Advanced Complex Analysis-Part 2
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Lecture 30
Introduction to the Montel Theorem – the Holomorphic Avatar of the Arzela-Ascoli Theorem and Why you get Equicontinuity for Free

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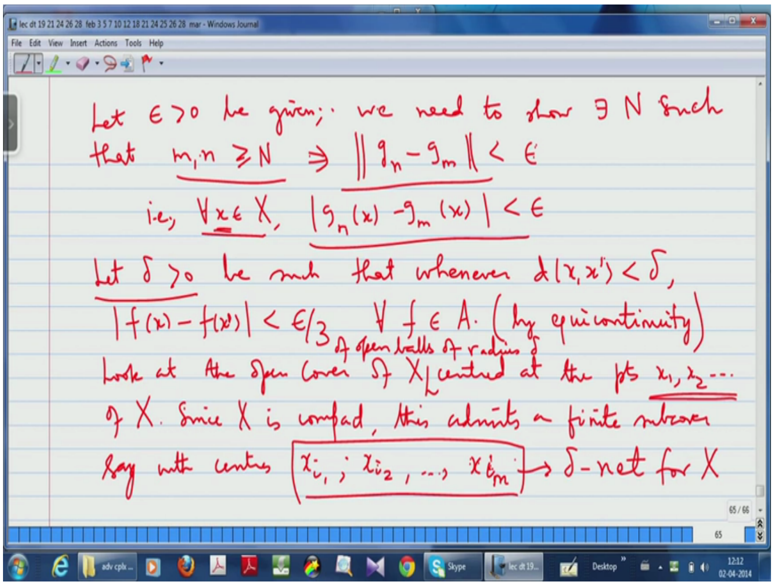

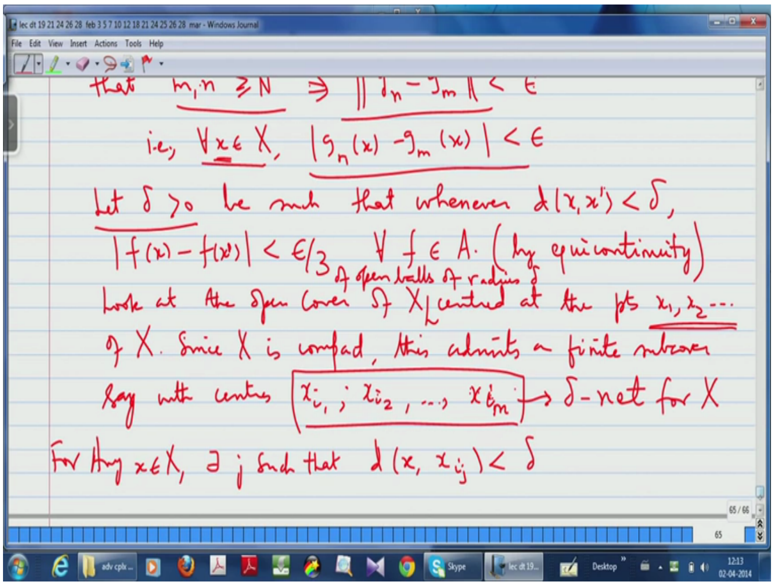

Okay, so we continue with this proof. So you know so let me point some let me write down something say here that I have probably said but I did not write down. See you are looking at the open cover of X centred at the points x 1, x 2 and so on of X so which open cover is it? It is the it is the open cover consisting of open balls of radius delta, okay that something I did not write down. So let me write it down here so look at the open cover of open balls of radius delta, see look at the open cover of this of the space X of the metric space X which is which we have assumed compact, okay.

And you are looking at the open cover which consist of open balls of radius delta that is this delta we have chosen that we have goten above corresponding to the epsilon or rather epsilon by 3, okay because of equicontinuity okay. So and you are for the open cover you are only looking at balls of radius delta and the centres are not ofcourse you could have taken centres to be all points of X but then you take the centres only among the points x 1, x 2 and so on that is you know the countable dense subset of the metric space that we have cooked up, okay.

And so this is actually in principle this is a delta net for X and you know X is X is compact so it is totally bounded so every I mean so what I am saying is it is not a delta net it will give rise to delta net because it is an open cover of open balls of radius delta, okay and because of compactness this open cover will give rise to a finite sub cover. So those finite sub cover will be centred at finitely many points which are among these x i's and we label those points by x i1, x i2, etc x im and you take the balls open balls of radius delta centred at each of these finitely many m points and you take the union you get X that is so this collection of points finite collection of points among the countable dense subset okay is this is delta net this is the delta net for X, okay.

