that is it is not uniformly convergent on your domain but it is uniformly convergent only when restricted to compact subsets of the domain that is normal convergence, okay so that is the difference.

So in the Arzela-Ascoli Theorem what you get is a convergence with respect to uniform convergence, okay and that means it is uniform convergence of the whole space, whereas in the context of the Montel Theorem what you will get is convergence with respect to only compact subsets that is normal convergence, okay so that is the settle difference that you have to understand.

And the whole but the punch line is that in the Arzela-Ascoli Theorem you need for this compactness or sequential compactness you need the two properties of the for these functions of the family to satisfy one is uniform boundedness, okay the other one is equicontinuity, alright. But this is for continuous functions in the Arzela-Ascoli Theorem case but if you come to the Montel case, okay equicontinuity is free it comes because you are looking at analytic functions because you are looking at analytic functions you know that the derivatives of the functions are derivatives are expressible using the Cauchy integral formula and then you can estimate the integrals and therefore you get the so called Cauchy estimates and the Cauchy estimates will tell you automatically that the derivatives are bounded and it is a natural fact that whenever the derivatives are bounded the original functions are themselves equicontinuous.

So equicontinuity comes for free it comes automatically if you are looking at analytic functions but if you are looking at that is in the context of Montel's Theorem but if you are looking at the context of Arzela-Ascoli Theorem you have to give equicontinuity as a extra condition because all you have is continuity to begin with of the functions, okay.

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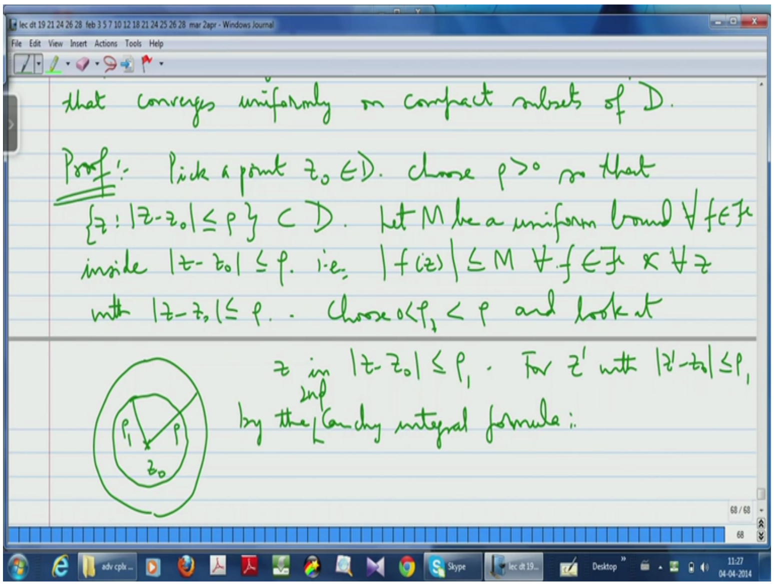

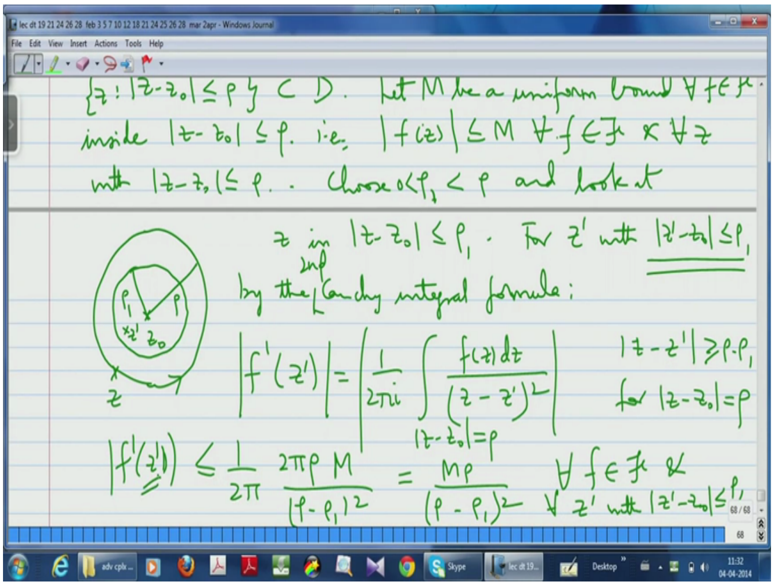

So the first point I want to show is that this family I am starting with the domain D in the complex plane and script f is a family of analytic functions on D and it is normally uniformly bounded so it is uniformly bounded when restricted to any compact subset of D and I have to show that given a sequence in the family I must be able to get a subsequence that converges uniformly on compact subsets, okay and the first thing I will show is that equicontinuity is automatic, okay.

