Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

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Lecture No 3

Recalling Riemann's theorem on removable singularities

Let us continue with whatever we were doing so you know let me recall few things, so you see we started by asking what the image of an analytic mapping is okay and of course we have already seen that you know because of open mapping theorem the image of an non-constant holomorphic map is an open set in fact it is an open mapping so it takes open sets to open sets, so the image of the domain is again a domain and then I told you that we wanted to know what we want to know more about the image set okay namely the set of values that the analytic function takes and you know somehow the open mapping theorem tells you that for example you cannot expect the image of a non-constant holomorphic map to lie inside a curve for example because there is it does not have the property that it can accommodate a sufficiently small disk okay.

So then of course we stated the so-called little Picard theorem or the small Picard theorem which says which deals with the case of an analytic function which is analytic on the whole complex plane so-called the entire function and the theorem says that the image will be either the whole plane or it will be the plane minus a single point again and interestingly the proof of this theorem which is usually stated only stated in the 1st course in complex analysis involves interesting amount of analysis and that is what we will try to do as part of our these series of lectures and I told you the key to this proving this little Picard theorem is the so-called big Picard theorem or great Picard theorem which is got to do with the image of or deleted neighbourhood of an isolated essential singularity under an analytic mapping okay and the great Picard theorem, what does it says?

Says the conclusion of the great Picard theorem amazingly is the same as that of the little Picard theorem, it says that you take a deleted neighbourhood of an isolated essential singularity of an analytic function then the image of that under the analytic function will be again the whole plane or it may be the plane minus at the worst 1 point one value that is missed, so it says that the plane is a punctured plane and I told you that the proof of the little Picard theorem that will give is going to be as a corollary of the big Picard theorem and to study the proof of the big Picard theorem or the great Picard theorem we need to study what are about Meromorphic functions okay and these are functions essentially these are functions…

These are analytic functions which have the only singularities as poles and so this lead us to understand and recall rather recall the notion of singularity, so I told you that singularities of analytic function are of 2 types namely the isolated and the non-isolated one okay and of course a classic example of a non-isolated singularity is are the points on the negative real axis which have to be cut out if you want a for example define an analytic branch of the logarithm of log Z logarithm of Z okay Z being complex variable and I told you that these non-isolated singularities require much deeper techniques for example the study of Riemann surfaces to deal with them but then we are going to be worried only about isolated singularities and I told you the isolated singularities come in 3 categories of groups and these are mutually exclusive, the 1st kind of isolated singularities called a removable because the idea is that the singularity can be removed in the sense that you can redefine the function so that it becomes analytic at that point that is why is why it is not really a singularity.

Then there are the poles which are thought of as zeros of the denominator okay and of course if the function does not have denominator it is not writable as numerator by a denominator then poles are just zeros of the reciprocal of the function okay and I told you that a function has a pole of a certain order if and only if the reciprocal function has 0 of the same order at that given point and then what we are left with this is are the singularities which are neither removable nor poles and they are cleverly called as essential singularities and they are very important as the for example the great Picard theorem tells you.

Now what I am going to do is 1st prove a certain weaker form of great Picard theorem the so-called Casorati Weierstrass theorem which says that you know if you take an isolated essential singularity and you take a small neighbourhood of that deleted neighbourhood of that matter how small the image of that will be dense in the whole complex plane namely it will you can always find a sequence of complex numbers in any neighbourhood such that the values approach any given complex value okay and that is a very deep that is already a very deep result but it can be you know deduce from the Riemann's theorem on removable singularities, so what I am going to do is now I am going to tell you something about the Riemann's theorem on removable singularities because it involves a lot of nice concepts okay.

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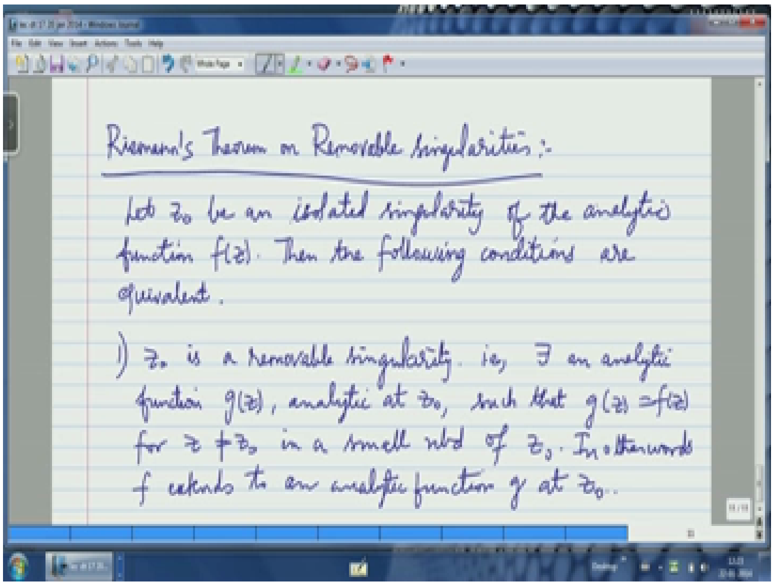

So here is Riemann's theorem theorem on removable singularities and it is a is a very deep theorem because you know it already uses some of the most basic theorems in complex analysis is proved okay so we will see that so let me state that, let Z naught be an isolated singularity singularity of the analytic function the analytic function f of Z. Then the following conditions are equivalent, so number 1 Z naught is removable singularity that is the 1st condition and so here you can give several definition of removable singularity but we will give the most natural one, the most natural definition for removable singularity that it can be removed namely that you can the analytic function can extend to an analytic function even at the point of singularity okay.