And well we will have to work with this so what we will need to show is that we need to show that you know we have to show that this the whole idea is to show compactness of a family of functions. So we started with an arbitrary sequence and then we tried to verify sequential compactness and we have got an hold of this subsequence, this g n’s okay by the diagonal argument and we have to show that this subsequence converges on all points of X at all points of X and uniformly and mind you the subsequence is already been cooked up in such a way that it converges on the countable dense subset, okay.

Now so I will have to show this for every small x in capital X I will have to show that for my given epsilon you can chose n and m sufficiently large namely greater than or equal to certain capital N such that mod I mean the distance between g n and g m at X is can be made less than epsilon, okay that is the we are just verifying that the sequence is the sequence of g n’s is uniformly Cauchy okay and uniformly Cauchy is means Cauchy with respect to the supremum norm the metric induced by the supremum norm, okay.

So so the point is that how do you get to an arbitrary point of X whereas the g n's are converging only on these points which are points among countable dense subset, okay you have to interpolate the x's is with the x i's alright that is what you will have to do and it is done very easily by the triangle inequality. So what will happen is that if now you take you any for any x in X, okay there exist a j such that the distance between x and x ij is less than delta this is true that is because these x ij's x i1 through x im that is delta net, alright. So there is such j so pick that j take the corresponding x ij and you interpolate with respect to that x ij.

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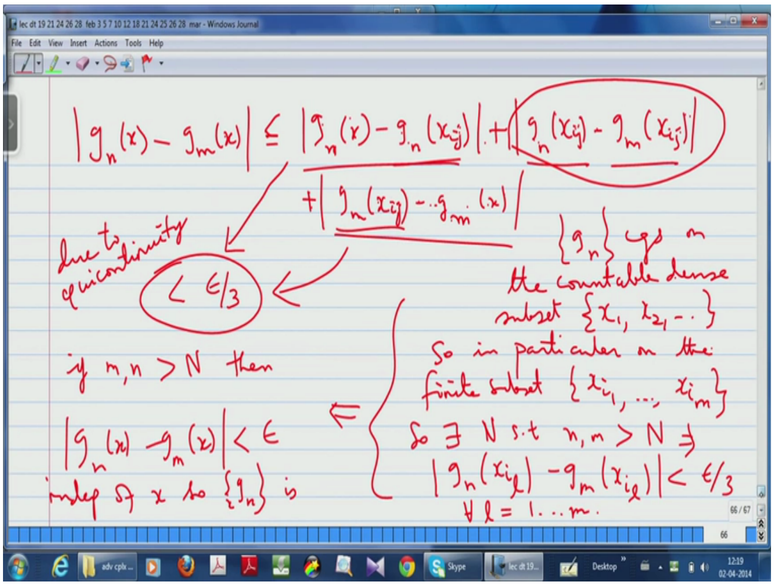