So for that pick a point z not belonging to D okay choose rho greater than 0 so that the closed disk centred at z not radius rho the set of all points in that closed disk is inside D okay you can do this because D is an open sets ofcourse I am working with a non-empty open sets, okay so D is an open connected set. So it give me a point of D by openness there is always a small disk surrounding that point which is also in D and I can take a smaller disk along with the boundary inside that disk, okay so I can find this rho, alright.

Ofcourse you know the reason why I am including the boundaries because I want the compact set, okay if you take a closed disk it is both closed and bounded so it is compact and then I can use the hypothesis that the family of functions is then uniformly bounded on this compact set. So let M be a uniform bound for all let M be a uniform bound for all f belonging to script F inside Mod z minus z not less than or equal to rho, okay.

So what does this mean? This means that mod f of z is less than or equal to M for all f in script F and for all z with mod z minus z not less than or equal to rho, okay and now what we what I would like to show is that I would like to show that in a smaller disk what happens is if you take a smaller closed disk then the family is actually equicontinuous.

So what I do is I chose a smaller closed disk so basically you know if I draw a diagram so it is going to be like this I am going to have this I am going to have this disk centred at z not radius rho and then you know now what you do is choose rho 1 which is less than rho rho 1 which is positive and less than rho, okay so that I can consider the smaller closed disk of radius rho 1 centred at z not, okay and look at z in mod z minus z not less than or equal to rho 1, okay.

So the situation is like this I take a smaller disk and this length is rho 1, okay and I am looking at the smaller disk alright. Now for z prime in the smaller closed disk by Cauchy integral formula well if you want second Cauchy integral formula what do you have you have if I write it out I will get mod well first let me write out the formula without worrying about the estimate in it.

So f dash of f dash of z prime is 1 by 2 pi i integral over mod z minus z not is equal to rho f of z d z by z minus z prime the whole squared. So this is the Cauchy integral formula, okay and mind you the contour of integration is this outer this outer circle taken with the positive you know positive sense, right. And now the point is that you have this but then I want to basically what I am trying to show is that I am trying to show that in the smaller closed disk all the derivatives are bounded, okay and because all those derivatives are bounded in the smaller closed disk the family is equicontinuous, okay.

So all I am saying is that pick any point there is a small closed disk around that point where the family is equicontinuous and since the point you pick those arbitrary this will tell you that the family is equicontinuous on all of D okay so you get equicontinuity for free, okay and ofcourse the reason is because you have the Cauchy integral formula, okay. So what is this modulus of this will be modulus of this and you know the modulus of the integral is less than or equal to the integral of the modulus it is less than or equal to the maximum of the modulus of the integrant as the argument variable of integration varies over the region of integration multiplied by the length of the contour.

So what I am going to get is I am going to get this is less than or equal to 1 by 2 pi that is what I will get this term outside 1 by 2 pi i and the length of this contour is going to be 2 pi rho that is just a circle radius rho alright and then what about the integrant the modulus mod f is going to be bounded by M because mod f is always bounded by M for all f in the family script F, okay so mod f is going to be bounded by M so I can put this M here.

And as far as the denominator is concerned you see look at this see z is lying on the contour of integration, okay and I am my z prime is see my z prime is lying in the smaller disk so my z prime is here, okay and the distance between z and z prime is therefore atleast rho minus rho 1, okay. So mod z minus z prime is greater than or equal to rho minus rho 1 for mod z minus z not is equal to rho, okay if you take z on the outer circles then the distance from a point to the inner circle has to be atleast rho minus rho 1 which is the difference of the radii, okay.

The closes z prime can get to z is along this radial line from z not to z and that closes distance is rho minus rho 1, okay. So you know why I need this inequality because now I want to I have to invert the inequality and then the direction of the inequality will change so I will get 1 by mod z minus z prime the whole squared mod will be less than or equal to 1 by rho minus rho 1 the whole squared. So I can put a rho minus rho 1 the whole squared here, okay.

So finally I will end up with just M rho by rho minus rho 1 the whole squared, okay and this is a bound this is a bound so this is true for all f all functions f in script F and for all z prime with lying in the inner disk mod z prime minus z not less than or equal to rho 1, okay so this happens. So in sum what have we shown we have just shown that the modulus of the derivatives of all your functions inside smaller closed disk they are uniformly bounded by this constant, okay and this constant has got nothing to do with any particular point see this constant that I have got on the right side it is M rho by rho minus rho on the whole squared that has got nothing to do with the points z prime except that the z prime should lie inside this in smaller disk and it has got nothing to do with also the function f that I choose from the family so it is uniform constant, okay.

So what you have shown is that the because of analyticity the functions the derivatives are bounded.