What it means is that you can find and analytic function which is also defined at the singular point and which equals the given analytic function outside that point okay that is what, that is what it means to say that the analytic function extends to an analytic function also at the singular point, so that means by extending it you have actually been able to remove the singularity alright, so of course you know the standard example you should think of is sin Z by Z at Z equal to 0 you can you know the limit and at Z equal to 0 is 1 so you know you can you can take the function that takes the value sin Z by Z at Z naught equal to 0 and at Z equal to 0 you can define it to be 1 and then this turns out to be an analytic extensive so let me write that down that is (())(9:45) an analytic function so g of Z analytic at Z naught such that g of Z is same as f of Z for Z naught equal to Z naught in a small neighbourhood of Z.

So this is what it means to say that the singularities removable, I am able to extend the analytic function to the point, to the singular point okay and to be able to extend the analytic function to the singular point means that I am able to find another I am able to find an analytic function which is also analytic at the singular point and restricts to the given function outside that point okay so let me re-write it, it is very important in mathematics to be able to say things in different ways verbally okay without using symbols or notations as far as possible because it will help you to get a good understanding of the ideas okay, so you should be able to state theorems at least as accurately as possible without using much notation and just using concepts okay.

This is something that you should try to do may be able to write technical mathematics namely you can write a theorem with all the technical symbols and so on then it is very important the purpose of communication and understanding should also be able to say things in a way that does not involve any rotation okay and so in that sense when you say something is a removable singularity, a point is a removable singularity, what it means is that the analytic function extends to analytic function at the single point okay, so let me write that down in other words f extends to the to an analytic function g at Z naught, so this is the I mean so this is actually the definition of what removable singularities okay.

So this is the 1st this is the 1st condition okay so you can see that what this theorem does is that it gives you various equivalent conditions for singularity to be removable, so what is the 2nd condition so you see in all these in all theorems connected with characterisation of singularities that usually at least 3 statements one is about one is essentially the definition of that singularity, the 2nd one is the behaviour of the limit of the function as you (())(12:54) the singularity okay and 3rd one is the behaviour of the Laurent series around that singularity okay so for example if you take a case of poles which I stated in the last lecture you see I stated that theorem in the last lecture and I wanted you to try to prove it and I do not know if we have done this exercise are not but essentially you see if you try to do that exercise at some point you might have to use Riemann's removable singularity theorem which I am going to actually prove now okay.

So but nevertheless the purpose of the exercise was to make you realize that you need to that you might need to use this okay, so you know in the case of a pole the condition for the 1st condition for a pole was that it is a pole namely which is the basic definition of a pole which is just that the it is the 0 of the reciprocal, the point is 0 of the function which is the reciprocal of the given function okay that is the 1st condition. The 2nd condition is behaviour of the limit of the function as you approach the poll and that condition turned out to be that they limit, the limit of the function turned out to be infinity okay and that is to be interpreted as the limit of the modulus of the function goes to plus infinity okay as we approach the singularity, right.

That is another characterisation of a pole and what is a 3rd condition, the 3rd condition involves the Laurent expansion and what does the condition based on the Laurent expansion, the condition based on the Laurent expansion was that the Laurent expansion contain only finitely many negative powers of Z minus Z naught okay, so you know Taylor expansion is something that contains only positive powers and 0 powers and this is of course the 0 power correspond to the constant term okay and the Laurent expansion is something that will also contain negative powers of Z minus Z naught okay where Z naught is the singular point and if you have a Laurent expansion which has only finitely many negative terms, I mean terms with negative powers of Z minus Z naught that is an indication that Z naught is a pole okay.

Now that these 3 conditions are equivalent was a theorem I stated last time so in the same way for removal singularities, the 1st condition I have stated is what removable singularities essentially and I will give you 2 other conditions so the 2nd condition is going to be a condition that has got to do with the limit of the function, so the 2nd condition says that limit of the function exist as you approach the singularity okay just the existence of the limit is a 2nd condition okay and why it is why it is significant is because if the limit exists what you are saying is that the function extends to a continuous function at that point okay.