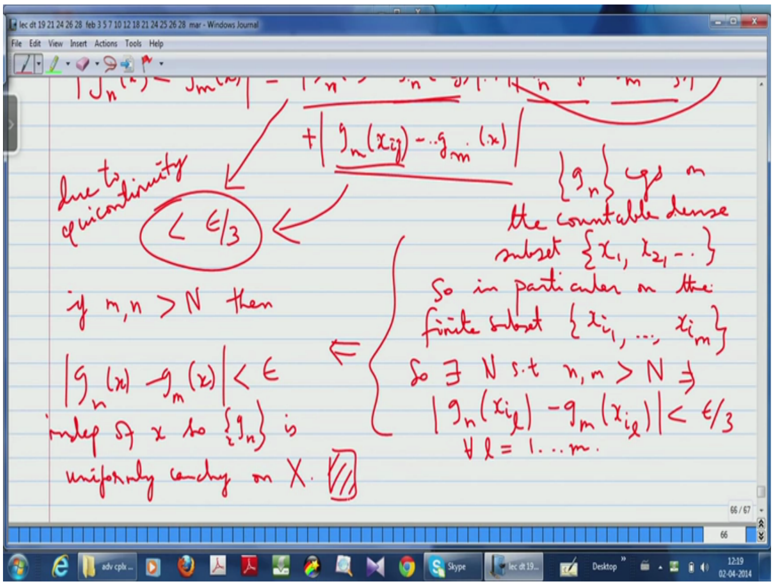

So now you can write out the triangle inequality what you will get is well you are going to get distance between g n of x and so I want to look at g n of x minus g m of x so that is what I am going to write out, the distance between g n of x and g m of x is that now you introduce that x ij which is to within a delta of x so you write this as this is less than or equal to g n of x minus g n of x ij plus g n of x ij minus g m of x ij plus mod g m of x ij minus g m of x so this is how you split the triangle inequality, okay you extrapolate I mean you interpolate the x back to itself by this x ij and then you introduce this minus g n of x ij plus g n of x ij minus g m x ij plus g m x ij, okay.

And now the point is that this now you see what you will have to understand is that the see because of you see in the first term it is the same function g n alright and by equicontinuity of the family, okay the distance between function values for any function of the family is going to be less than epsilon by 3 if the distance between the arguments, the variables is less than delta.

So you see this is already less than epsilon by 3, okay and so is this so the first and the third term are already less than epsilon by 3 that is because of this is just because of equicontinuity due to equicontinuity. Mind you the equicontinuity is has already been assumed for all for all functions so these are certain functions in that family so it holds for them as well.

And then the so the first term and the last term are fine and they are going to give me an epsilon they are going to be lesser than epsilon by 3 each the problem have to worry about the middle term but the middle term is not the big problem because actually you know the g n's the sequence of g n’s it converges on the countable dense subset of the x i's which is labelled by the points.

Therefore these g n’s are going to converge at x ij certainly, okay so you see so how do you deal with this you see g n’s converges on the countable the countable dense subset subset consisting of x 1, x 2 and so on. So in particular it will converge on you know this finite subset okay so in particular on the finite subset which is given by x i1, etc upto x im and ofcourse the x ij that I have chosen is inside this finite set okay.

But then the point is that you know by definition of convergence of the sequence I can certainly find an N the capital N such that small n and small m greater than capital M will make sure that this quantity is always less than epsilon by 3, irrespective of the x ij that I chose that is because there are only finitely many you may get you will get one bound one subscript for each i each j and then you take probably the maximum amongst all those, okay.

So there exist N such that n, comma m greater than N implies that the distance between g n of x il and g m of x il can be made less than epsilon by 3 if for all for all for all l for all l varying from 1 to m, okay so this can be done because it can be done for each x i it can be done for x i1, it can be done for x i2, and so on and for each you get a you get an integer and then you take the maximum amongst all those, okay then only finitely this can be managed so that is it then you are done.

So what this will tell you is that if m and n are greater than N okay then what you will get is that the left hand side which is g n x minus g m x is less than epsilon, okay and mind you this is independent of x this has got nothing to do with x, x did not matter, right so this is independent of x so this sequence g n is uniformly Cauchy on X and that is the proof of the theorem then ends the proof of the theorem, okay.

So we have demonstrated sequential compactness, so if you just to complete the proof accurately what we have shown is that we have shown that g n is uniformly Cauchy, okay but the g n's come from this subset which is already a closed subset, okay but it is a closed subset of a complete metric space therefore it is also complete and therefore showing that is the g n's is Cauchy is same as showing that the g n's are convergent the sequence of g n is convergent and that is what you wanted.

We started with an arbitrary family arbitrary sequence of functions from the subset and we have produced subsequence which converges, okay so that is the Arzela-Ascoli Theorem, alright.

Now what we need to do is that we need to now go back to complex analysis and look at our the kind of functions that we are interested in we are interested in you know analytic functions and then we are interested in ofcourse meromorphic functions that is our final aim. So what does it mean for meromorphic functions? So the beautiful thing is that you know I mean for to begin with atleast let us say analytic functions the beautiful thing is that you know see the main point is the following.

What does Arzela-Ascoli Theorem say? It says that if you have for example looking at continuous bounded real valued functions on a compact metric space then if you want the compact if you want a compactness of a subspace that is a collection of functions, okay then that is equivalent ofcourse that subspace has to be closed and bounded because compactness always implies closed and bounded, okay but what you need to extra put extra is equicontinuity, okay.

So basically you need boundedness, you need a (closed) for a closed subset to be compact, you need it to be bounded and it has to be equicontinuous that is what you want, okay and mind you this is the this is a nice thing because it is easy to equicontinuity is more friendlier to check rather than checking something like total boundedness which is very very difficult to check, okay for a family of functions it is not so easy, okay.