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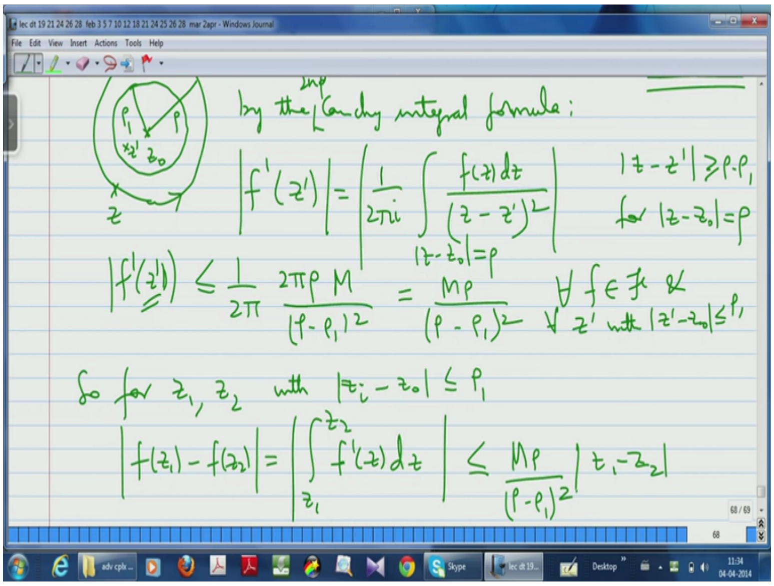

Now because the derivatives are bounded you get equicontinuity and how is that you get that by actually integrating these functions. So well you know if I had taken so so for if I take two points z 1 and z 2 with mod (z 1) mod z i minus z not less than or equal to rho 1 that is if you take two points in the smaller closed disk, okay you take two points z 1 and z 2 in the smaller closed disk then if you f z 1 minus f z 2 is nothing but the result of integrating from z 1 to z 2 of f dash of z d z okay this is because of just because of this fundamental theorem of integral calculus the moment the function has an anti-derivative then the integral is just the difference of evaluation of the function at the final and initial points, okay.

So you have this but ofcourse the path of integration here really does not matter so long as the path lies inside this closed disk and it is well you know I am taking the path to be the straight line segment form z 1 to z 2 if I take that straight line segment path then what will happen is that well if I put a mod to this I will get this is less than or equal to mod f dash of z okay which is now I have got a bound for that and mind you this z is lying on the path of integration and it therefore it is inside this smaller closed disk so this bound applies. So I will get this M rho by rho minus rho 1 the whole squared this is the bound for f dash of z okay and the length of this contour from z 1 to z 2 is just the length of the line segment from z 1 to z 2 that I am choosing so it will be a mod z 1 minus z 2 this is what I get, okay.

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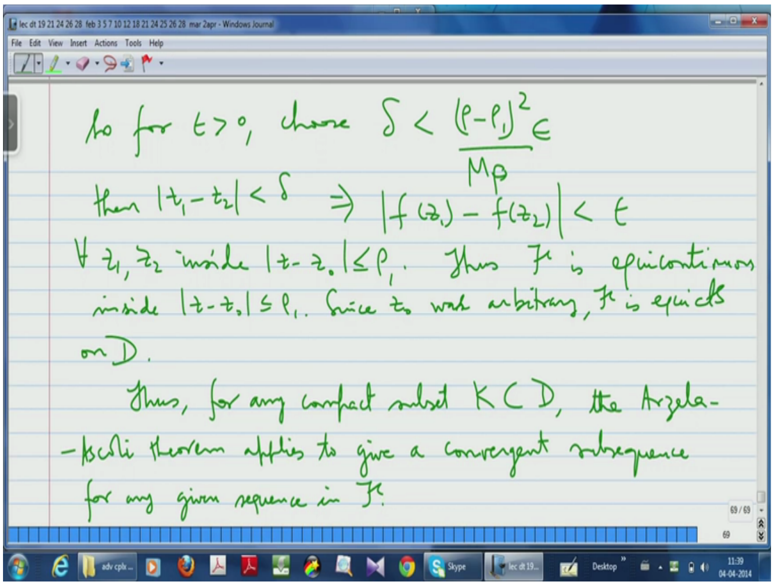

