Advanced Complex Analysis-Part 2
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Lecture 29
Proof of the Arzela-Ascoli Theorem for Functions - Equicontinuity Implies Compactness

Alright so now what we are going to do is we will continue with the proof of the Arzela-Ascoli Theorem, okay.

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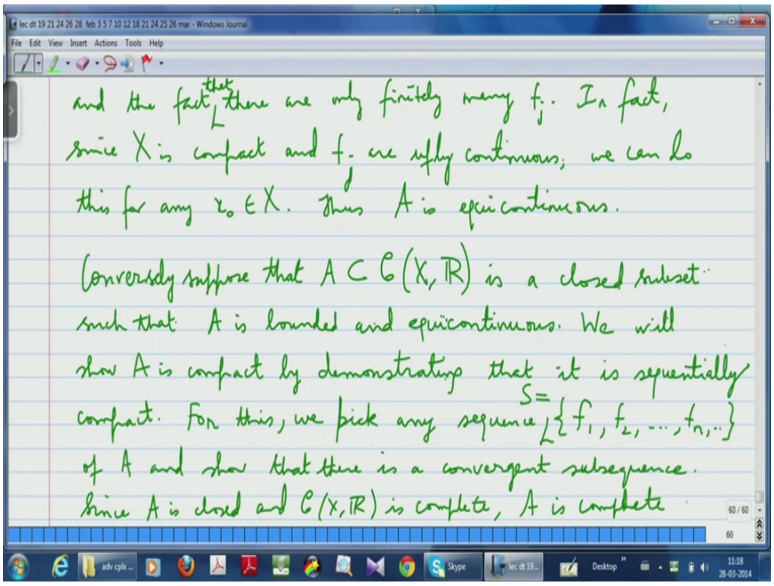

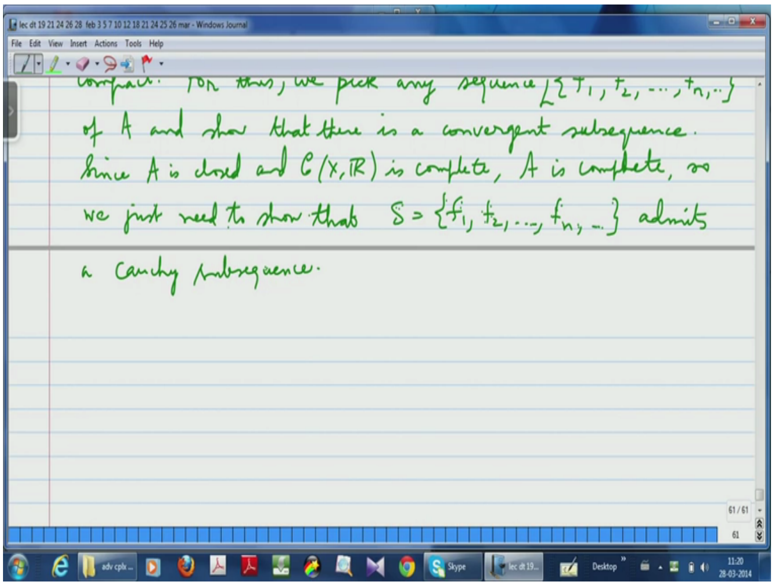

So so conversely suppose that A in C X, R suppose A in C X, R or ofcourse you could have taken C X, C okay instead of real valued functions you should have taken complex valued functions is is a closed subset such that A is bounded and equicontinuous, okay we will show show A is compact and how we will show that A is compact we will show A is compact by showing that A is sequentially compact, okay because you know compactness is equivalent to sequential compactness for a metric space. So we will show A is compact by demonstrating that it is sequentially compact compact, okay.

Now so so what you have to do is that you know we will pick a sequence from A, okay this is sequence of functions in A and we will assume and then we will try to extract the subsequence which is convergent that is what sequential compactness means, sequential compactness means that every sequence admits a convergent subsequence, okay. So for this we pick any sequence f 1, f 2, etc f n and let me call it as S of A and show that there is show that there is a convergent subsequence, okay.

So we start with any given sequence in A and we try to show that there is a convergent subsequence but now notice that C X, R okay the space of continuous bounded real valued functions on X mind you is is a Banach algebra it is complete as a it is a complete norm linear space with the norm given by the sup norm, okay the supremum norm. So any close subset of a complete space is also complete, okay.

Therefore since A is closed A is also complete this means that every Cauchy sequence converges ofcourse you know that convergent sequences are Cauchy, okay. So in order to show that there is a convergent subsequence it is enough to show that there is only a Cauchy subsequence. So finally what we will do is we will use the closeness of A and we use the completeness of the bigger space the Banach algebra to reduce to showing just that we will be able to extract a Cauchy subsequence, okay so we have to just check that there is a subsequence which satisfies the Cauchy property, okay.