So now when you go to the context of complex analysis and if you are looking at analytic functions what happens is something very beautiful happens. See the what you get is you know analytic functions for analytic functions you have the Cauchy integral formula you have the integral formulas. See this integral formulas if you if you apply the so called integral inequality the ML inequality which says that the modulus of an integral is bounded by M times L, where M is the maximum modulus of the integrand on the contour of integration, L is the length of the contour of integration, okay this is the ML inequality.

If you apply this ML inequalities to the Cauchy integral formula what you get is what you get are called the Cauchy estimates, okay. So the beautiful thing is that for analytic functions you have Cauchy estimates, okay and what these Cauchy estimates will tell you that is that if you are working on a compact set, okay it will tell you that the derivatives are bounded okay because mind you the derivatives of the analytic function are given by the Cauchy integral formulas the general Cauchy integral formula will give you the nth derivative.

So if you therefore you know the bounds for the derivatives are given by applying the ML inequalities to the Cauchy integral formula. So what happens is that if you have an analytic function on a compact set for example on a closed disk if you want then all the derivatives are all bounded okay all the derivatives are bounded. So what happens is that in some sense you get boundedness of derivatives but the beautiful thing is once you have boundedness of derivatives that always implies something stronger for the original functions it imply equicontinuity.

So what happens is if you are working with analytic functions equicontinuity is automatic, okay equicontinuity is just automatic. So you know therefore what happens is that you know just uniform boundedness will give you sequential compactness that is the big deal the big deal is equicontinuity is comes for free if you are going to work not just with continuous functions but if you are working with analytic functions, okay that is the philosophy, okay that is the direction in which I am going to explain how these things work.

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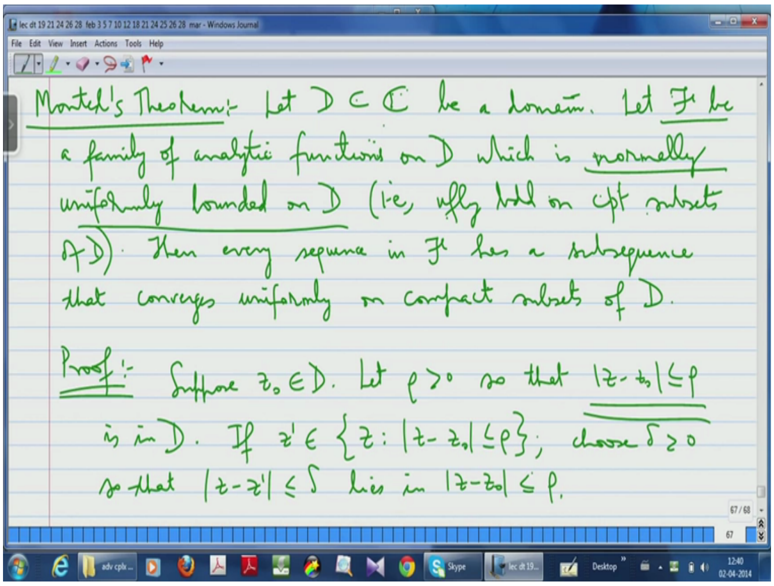