So mind you it is it is certainly weaker than the 1st condition because the 1st condition namely the definition of a removable singularities that it extends to an analytic function of that point whereas the 2nd condition is the condition on the limit that the limits exist that point only tells you that it extends to only a continuous function at that point okay. So now what you are saying is that a fact that you can continuously extend the function to the point already makes it analytic at that point this is the characteristics of the removable singularity okay.

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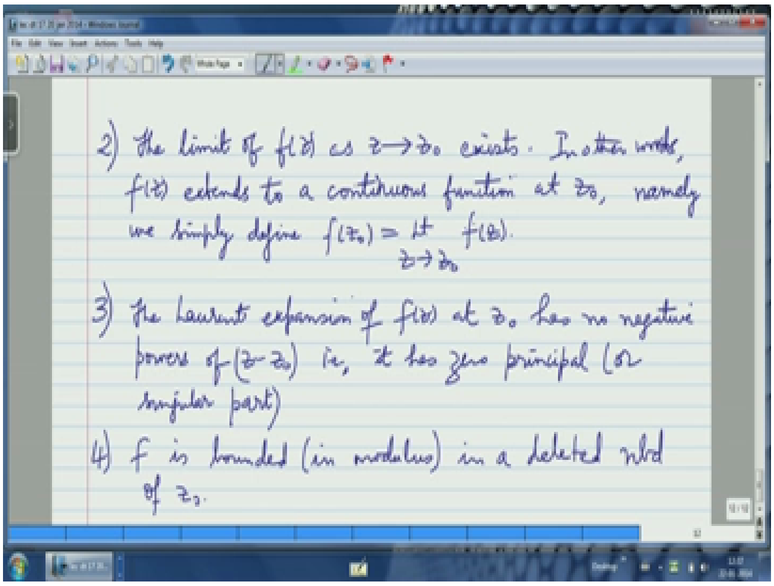

So let me write the 2nd condition, so here is the 2nd condition the limit of f of Z as Z tends to Z naught exists this is the condition okay, so in other words in other words f of Z extends to a continuous function at Z naught namely we simply define f of Z naught to be the limit as Z tends Z naught of f of Z okay you define you redefine in fact Z naught and in the resulting function becomes also continues as Z naught but what you do not have immediately is that it is also analytic at Z naught and that is the serious that is a serious consequence that is the Riemann singularity, removable singularity theorem okay.

Just continuity at that point is good enough for analyticity okay that is the crux of the Riemann removable singularity theorem, so this is the 2nd condition and here is the 3rd condition, so I told you that the 3rd condition is something that has got to do with the Laurent expansion and you know that so I want to give you some background on this, you know if you have a function with this analytic at a point then you have so-called Taylor expansion at that point okay. If the point is Z naught then you have power series in Z minus Z naught which involves 0 and positive powers of Z minus Z naught with some coefficients and of course these coefficients are you know a are just the related to the nth derivatives.

The derivatives of the function at Z naught okay and so this is the Taylor theorem that you can have a Taylor expansion for the function at the point Z naught where it is analytic and extension of Taylor's theorem is the Laurent theorem is an amazing extension because it deals with the case when Z naught is not a point of analyticity but it is a point there is an isolated singularity, what Laurent theorem says is that you can still get a series but now this time we will also have to allow negative powers of Z minus Z naught that is the Laurent series okay and Laurent theorem says that you can get that series and the coefficients are again given by integral okay.

See in the case of Taylor series the coefficient are given by integrals and these integrals are essentially connected to the derivatives by the general Cauchy integral formulas okay and in the Laurent expansion the coefficients of the negative terms the coefficient are all anyway given the integrals okay there is no question of derivatives at the point Z naught because at Z naught it is not a point where the function is analytic, so the coefficient of the Laurent series given in terms of integrals okay. Now so you see the Taylor series a very special case it is special case of Laurent series okay and the fact that…so this is the point if you have if you have an analytic function at a point okay if you try to write the Taylor series at the point you will get the same result as if you try to write a Laurent series at that point okay.

See the fact is that if you have an analytic function we know that it is given by a Taylor expansion at that point but if you throw that point away okay then you get deleted neighbourhood of the point and in a deleted neighbourhood of a point you always have a Laurent expansion or any function regardless of whether the function is analytic at that point or not, okay and the point is that if you write out the Laurent expansion for an analytic function at a point okay you will get only the Taylor expansion, you will not get the negative terms in the Laurent expansion the negative terms in the Laurent expansion constitute what is called the singular part or the principal part of the expansion okay and the principal part will not exist that is the sign of the fact that the function that you are actually expanding into a Laurent series is actually analytic at that point okay.

So that is a 3rd condition there is a 3rd condition in terms of Laurent series or a removable singularities that when you write the Laurent series for the function at the removable singularity centred at the removable singularity will see that the negative terms, the principal part is an exists it is 0, so that is the 3rd condition which is based on the Laurent expansion. So here is the condition, the Laurent expansion of f of Z at Z naught has no negative powers of Z minus Z naught that is it has 0 principal part principle or singular apart okay this is the condition in terms of the Laurent expansion and then so these are the 3 conditions that you will always have as equivalent in any theorem on singularities, characterisation of singularities.