So let me write that down since A is closed and C X, R is complete A is complete so you know we just need to show that S the sequence S which consist of f 1, f 2, etc f n admits a Cauchy subsequence, okay. So so this is what we have to show, alright and what is it that we are given we are given we are given that the family A, the collection of functions A, the subset of functions A is equicontinuous and it is bounded, okay.

Now let us let us analyse what these condition mean, okay and ofcourse we are also given the condition that X is compact mind you X is a compact metric space so that also needs to be used, okay and mind you because X is a compact metric space automatically all the continuous functions on X are automatically bounded, okay normally if you take a non-compact space if you look at continuous functions but if you want to define the supremum norm you will have to restrict to bounded functions, okay so that you get a finite norm, alright for every function.

But then in this case you do not have to restrict to bounded functions among continuous function because every continuous function is bounded and that is because X is already compact, alright.

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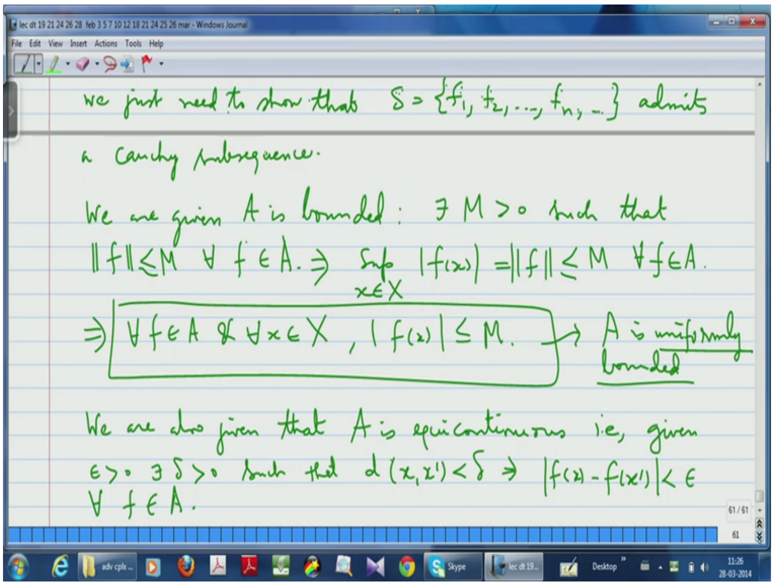

Now so what I am going to do is that let us analyse the condition that that A is bounded, okay we are given A is bounded what does this mean? This means that there exist an M greater than 0 such that the norm of f is less than M for all f in A, okay. So basically A is bounded means that A is bounded as a subset of a the of C X, R as a bounded as a metric space, okay bounded as a metric space means it is contained in a finite open disk I mean an open disk of finite radius centred at some point, okay but you can choose the point to be the origin because this is a norm it is not just a metric space it is a norm linear space it is a actually a vector space.

So there is a 0 element and every vector every element there AB vector has a norm and it is the norm that induces the metric because it is a norm linear space. So saying that a subset is bounded is a same as saying that all the norms the lengths of the vectors in the subset they are all bounded by a positive value that is what I have written, okay. So mind you what does this mean but what is norm f but norm f is what it is supremum over all small x belonging to capital X mod of x this is what norm f is this is the supremum norm and this is less than M and this is true for all f in A, okay and what does this mean this implies that for all f in A and for all x in A for all x in X okay mod f x is less than or equal to M sorry is less than M, I could have used less than or equal to but probably it does not matter let me stick to less than or equal to if you want does not matter, okay.

Now you see look at this condition if you look at this condition now I have rewritten this condition with with the points of the the domain x of the functions. So if you look at it like this you see that what you are seeing is that for all the functions in A okay M is uniform bound for the function values and that is why this boundedness as a subspace of C X, R is actually uniform boundedness that is why this Arzela-Ascoli Theorem is called the uniform boundedness principle namely it says that uniform boundedness plus equicontinuity will imply compactness, okay that is the that is the other way in which this theorem is often stated so the uniform boundedness comes from this.

So what this tells you is that A is uniformly bounded uniformly bounded and this uniform this uniformness is a kind of double uniformness because it is uniform bound for all the functions in A and for each each of those functions it is uniform bound for all points of x so it is a double uniformness it is uniform in the f, it is uniform in the x, okay in that sense it is uniform bound, okay.

So that is one thing that is given to us than the other thing is the other thing that is given to us is that this family or this collection A subset A of function is equicontinuous and what is that so let me write that down we are also given that A is equicontinuous i.e, given epsilon positive there exist delta positive such that d distance between x and x prime lesser than delta will imply that the distance between f x f x and f x prime which is just the distance function is just the modulus of the difference of the function values since these are real values less than epsilon for all f in A so this is the equicontinuity, the equicontinuity is that it is just the same the equicontinuity for a family is just the same as the epsilon delta definition for a single function but the only thing is that the for a given epsilon the delta works for all functions, okay at any given point, okay.