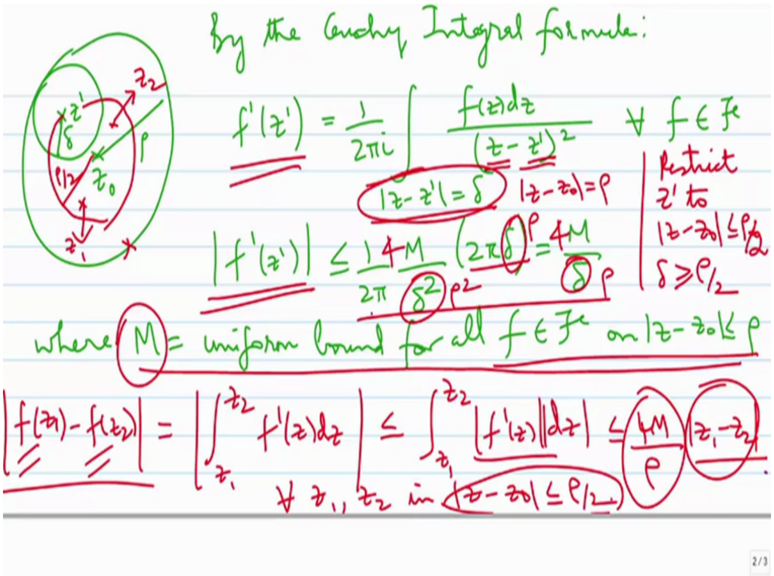

So let me make this so in this in that in that generality the theorem the version of the Arzela-Ascoli Theorem is very important theorem is called Montel's Theorem, okay. So we go on to Montel's Theorem which is the key ingredient for proofs of many important theorems in compression analysis including ofcourse way the including ofcourse the Riemann mapping theorem and you know the Picard theorems and so on so let me put this Montel's Theorem so here is Montel's Theorem.

So let D in the complex plane be a domain ofcourse I am always assuming it is an open connected set and it is non-empty ofcourse, okay. Let so let script F so let script F be a family you can call it as family or collection whatever you want be a family of analytic functions on D which is normally uniformly bounded on D, okay so what is this normally uniformly bounded? Whenever you say for a certain property if you say normally that property it means that that property is to be verified not for all sets but it is to be verified only for compact subsets okay. So when I say normally uniformly bounded on D it means that uniformly bounded on compact subsets of D, okay.

So let me write that that is uniformly bounded on compact subsets of D so I am using abbreviations ufly for uniformly, bdd for bounded, cpt for compact, okay. So suppose it is normally uniformly bounded, okay then you have sequential compactness, okay then you have compactness basically okay but the only thing is that you should see compactness a sequential compactness okay and the sequential compactness is normally what does it mean? It means that given any sequence you have a convergent subsequence but the only thing here is not just convergent on the whole, it will be you will get a convergence okay which first of all it is a convergence of sequence of functions so it will be normal it will be a uniform convergence, okay.

Mind you whenever you are talking about convergence for functions it is always a kind of uniform convergence, alright. For example that is how it is if you are looking at the continuous complex valued or real valued functions on a compact metric space, okay. So it is uniform convergence but it is not again just uniform convergence but it is uniform convergence restricted only to compact subspace. So it is uniform convergence in the you know normal sense, okay so that is the kind of sequential compactness that you will get so that is this result, okay.

Then every sequence in F has a subsequence that converges uniformly on compact subsets of D, okay there are several (())(20:28) in this in the statement of the theorem which I will try to explain, okay. Now so let me give you let us go to the proof of this okay so what I want to tell you is that you see in how do you how do you contrast this with respect to the usual Arzela-Ascoli Theorem so usual Arzela-Ascoli Theorem is for functions defined on a compact metric space, okay that is the first thing.

Then the second thing is in the usual Arzela-Ascoli Theorem whenever you are talking about convergence it is uniform convergence, okay it is just uniform convergence, it means uniform convergence on the whole space, right and the third thing is that the Arzela-Ascoli Theorem there is that if you want compactness which is same as sequential compactness, okay that is every sequence has a convergent subsequence that for that you will have to give boundedness which is actually uniform boundedness okay plus you have to give equicontinuity, okay.

Now the big deal is when you come to complex analysis, when you come to analytic functions I have already told you that the problem with analytic functions is that the convergence always is never uniform on a domain it is only uniform when you restrict it to compact sets. So you have to change the convergence to normal convergence so you must not require always convergence you should require convergence only on compact subsets, okay that is the first change you have to make.

The second change is that you can get rid of you do not need equicontinuity, okay and so you just get in a sense you are saying that uniform boundedness implies sequential compactness that is what you are saying, okay but the beautiful thing is that the sequential compactness is with respect to normal convergence, okay that is one important thing, the other thing is the functions are not defined on a compact set they are defined on a domain, okay that is another difference.

The usual Arzela-Ascoli Theorem you are looking at functions they are defined on a compact metric space, whereas here you are looking at analytic functions which are certainly continuous but they are defined on not a compact set they are defined on an open set open connected set that is the difference these are the differences. So now you but you can see that the point is that you are able to when you come from the topological side to the complex analysis side, okay you replace convergence uniform convergence by normal convergence you replace you forget equicontinuity because it comes for free, okay.