Now there is one more condition which is pretty interesting and it is a rather remarkable condition, the 4th condition is the following is the following if you have a removable singularity okay at a point you see let us believe that is a removable singularities to understand this condition then it becomes analytic at that point okay and if it is analytic at that point it is also continues that point okay and you know a continuous function on bounded set okay if you take a continuous function on a closed and bounded set it is going to be bounded in modulus okay because if you take a closed and bounded set it is compact okay and if you take the real values continues function on a compact that the image is going to be also compact set, topologically a continuous image of a compact set is compact set and a compact subset of the real line is going to be bounded okay.

So the point is that if you really believe that Z naught is an essential singularity is a removable singularity for the function the function should extend to an analytic function at that point in any case it is going to extend to a continuous function at that point for example that is what condition 2 says then in the neighbourhood of that point the function should be bounded in modulus of course. See whenever we say bounded or a complex valued function, we mean we mean a bounded in modulus okay that goes without saying okay. So that is the next condition so it is an amazing condition you just assume that there is a small deleted neighbourhood of the point Z naught where your function in modulus is bounded by positive constant okay that is also as strong as saying that it can analytically extent to that point that is the that is the really amazing hypothesis because it is very weak hypothesis, you see the 1st hypothesis, the 1st condition is that the function is the has a removable singularity namely which by our definition is that the function extends to an analytic function at that point okay.

The 2nd condition is slightly weaker, it says that it does not extend van analytic function at that point but it extends to a continuous function at that point okay that is a slightly weaker condition and of course the 3rd condition has got to do with the Laurent series which says that essentially the Laurent series is the Taylor series but the condition that I am going to state now is a very weak condition, it just says that there is a deleted neighbourhood of the singular point where the function is bounded in modulus, you see the boundedness is a very weak condition okay but that is strong enough to make it analytic at the point, so that is the amazing power of the Riemann removable singularity here, so let me write that down, f is bounded and of course bounded means in modulus in a deleted neighbourhood, so I am using nbd for neighbourhood as an abbreviation of Z naught, so this is the 4th condition which is by for the weakest condition on a removable singularity okay.

So you know why it is weak? If you want if I might say a little loosely is that you know continuous function is always bounded but there are bounded functions which are which could be very highly discontinuous okay, so the moral of the story is that bound and is giving rise to continuity is already something that is very hard to expect, you should not expect that and in this case bounded is giving me analyticity, so you can imagine analyticity is a terrific condition because you know analyticity at a point means that you know not only that this differentiable in the neighbourhood of the point including that point but it is also infinitely differentiable there, so it is a terrific condition okay that you are able to get this from boundedness is an amazing thing that is what you should appreciate okay. So alright so what I am going to do is I am going to you know prove that these various conditions are equivalent okay and in the process help you revise some basics of complex analysis okay, fine.

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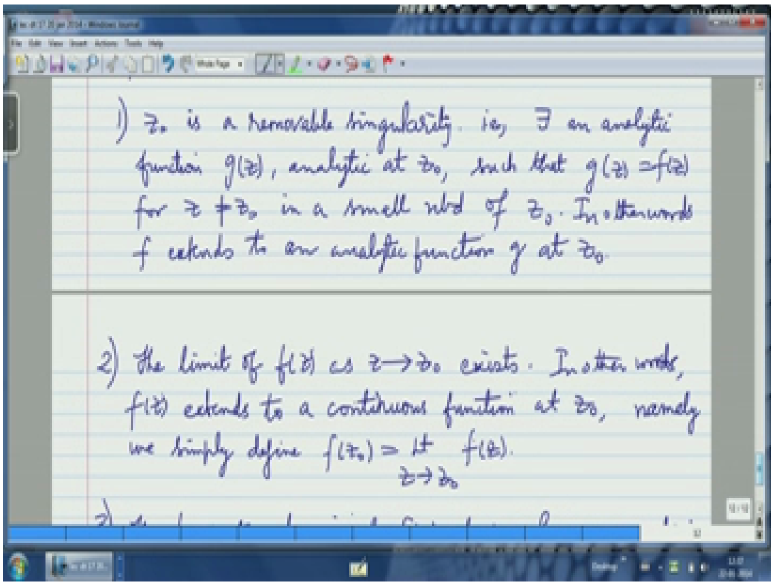

So let us look at these conditions so let us look at condition 1 and 2 you see it is very clear that one implies 2 okay if the analytic function extends to an analytic function at that point then certainly it extends to a continuous function at that point because analytic function is continuous okay differentiability implies continuity, so one implies 2 is very trivial alright and uhh. So you can see so I am just trying to see which of these are very easy to deduce, which of the equivalence are easy to deduce, so one implies 2 is pretty easy okay.