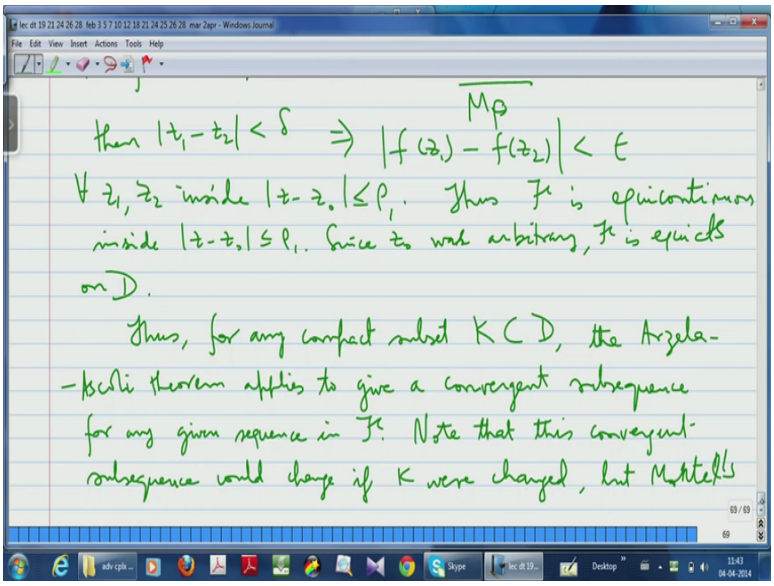

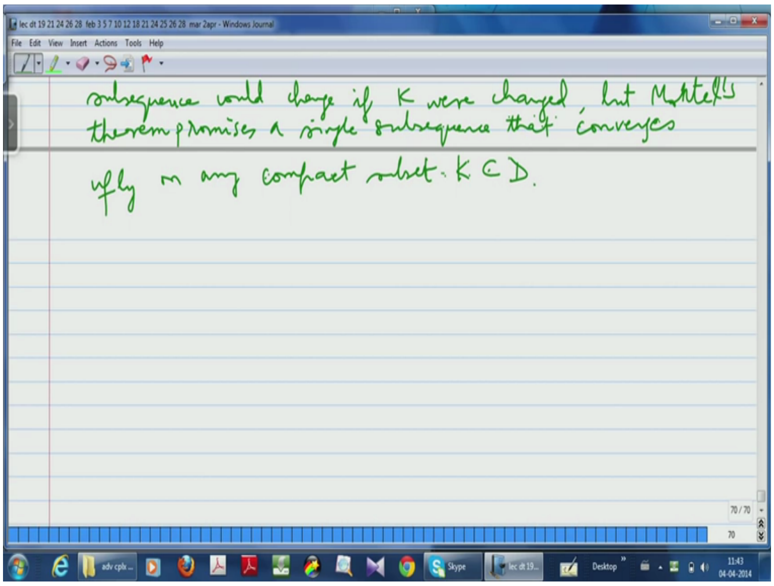

So now that is it this gives me equicontinuity because so for epsilon greater than 0 chose delta to be less than you know what to do you have to choose it to be less than rho minus rho 1 the whole squared by M rho times epsilon you choose this okay then mod z 1 minus z 2 less than delta will imply that mod f z 1 minus f z 2 will be less than you look at this quantity I have to replace this mod z 1 minus z 2 by of the bigger quantity rho minus rho 1 the whole squared by M rho epsilon, okay and I will get less than epsilon, okay. So basically this is true for all z 1 and z 2 inside mod z minus z not less than or equal to rho 1, okay.

So what you have shown is that given any two points the moment two points inside the smaller closed disk are lesser than delta the function values are lesser than epsilon and this works for any two points and for any function so you have shown kind of you have shown uniform continuity of the family inside the smaller closed disk, okay. So thus script F is equicontinuous in inside mod z minus z not in the smaller closed disk. Since z not was arbitrary, F is equicontinuous on D so I am writing equicts for equicontinuous on D. So this is the important point I mean this is the distinguishing feature between Montel's Theorem and Arzela-Ascoli Theorem in philosophy.

While in the Arzela-Ascoli Theorem you need equicontinuity, in the Montel Theorem you get equicontinuity for free because you are already working with analytic functions and so the moral of the story is if you have a uniform bound for your functions that gives rise to a uniform bound for the derivatives okay that is because of the Cauchy integral formula and estimation and the uniform bound for the derivatives gives you equicontinuity for the original functions.

So just the uniform boundedness of the original functions for the analytic functions is enough to give you equicontinuity also, okay so we are now in good shape now you see now I can actually apply the Arzela-Ascoli Theorem for example on this if I take any you take any compact subset of D okay take any compact subset of D then these continuous functions will restrict to continuous functions on that compact set ofcourse okay any continuous function on a topological space if you restrict it to a subspace it will remain continuous if you take for the subspace the subspace topology, okay.

So if you take a compact subset of the of D if you restrict these family of functions you are going to get continuous complex valued functions on that compact set and they are also going to be equicontinuous because you have already checked the equicontinuity, alright. So and mind you already assume that the family is normally bounded normally uniformly bounded therefore they are also going to be uniformly bounded on the compact set therefore the usual Arzela-Ascoli Theorem applies and given any sequence I can extract a subsequence, okay. So I have brought myself into the for view of applying the usual Arzela-Ascoli Theorem, okay so let me write that down.

Thus, for any compact set for any compact subset K in D, the Arzela-Ascoli Theorem theorem applies to give a convergent subsequence for any given sequence in script F, okay. So you are able to apply the Arzela-Ascoli Theorem but you are still but we are still not I must say we are only half way through the Montel Theorem see there is a small set again there is a small set you have to notice.