So how does one proof this? So the proof is very very simple the first thing I want to tell you is that if you are see if you are looking at a compact subset of the domain then there is nothing great because you can directly apply Arzela-Ascoli Theorem okay and you have to use the bounded the Cauchy estimates, okay. So for example let me tell you so suppose suppose z not is a point of D, okay let rho be greater than 0 so that the disk mod z minus z not less than or equal to rho is in D, okay you choose sufficiently small radius so that the closed disk is inside D alright ofcourse I can always find since z not is a point of D which is an open set I can always it is an interior point so there is always a disk open disk surrounding z not is also in D.

Now you take disk of slightly smaller radius, okay and that close disk will also be in D you can take that as your rho, okay. So the reason for taking the boundary also is you know pretty well because then I get a compact set because the closed and bounded set so it is compact and once it is compact I can apply all the hypothesis I have. So now what happens is watch that you know if I take this if you take the family if you look at a family of analytic functions on D and you restrict it to this this closed disk, okay.

What you are getting is a family of continuous functions complex valued functions mind you analytic functions are continuous ofcourse, okay they are complex valued continuous functions and you are restricting them to a closed disk which is a compact subset is also compact metric space. So actually you know Arzela-Ascoli you are in the situation in Arzela-Ascoli Theorem. The Arzela-Ascoli Theorem actually needs only continuous real or complex valued functions defined on a compact metric space, okay so you are in that situation alright.

So since you are in that situation you are already given that this family is normally uniformly bounded, it means that it is uniformly bounded on compact subsets therefore this family is bounded on this closed disk because it is a compact subset. So you already have boundedness you have boundedness of the family which is actually uniform boundedness you have that already.

Now what more do you require for extracting a convergent subsequence from a given sequence what you require is that you require equicontinuity, okay but the point is that because of the analyticity and the Cauchy estimates equicontinuity is automatic. So let us see why that is true, you see that so so let me say the following thing well if z prime is in the set of all z such that mod z minus z not is less than or equal to rho, okay you take a point here and then you see what will happen is so you know if you want let us take let us take something that that is right so what you do.

So let me draw a diagram so that it is easier to visualize, so here is my z not and here is my closed disk centre at z not radius rho and here is my z prime, okay now what you do is that well so notice so you can see you know you can choose delta such that the closed disk centred at z prime radius delta lies inside this closed disk, okay so I can chose a delta like this, okay choose delta greater than 0 so that mod z minus z prime less than or equal to delta lies in mod z minus z not less than or equal to rho you can do this, okay.

And then now you do the following thing you look at what is look at the Cauchy integral formula see by the Cauchy integral formula well this is the well this is the the second Cauchy integral formula which is for the derivative first derivative f dash of z prime okay is what it is 1 by 2 pi i integral over integral the positive sense over this circle mod z minus z prime is equal to delta, okay of f z dz by z minus z prime, alright and I am going to get the whole squared, okay this is the first Cauchy integral formula or rather second Cauchy integral formula, okay.

The first Cauchy integral formula is for the function (())(29:19) which is a 0 think of it is a 0 derivative. So this is the Cauchy integral formula this is true for all functions f in F this is fine this is because after all f is a family of analytic functions so this is true. Now apply this I am just trying to write out the Cauchy estimates. So mod f prime of z prime is what? This is going to be modulus of this integral but that is less than or equal to the maximum value of the integrand multiplied by the length of the contour which in this case is the circle of radius delta, okay centred at z prime.

So what I am going to get is so this is less than or equal to I am going to get 1 by 2 pi is what I am going to get if I put a mod here and the length of the contour is well it is now if you put a square then it is an f prime, if you put so what is the Cauchy what is see f not is if you put a this is the first formula. So if you want the nth derivative you have to put n plus 1, okay. So what is this so I will get see if I calculate the modulus of this I will get mod f z okay so you know let me put let me put M here.

So for the modulus of f z I am going to get an M, okay and then for the mod z minus z prime the whole squared see that is mod z minus z prime is delta because the variable of integration is z, the variable of z lies on the region of integration which in this case is this contour which is positively oriented this is the orientation the usual positive orientation. So this is M by delta squared, okay.

And I am going to get a and I am going to get the the length of the contour and that is going to be 2 pi delta, okay. So basically I am going to get M by delta and what is this M? See this M is the common bound for all the functions in your family on this closed disk on this big closed disk that is because that is given. See you are given that look at go back go above and look at this see you are given you are given that this family F, A is family of analytic functions it is normally bounded normally uniformly bounded on D so it means that it is uniformly bounded on every compact set on every compact subset of D.