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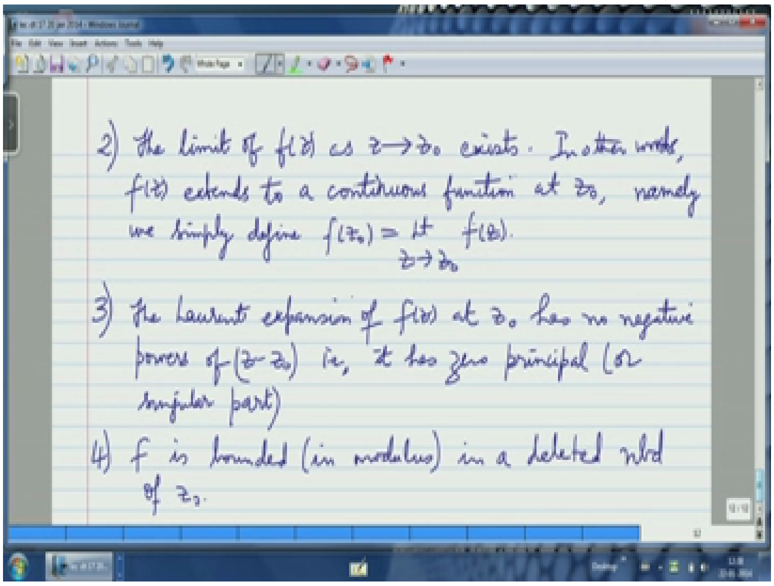

And well I think if you look at 2 and 3, 2 and 3 are equivalent okay 2 and 3 are equivalent that is very easy to see because you know you take the…so let me first tell you in words suppose you assume 2, suppose the limit of the function at Z tends to Z naught exist okay then I can take the limit of limit as Z tends to Z naught in the Laurent expansion and as limit Z tends to Z naught in the Laurent expansion exists means that cannot be any negative power of Z minus Z naught in the Laurent expansion because if you have one negative power of Z minus Z naught in the Laurent expansion it will be a term of a form a n by Z minus Z naught power n and that as Z tends to Z naught will go to infinity okay and you cannot get a finite limit.

So the moral story is that 2 implies 3 is obvious and 3 implies 2 is also obvious because you know if the Laurent expansion does not have any negative terms then I can take limit as Z tends to Z naught in the Laurent expansion and essentially what I will get is the constant term okay. If I take limit Z tends to Z naught I am just sitting Z minus Z naught equals to 0, Z equal to Z naught and then since it is already power series in Z minus Z naught if I put Z equal to Z naught I will get the constant, and that will be the limit okay.

Now so 2 and 3 are clearly equivalent alright, now there is only one technical point which I want you to notice because this is an advanced this is a course in advanced complex analysis, the technical point is that you know how can you take a limit in the Laurent series okay, how can you take limit Z to Z naught, Z tends to Z naught in the Laurent series see so basically we are arguing in a following way, we are arguing as if we can take taking the limit as Z to Z naught, Z tends to Z naught in the Laurent series is same as taking the limiting in each term and then summing it up a that is the way we are arguing and why is that correct that is because the practice because of the fact that you know you see the Laurent series as it is you know that is its convergence is uniform okay and whenever the convergence is uniform okay you can take a limit okay and so that is used.

You can take a term wise you take a series or you take a function of series okay and you take limit Z tends to Z naught of the functional series okay that is the same as taking limit Z tends to Z naught of each of the terms of the functional series and then taking the limit of the resulting numerical series. This is allowed provided the functional series converges uniformly and basically I am just using the fact that if you have a convergent, uniformly convergent series of continuous functions than the limit is also continuous okay that is all I am using so that is little bit of technicality that is used when you want to prove 2 and 3 are equivalent okay and of course you know 1, 2 and 3 all the 3 will imply 4 because you know continuous function is bounded, containers function on a compact set is bounded.

So a difficult part is to go from 4 to 1 okay. 4 is a weakest condition, the condition for is the weakest condition it is a condition that says at you just tell me that near Z naught, near that singularity a function is bounded and low behold it becomes analytic at Z naught that is the that is the most tremendous observation okay, so 4 implies 1 is the most difficult part that is the crux of the theorem which we will try to prove and essentially one can one will again essentially use Laurent expansion is okay, so let me write down all of this.