What have we achieved so far give me a compact set K, suppose I start with a sequence in the family F, give me a compact set K I will be able to get a convergent subsequence but you see if I change the compact set I make it a different convergent subsequence, okay whereas what does the Montel Theorem say the Montel Theorem say you start with a sequence you can get a subsequence the same subsequence which will work on every compact set you see that is the settle difference so we are half way through for given a sequence in F in the family script F you can always extract a subsequence which converges on that compact set but if you change the compact set the subsequence can change.

What Montel Theorem promises is one subsequence that will work for all compact sets, okay and the way to get that it is a very very clever thing it is again a diagonalization argument, okay. If you go back to the Arzela-Ascoli Theorem we use a diagonalization argument actually in the proof of the Arzela-Ascoli Theorem one way where we tried to show that you know if a family is equicontinuous and uniformly bounded then it is sequentially compact or compact.

So what we did was we took a family of continuous functions it mean that family and we tried to extract the convergent subsequence but since we were in a closed subspace we just contented ourselves with extracting a Cauchy subsequence, okay but the point is how did you get this Cauchy subsequence we knew that the metric space was on which the functions were defined was compact so we knew it is separable so we took a separable dense subset, we took a countable subset of points x 1, x 2, x 3 and so on which was dense and then what we did was we took this sequence first apply it to x 1, okay and by boundedness this gave you a subsequence of functions which converge that x 1 and then we apply that subsequence of functions to x 2, okay and again being bounded we got a further subsequence which converge that x 2 and then we did this process ad infinitum we got a matrix of functions and then we took the diagonal sequence and that diagonal sequence was a sequence of functions that converged on this countable dense subset and then we used equicontinuity to interpolate and to conclude that therefore this diagonal sequence converges on all of x and uniformly, okay that.

So you see the same kind of diagonalization argument we use now, okay and the key to that is that is a special construction what you do is that you construct a sequence of increasing sequence of compact sets which fill out your domain D, okay. So it is more like you know if your domain D for example is the whole complex plane okay and increasing sequence of compact sets which fills out the complex plane may be just sequence of disks closed disks of increasing radii, okay. So it is just that but you will have to take care about the boundary, okay so we will do that.

So let me write this down note that this convergent subsequence could change if K were changed, but Montel's Theorem theorem promises a single subsequence that converges uniformly on any compact subset, okay so that is a settle difference.

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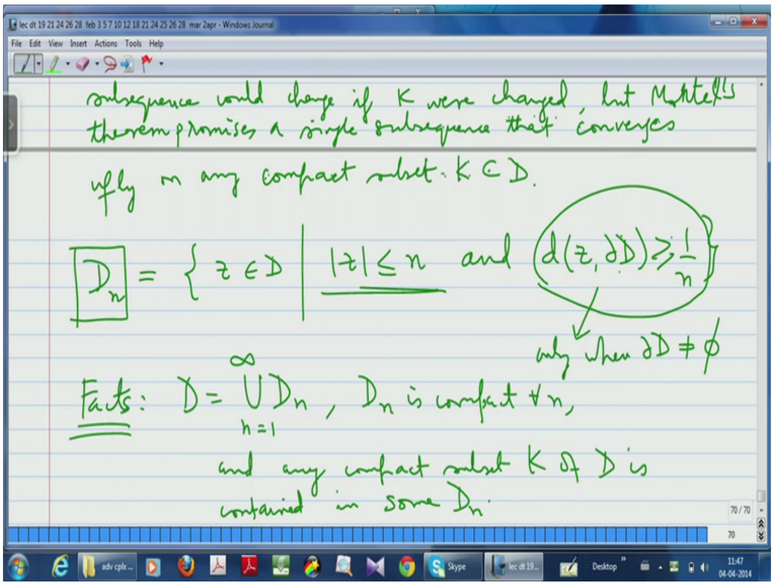

So what we will do is that so I am going to look at these sets D n so here is my set D n here is how I define it it is a set of all points in D with the following property mod z is less than or equal to n, okay so it is a subset of the closed disk centred at the origin radius n, alright. So it is a anyway that this first condition tells you that each D n is automatically bounded because that is this bound, alright and then and here is the second condition which is the distance from e z to the boundary is atleast 1 by n, okay. So it is rather you know it is rather a nice condition, okay.

So what I want you to note is that you know there are only two essentially two cases that you have to understand see if dou D is empty what does it mean? If dou D is empty it means D is the whole space, okay dou D can be empty only if and only if D is the whole space because you know after all dou D put together dou D is a boundary of D dou D and that is a closed set you know if you take dou D and take the union with D you get D closure, okay.

Therefore dou D being empty is same as D equal to D closure but if D is equal to D closure you are saying D is closed but D is already open D is both open and closed okay and if it is non-empty it has to be the whole space because whole space is connected, okay so d only so d situation that dou D is empty corresponds to D is d whole context plane, okay that means actually you are looking at a family of entire functions, okay and in that case this D n is just the sequence of disks of increasing radius and this condition will become useless the distance between D z and dou D is greater than or equal to 1 by n is super force, okay.