So on this closed disk of centred at z not and radius rho which is ofcourse a compact subset of D it is bounded, okay all the functions are bounded I am taking that bound to be M. So let me write this where M is the uniform bound for all f in functions f in this family script F on this disk centred at z not radius closed disk with radius rho, okay we can put this bound independent of delta also, okay.

So this delta that I choose seem to depend on the z prime alright but then you can get rid of the delta so that I can get a uniform bound for the derivative f dash is as follows. See the first thing you can notice is that you know I can change this contour of integration which is the this smaller circle centred at z prime radius delta to the larger circle which is mod z minus z not is equal to rho and I can do that because of the fact that f is analytic in the bigger closed disk and also the point z prime is also enclosed by the bigger circle, okay. So you know the therefore this (())(33:31) f dash of z prime is valid, okay.

So in this way I have gotten rid of the delta in the integrating contour, okay then the other thing is that I will have to get over the delta here appearing the bound here and so for that what I will do that I will just have to restrict z prime to be with inside a you know a circle of radius rho by 2, okay from I mean centred at z not. So restrict you know is a prime to mod z minus z not less than or equal to rho by 2, okay.

If you do this then you see this effectively makes this delta which is supposed to be you know the distance between the point z which is on the contour of integration and the point z prime which is inside the contour this distance from z to z prime what it will be, you see this z is not going to lie on the outer contour the outer circle and there is z prime inside, okay and you see the minimum distance between z and z prime is rho by 2 and therefore you know so this delta here will essentially be you know this delta is supposed to be the distance between it should be modulus of z minus z prime, okay and that distance is atleast rho by 2 alright and ofcourse therefore you know 1 by delta squared will become less than or equal to 4 by rho squared.

So basically instead of this delta squared I will get a rho squared and I will get a 4 here okay the inequality will get reversed if you take a (())(35:26) and ofcourse this 2 by delta will become 2 pi rho because that is the length of the contour which is the length of the outer circle. So in effect this bound will become 4 M by rho, okay and that will become a bound that has got nothing to do with delta, alright.

Now you see ofcourse M is the uniform bound for all the functions in the family on the bigger closed disk centred at z not and radius rho but what does this give you. You see now you (())(36:00) what is f of z 1 minus f of z 2 okay what will this be if you take z 1 and z 2 inside this to be two specific instance of z prime. So here is z 1 if you want and here is z 2, okay and ofcourse you have this integral is going to be independent of the path so long as the path is you know inside this closed disk that is because that is analytic there and what will happen is that you know you are going to get this is by the ML inequality this is less than or equal to integral z 1 to z 2 mod f dash of z (d z) mod d z.

And now you know this now you can apply this bound this mod f dash of z is less than or equal to 4 M by rho times this mod d z is going to give you mod z 1 minus z 2 and that is for example if you take the straight line segment from z 1 to z 2, okay and this is valid for all z 1 and z 2 in this closed disk mod z minus z not less than or equal to rho by 2, okay. So you see what does this tell you? This tells you that the you know I can make the distance between f z 1 and f z 2 you know small the moment small enough the moment I can make the distance between z 1 and z 2 small enough, okay.

And this is in a way that is independent of the particular choice of z 1 and z 2, okay and also in a way that has nothing to do with the function f because this M is uniform bound for all the functions, okay and that is exactly saying that f all the functions f the whole family of functions script f okay that is equicontinuous on this disk centred at z not and radius rho by 2, okay and that is how you get equicontinuity for free, okay.

See the last inequality tells you mod f z 1 minus f z 2 is less than or equal to some constant time z 1 minus z 2, okay so what that gives equicontinuity that is like (())(38:33) condition you see given an epsilon you choose carefully the delta and the way you choose delta independent of z 1 and z 2 so independent of z 1 and z 2 you are saying the distance between f of z 1 and f of z 2 can be made lesser than epsilon whenever z 1 and z 2 are within a delta an independent of z 1 and z 2 that is equicontinuity you are able see it is actually it is beautiful it is uniform, it is a kind of uniform continuity because you are it does not you do not worry about whether z 1 and z 2 which z 1 or z 2 it is.

And it also works for all functions f so it is a kind of uniform continuity that is what you get I mean that is the power of this of the Cauchy integral formula that you are using, okay and this is what you get if you assume analyticity you get this you get equicontinuity just like that, okay so the only thing that is left is uniform boundedness, okay but that is already assumed. So you get sequential compactness but in the normal sense that is the point that is Montel's Theorem, okay alright.