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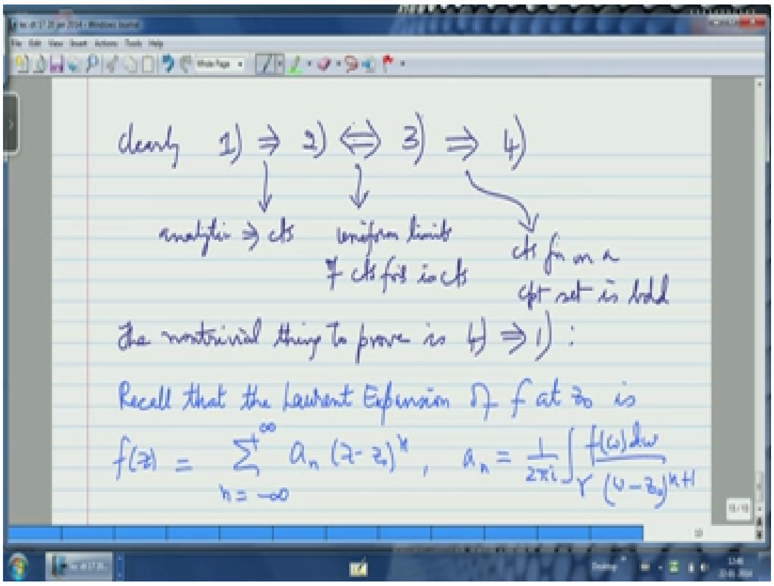

So clearly 1 implies 2 which is equivalent to 3 and they all implies 4, so this is what we have seen. The nontrivial part is 4 implies 1 okay, so you know if you want 1 implies 2 basically uses analytic implies continuous analytic implies continuous okay and 2 implies 3 is going to use well uniform limit of continuous functions is continuous okay and of course 1, 2 and 3 implies 4 all these all these they basically use the fact that a continuous function on a compact set is bounded okay continuous function on a compact set is bounded okay so the nontrivial, the nontrivial thing to prove is 4 implies 1 namely that boundedness in the neighbourhood of the point use the analyticity at that point which is an amazing thing okay and interestingly the way you prove it is again using Laurent expansion a just use the Laurent expansion.

So here is a proof recall that so let me let me go to a different color recall that the Laurent expansion of f at Z naught is f of Z is equal to Sigma n equal to minus infinity to infinity a n Z minus Z naught the power of n where a n is 1 by 2 pi i integral over gamma fw dw by w minus Z naught to the power of n plus 1 this is the Laurent expansion okay and where gamma is a simple close (())(35:02) which goes ones around the point singular point Z naught in the anticlockwise or positive signs okay. We can very will take gamma to be circle centred at Z naught sufficiently small radius okay, so the point is that the point is to you know let me explain the idea how to prove?

See what you trying to show you are trying to show that the function is analytic, one way to show that the function is analytic is that is Laurent series is actually Taylor series because you know Taylor series always represents an analytic functions okay convergent power series within its disk of convergence always represents an analytic function okay so incidentally that is also that is also probably used in one of the (())(36:01) okay that is something that you have to remember, if you have a convergent power series okay then any power series converges in a disk okay, that disk could have possibly in finite radius in which case it is the whole plane as happens in the case of a polynomial (())(36:20) exponential function and in that disk the convergence of the power series is always normal namely it is uniform on compact subsets okay and it is also absolute okay.

So this is something that you should have come across in the first course in complex analysis where essentially you make use of the Weierstrass M test okay, so at a convergent power series by Abel's theorem is that it can be differentiated term by term and what you get is again a power series of the same radius of convergence and that is the derivative of the original power series and this is one way of showing that the derivative of a power series exists and it is gotten by differentiating it term by term, the term by term differentiation is justified because of the uniform convergence okay and if you apply this ad infinitum what you get is the power series is infinitely differentiable it is actually analytic and it is infinitely differentiable.

So a power series whenever you are looking at a power series inside its disk of convergence you are actually looking at an analytic function and what has the analytic function to which it convergence got to do with power series, this power series is nothing but the Taylor expansion of that limit, that limiting function, so you start at the power series it converges within its disk of convergence to a certain function that function is analytic function and if you expand it as Taylor expansion at that point you will get back the power series, so the moral of the stories that whenever you are looking at a convergent power series you are actually looking at the Taylor series of an analytic function okay and what is that analytic function, it is exactly the function to which this power series converges, okay.

So what we trying to show is that we are trying to show that this point Z naught is basically a point where the function is analytic, so what you expect is that you expect the Laurent series should actually be a Taylor series that means all the coefficients of the negative powers of Z minus Z naught in the Laurent expansion would be 0, so you try to show that all those coefficients are 0 then you are done okay and how do you show those coefficients are 0, the coefficients are given by integrals and integrals can always be estimated by the so-called ML inequality okay so what we will do is we will show that all the negative coefficients in the Laurent expansion they are all 0 and we are done that is what exactly I am going to do.