You write this condition only when it make sense only when dou D is non-empty, okay if dou D is empty if you want define d distance for d infinity so that infinity is always greater than or equal to 1 by n if you want but write think of this condition only when dou D is non-empty, okay. So you know what is it that you are doing the reason why we put this condition is because you know you cannot include everything in D, okay.

If you try to include everything in D you will have issues if you try to because D is open, okay because D is open basically I want all these D n's to be compact, okay. So that the fact is that D is actually the union of all these D n's D is the union of all these D n's so it is 1. The second thing rather facts second thing is D n is compact for every for each n, that is the second condition and third this is the very important condition any compact subset is contained in some D n and any compact subset K of D is contained in some D n so this is the beauty about this collection D n.

And you see it is this compactness that I am worried about, see each D n is automatically bounded there is no problem about that each D n is automatically bounded. So the only thing that prevents it from being compact is it not being closed, okay if each D n if you check that each D n is also bound is also closed then each D n will become both closed and bounded so it will become compact, okay.

Now the big deal is it is for making sure that each D n is closed that I have put this extra condition that distance from z to dou D is greater than or equal to 1 by n that is the reason I have put it that is because you see suppose my reason is just unit disk, suppose D is just the unit disk, okay suppose D is just set of all z such that mod z less than 1, okay then what is D 1? D 1 is D, what is D 1 is set of all points which is whose modulus is less than or equal to 1, okay that is if I forget the other condition, okay forget the second condition that the distance from the point to the boundary of D is greater than or equal to 1 by n suppose I do not put that condition and if D is the unit disk then D 1 is D so D 1 become the unit disk and it is not compact because it is not closed, okay.

So therefore for the closeness of each D n that you put this extra condition the distance you are throwing away part of the boundary, okay you are throwing taking points from certain particular distance 1 by n from that boundary and you are not allowing points to go very close to the boundary, you allow upto a certain distance minimum distance which is 1 by n and that condition makes each D n closed I mean that is the significance of this condition that I have circled, okay.

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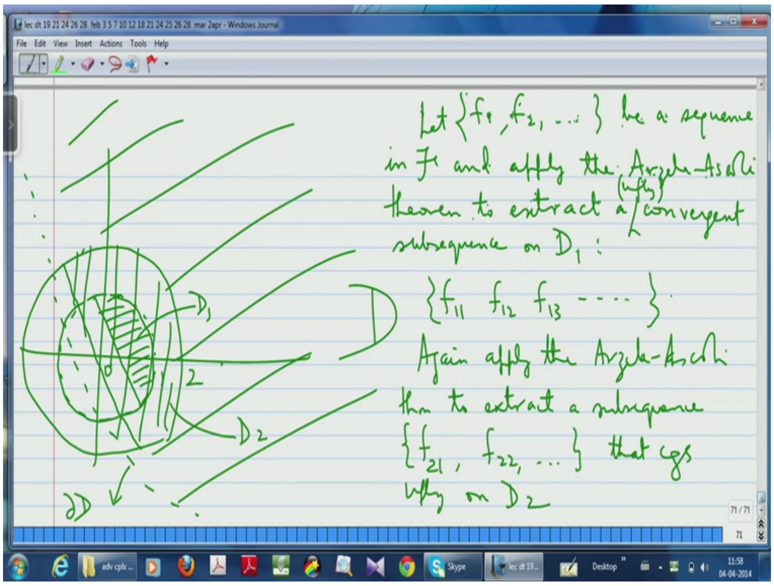

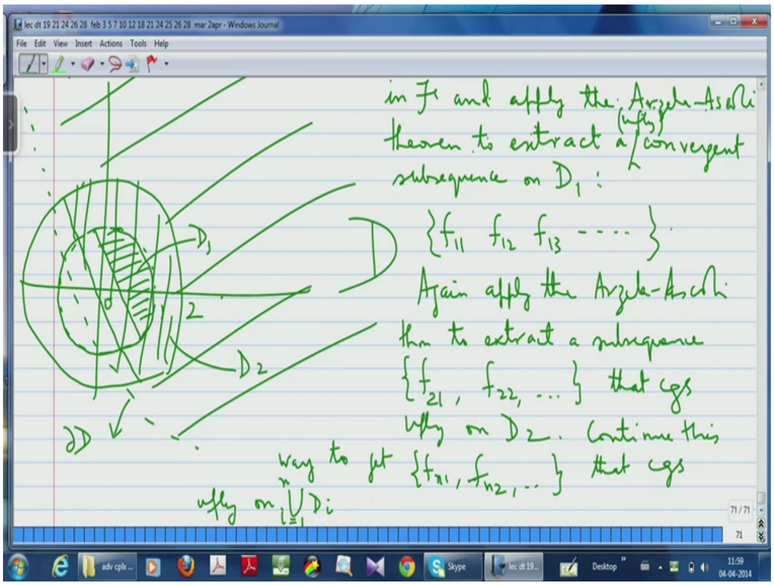

So if you draw a diagram you know it is more like this it is like you know so so let me draw somewhere here so you know suppose this is the plane and suppose this is your domain D alright then you know what is D 1? D 1 is all those points which are lying in the so let me draw this this is too big so you know so these are all the points the unit circle, okay this is mod z less than or equal to 1 and so I will also include the boundary because I have allowed less than or equal to 1, okay.