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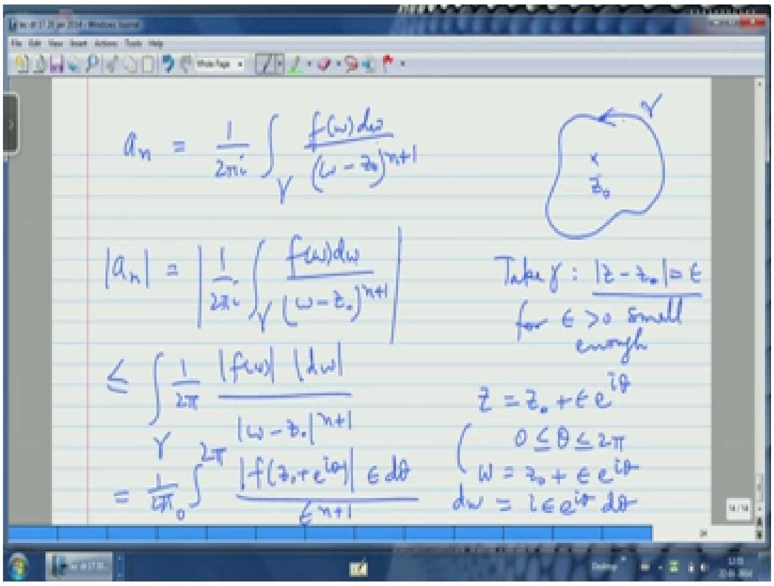

So what is a n? a n is 1 by 2 pi i integral over gamma fw dw by w minus Z naught to the power n plus 1 where you know the picture is like this you have Z naught and you have gamma a simple closed curve going around Z naught sufficiently close to Z naught going around 1, now you take for take for gamma to be the circle okay the shape of gamma really does not matter because of course this is theorem actually, take gamma to be the circle mod Z minus Z naught is equal to Epsilon for Epsilon as positive small enough small enough so that the circle on the circle and inside the circle except at the point Z naught the function is analytic okay and well you know when you can…

So the equation Z mod Z minus Z naught equal to Epsilon you can use that write it as a parametric equation and do an integration okay so this is the same as writing it as mod Z is equal to Z naught plus Epsilon e to the i theta where theta varies from 0 to 2 pi okay and so this integral becomes so this integral so if I calculate the modulus of a n mind you I am trying to show that the modulus of a n is 0, I am trying to show that the a n are 0 for negative n this formula is valid for all values of n. I am trying to show a n is 0 for all negative n that is good enough to say that the Laurent series is the Taylor series and that will tell me that the function is actually is analytic at the point okay.

So I have to calculate mod a n, mod a n is going to be the modulus of this integral okay and then you would have come across this estimation formula in the 1st course of complex analysis which is used all the time we says that the modulus of the integral is less than or equal to the integral of the modulus, so this is less than or equal to integral over gamma of if I take the modulus I am going to get 1 by 2 pi mod fw mod dw by mod w minus Z naught to the power n plus 1 okay and this integral becomes therefore just integrals 0 to 2 pi because now I change the variable of integration from w to theta mind you, you should always remember that whenever you write an integral then the then you have an integrant and you have a variable of integration okay and the integrant is the function of the variable of integration and the variable of integration are always with me on the area on the region of the integration okay.

In this case the region of integration is the is the curve gamma which we have taken to be a circle, so your w is actually varying on the circle okay, so w should be written as Z naught plus Epsilon e power i theta okay so what I will get is let me put this mod f Z naught plus e to the i theta and then I have write out so I should change this Z to W, so it will become w equal to Z naught plus Epsilon E to i theta, dw will be i Epsilon e to the i theta d theta and if I take mod dw I am going to get Epsilon d theta okay and then I am going to get here I am going to get Epsilon mod w minus Z naught is Epsilon e to the i theta, it is modulus is Epsilon so I will get Epsilon to the n plus 1 okay, so this is what I am going to get alright and you know what I am going to do next you see whatever I assumed I have assumed condition 4, condition 4 is that the function is bounded modulus in a sufficiently small neighbourhood.

So you know this term mod f of Z naught plus e power i theta I am going to remove that and put an M because model of f Z is going to be lesser or equal to M in a sufficiently small disk and I am assuming that Epsilon the small enough so that this circle lies in that disk in that deleted neighbourhood of Z naught okay, so what I am to going to get the next step is I am going to get rid of this mod f Z naught plus e to i theta I am going to pull that out and instead of that I am going to put M. I am going to get I am going to get I forgot 2 pi there is 1 by 2 pi outside and then so you know what I will get is, I will get basically I will get M times Epsilon times 2 pi divided by 2 pi times Epsilon M plus 1 okay. Mind you when I integral 0 to 2 pi d theta I am going to get 2 pi okay and that 2 pi is going to cancel with the 2 pi outside.