And if I and then I am looking at all those points of D whose distance from I think I will have to slightly modify this diagram so that I can get a non-empty set so let me draw something like this, okay so suppose this is my so this is my so this is my suppose this is my D okay and mind you D does not contain the boundary so you know I should redraw this I should draw it like this. So this is my dou D this is the boundary of D, okay it is like a half plane except that the boundary is not allowing if you want, okay.

Now what is D 1? D 1 is all those points lying in the closed unit dis, okay whose distance from this boundary is atleast 1. So if you actually calculate it see what you will get is you will get this so with this distance 1, okay this will be D 1 this is what D 1 will be okay and this boundary is included, okay because these are the points whose distance from the boundary dou D is atleast 1 and points less than with distance less than 1 are all not included, okay.

So this is D 1, next you know if you take D 2 okay then what will you get? So I have this disk now this is a bigger disk radius 2, alright and what am I going to get I am going to look at all those points inside this closed disk also for which the distance from the boundary is greater than or equal to half, okay. So what I will get is I will get this I will get this so basically I will get this this region that I have shaded with vertical lines that is going to be D 2, okay.

So like this you can see that this on the one hand the D the D n's are becoming bigger and bigger so as to cover D on the other hand they are coming closer and closer to the boundary of D so that you do not lose any points very close to the boundary, okay this is what is (()) (34:15). But keeping all the D n's away from the boundary by a distance of 1 by n is the condition to make them a closed subset make each of this closed, okay.

So that is the reason each D n is therefore closed and bounded so each D n is compact and the union of all the D n's is D okay so you can make this you can write out all the details basically using properties of the fact basically using the fact that you the distance function the metric function is a continuous function that is what you have to use if you write down everything in detail, okay the distance function is always a continuous function so you have to use that, alright.

And you can see several things it is very clear that all the D n's will cover D okay because you give me any point of D okay it has to lie in some enough closed disk some mod z less than or equal to n, okay and if the point is and it has to be at some distance away from the boundary because D does not contain its boundary it is an open set so every point in D has to be away from the boundary so you can always find every point of D in some D n, okay so the union of all the D n's cover D, okay and all the D n's are compact.

And what is more beautiful is you give me a compact subset of D if you take a compact subset of D such a compact subset will be in some D n certainly because it will be contained in some big enough closed disk, okay and it will have because it is compact its distance from the boundary of D will be some finite quantity, okay. So it will be contained in D n for n sufficiently large.

So given any compact subset of D it has to be contained in D n for some sufficiently large n, okay therefore the beauty is that you constructed this increasing sequence of compact subsets which cover the whole of D and now comes the trick what you do is each of these for each of these subsets you can apply the Arzela-Ascoli Theorem, okay and then use a diagonalization argument and pick out a diagonal subsequence which will now converge on all compact subsets of D and that is the sequence that is promised by Montel's Theorem so that is what I am going to write down.

So let me write this down let us go so let me write it here. Let f 1, f 2, etc be a sequence in script F and apply the Arzela-Ascoli Theorem to extract a convergent subsequence on D 1, okay mind you D 1 is a compact set okay it is a compact subset of D so on D 1 it is compact subset so it is a compact metric space and therefore you know I can apply the Arzela-Ascoli Theorem and on D 1 I can get a convergent subsequence by and here when I say convergent subsequence is uniformly convergent, okay because it is convergent in the space of the functions.

So extract a so here let me put uniformly, okay so call that subsequence as f 11, f 12, f 13 now you see that you are in the diagonalization business, okay. Now what will you do now you take this subsequence f 11, f 12, f 13 and again apply the Arzela-Ascoli Theorem to get another subsequence which will converge uniformly on D 2 and call that as f 21, f 22, f23 okay and proceed in this manner and then you take the diagonal sequence the diagonal sequence will be a subsequence of the original sequence which will converge on D which will converge compactly which will converge uniformly on all D n's and since any compact subset of D is contained in some D n therefore it will also converge uniformly on all compact subset of D and I am done, okay so that is it.

So let me write that down again apply the Arzela-Ascoli Theorem theorem to extract a subsequence (f 22) f 21 that converges uniformly on D 2, okay. Continue this way to get f n1, f n2 that converges uniformly on I should say D 1 D 2 etc D n so in fact I should not put comma I mean instead of putting comma I can actually put union, union D i i equal to 1 to n, okay and that is it.

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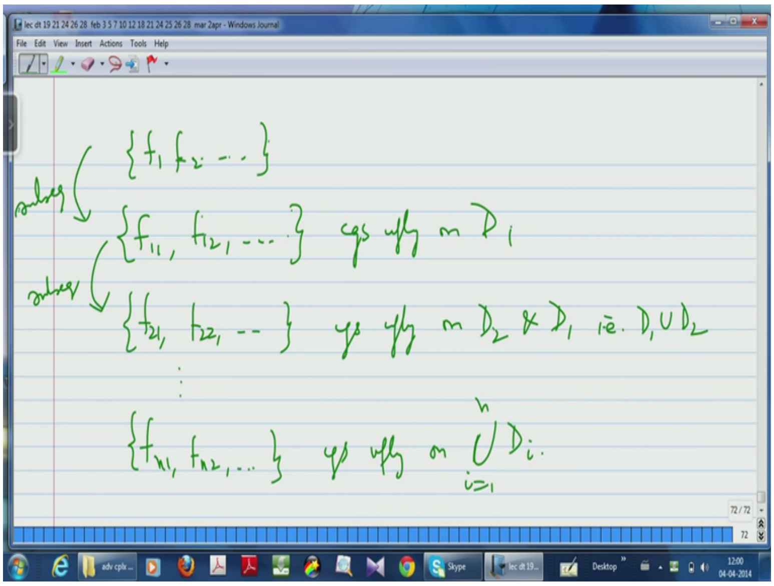

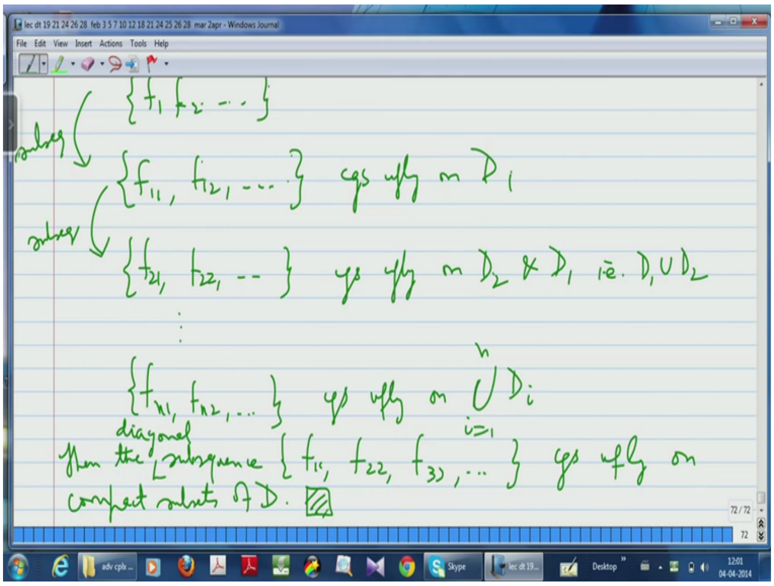

So we are in the following situation so you have this original sequence f 1, f 2 and so on then you have this f 11, f 12, and so on this is subsequence and the point is this converges uniformly on D 1 and then you have f 21, f 22 and so on this is also a subsequence this that converges uniformly on D 2 and D 1 so that is D 1 union D 2, okay and you go like this ad infinitum you end up with f n1, f n2 that converges uniformly on union i equal to 1 to n D i, okay.

Then the subsequence the diagonal subsequence so this is a diagonal trick the diagonal subsequence f 11, f 22, f 33 converges uniformly on compact subsets of D that finishes the proof, ofcourse I must tell you that D 1 union D 2 is just D 2, okay because D 2 is bigger than D 1 and this union i equal to 1 to n D i is just D n each D i includes the D j's for j less than i because it is an increasing collection of compact subsets, okay and any compact subset of D is contained in one of the D j's okay and therefore this diagonal subsequence will converge uniformly on that compact subset.

So starting with a sequence of analytic functions is defined on the domain D okay just putting the condition that this analytic this sequence is uniformly bounded on compact subsets of D namely that is normally uniformly bounded just that condition is good enough to guarantee that you get compactness in the sense that any sequence from this family will admit a convergent subsequence and that is the point of Montel's Theorem, okay.

So what we will need to do is that we will have to you see somehow the derivatives are entering into the picture and the trick is that do you want to extend this to meromorphic functions but you know for meromorphic functions the derivatives are not defined at the poles but then we have replaced but for that we have the spherical derivatives. So the trick is that you try to get another version of this Montel Theorem which will work for meromorphic functions and you try to use the spherical derivative and still everything works, okay. So you get a Montel's Theorem for meromorphic functions, okay and that is what we are going to do next because finally we have to worry about meromorphic functions for the proof of the Picard theorem, okay.