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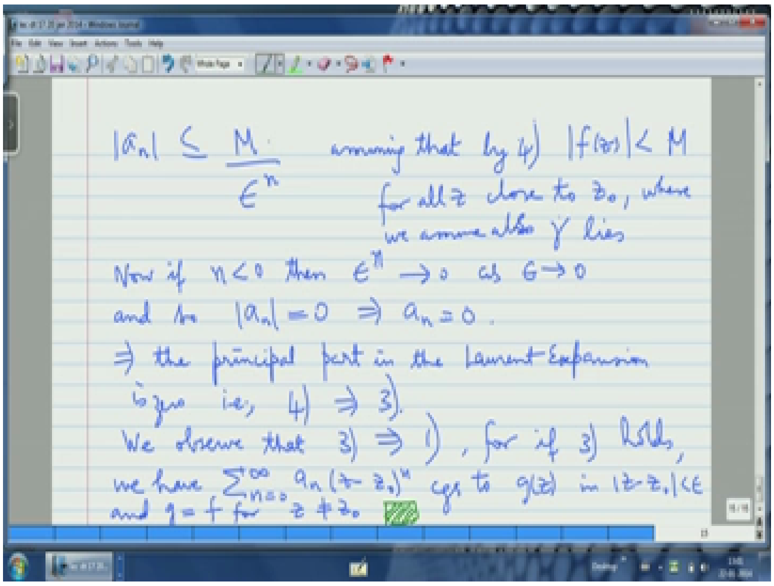

So basically what I am going to get is I will get mod a n is less than or equal to M times if M by Epsilon Power n this is all I am going to get assuming that by 4 mod f Z is less than M for all Z close to Z naught where we assume also gamma lies okay, so I am going to get this. Now watch see Epsilon is a small quantity okay, Epsilon can be made as small as I want I can make Epsilon smaller I can make Epsilon tends to 0. Now if n is negative if n is negative mind you I am trying to show that the a n for n negative are 0, I am trying to show that all the negative terms in the Laurent expansion do not exist okay, so all the negative Laurent coefficients are all 0 I am trying to show, so if n is negative that is the case I have look at then this Epsilon Power n will go to the numerator. So I will get a small quantity to a positive power okay and if I now let a small quantity go to 0 its positive power will go faster to zero so numerator will go to 0 and since this is valid for all Epsilon greater than 0 mod a n has to be than or equal to 0 and that will force mod a n is 0 and that will force at a n is 0 and we are done okay and that is the end of the proof okay.

So let me write that down, now if n is negative then Epsilon power n tends to 0 as Epsilon tends to 0 and so mod a n is equal to 0 implies a n is equal to 0, so this implies that the principal part in the in the Laurent expansion is 0 that is what we have actually proved is that we have proved that you know 4 implies 3 in fact we have actually proved 4 implies 3 and so 4 implies 3 alright and of course 3 mind you is equivalent to 3 implies 1 because if the Taylor if the Laurent expansion is a tailor expansion okay namely if it has no principle part then it is a power series it will converge to a function and that function is going to be equal to the given function outside that point and therefore what happens is that you have extended that function analytically to that point also okay.

So let me write that down there is a little bit of there is a little bit of technicality, so let me write it down, we observed that 3 implies actually 1 for if 3 holds we have Sigma n equal to 0 to infinity a n Z minus Z naught power n converges to g of Z and converges to g of Z in mod Z minus Z naught less than Epsilon and g is equal to f for Z not equal to Z naught so see let me repeat this if you assume 3 what does 3 say, it says that f of Z has a certain Laurent expansion in which there are no negative terms. f of Z what does it mean? It means that you have 1st of all a Lauren expansion which converges to a function that function to which it converges is none other than f of Z okay and this is valid whenever Z is not equal to Z naught but what is a Lauren expansion at converges to f of Z it is actually a Taylor expansion but you know namely it is just it is just the convergent power series and you know the convergent power series is actually a tailor expansion of the analytic function to which it converges, so you take only the Laurent expansion which has 0 principle part, it will converge to an analytic function call that function is g of Z.

Now that function is going to go inside with f outside Z naught by definition because already know that the Laurent expansion also converges to f outside Z naught, so in principle what has happened is that you have found an analytic function g of Z which is analytic at Z naught and outside Z naught it coincides with f okay so finally this proves 4 implies one right and that completes the proof of Riemann removable singularities theorem, so that is the end of the proof which I will signify by putting a usually in books you see that people put a shaded square I will put something like this to indicate end of proof okay but there is a remark that I want to make, so the remark use to fix some loose ends in the statement of the theorem, so I am going back to the 1st condition okay, the 1st condition namely the definition of removable singularity, what is the condition? The condition is that the singularities removable (())(50:56) there is a continuous function there is an analytic function to which this function converges.

What I want to say is that you see condition only says the function converges to an analytic function at that point but it does not say that this analytic function is unique okay, so the 1st condition which is just the definition our definition of removable singularities that you function can be extended to an analytic function at that point okay but I say it can be extended to an analytic function I am not saying it can be extended to a unique analytic function and the fact is that it can be extended to a unique analytic function and the reason is that there is a deeper theorem behind this.

Suppose it extends to 2 analytic functions at that point okay then you use the identity theorem which you should have studied in 1st course in complex analysis which says that if 2 analytic functions coincides on an open set nonempty open set or for example even if they coincide on sequence of point which has a limit point at which both of them are analytic then they have to be identically equal, so that identity theorem will tell you that if the singularities removable then the function to which the analytic function to which the given function extends at the single point is actually a unique function okay, so there is identity theorem there okay that you should remember okay so I will stop to that.