And usually the delta will also change as you change the point but here you can even make the delta independent of the point also because the functions are all continuous functions on a compact space and therefore they are uniformly continuous, okay.

So so you have this so this is so these are things that we have we had given and we have to use, okay. So the first step of the proof will be to so the so let me tell you the idea of the proof the idea of the proof is that the first thing we will do is that we will show that see basically we have a sequence, okay we are given a sequence of functions from A, we have to pick out a Cauchy subsequence, okay. So mind you saying that this Cauchy that you have a Cauchy subsequence is means this is Cauchy in the metric for the space of functions but that means uniformly Cauchy with respect to points of x, okay.

See what you must understand is that whatever you say for functions on with respect to the metric on C X, R okay that will translate if you because if the sup norm if you translate it to an equivalent statement concerning points of x it will become a uniform statement, okay it will be a property which does not depend on the particular points of x so it will be uniform on x, okay.

So for example convergence in the space of functions will correspond to uniform convergence on x, boundedness in the space of functions as we just now saw corresponds to uniform boundedness, okay. So whatever you say in the space of functions that will correspond to a property which is uniform with respect to points of x that is what we have to remember, when you translate from function theoretic property to the space theoretic property the property you will get a uniform version it is a uniform version because it does not depend on the points of x value of verifying it, okay.

So basically what we have to do is from the given sequence we have to pick a Cauchy subsequence but this Cauchy this is the Cauchy subsequence in the in the with respect to the metric on the space of functions but that will transform that will translate to a uniformly Cauchy sequence of functions with respect to x, okay that is what we want, alright. Now the trick is that what we will do is that we will use we will make use of the fact that because x is compact it is a compact metric space x is actually separable that is it has a countable dense subset, okay.

Then what we will do is that we will be using this countable dense subset we will find we will extract a subsequence, okay using a diagonal argument from the original sequence of functions such that this subsequence will converge at all points of this countable dense subset and then we will use the equicontinuity to show that that is good enough to say that the subsequence is Cauchy on all of x, okay because because it is already uniformly Cauchy on you know a dense subset, okay so this is the trick.

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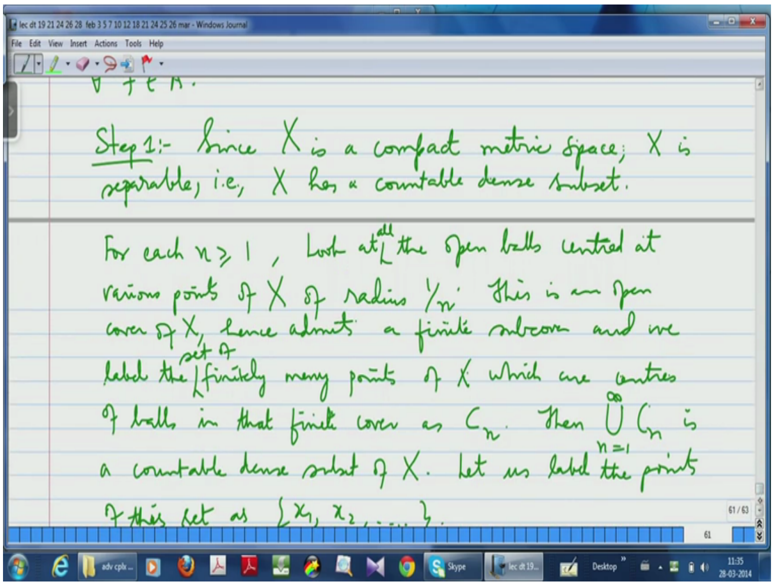

So the first step that we need is a compact metric space is separable we need that fact, okay. So let me say state that here step 1 since X is a compact metric space, hence X is separable that is X has a countable dense subset and how does one prove this? This is just an exercise saying that a compact metric space is separable the method is very very simple so what you do is basically for each for each n for each n greater than 1 actually you could have taken any sequence of numbers going to 0 which is any sequence of rational numbers going to 0 but I will choose 1 by n for each n greater than or equal to 1, look at look at all the open balls centred at various points of x of x of sorry of capital X of radius 1 by n, okay.

So you take n equal to 1, 2, 3 and so on and for each fixed value of n, look at all the open balls centred at various points of X, radius equal to that n 1 by n reciprocal of that n, right. So in the first case I will look at all balls open balls at various points of X centred at various points of X with radius 1, then I look at all open balls of X centred at various points of X with radius half and then I will next look at radius 1 by 3 and so on, okay.

Now each of these for any fixed n each of this is an open cover and X being compact admits a finite sub cover, okay and from that finite sub cover you pick the centres that will give you a finite set of points. So for every n you get a finite set of points and you put together all this all the sets of points as n goes to infinity you will get a countable union of countable sets and that is that will serve as the countable dense subset of X that you are looking for, okay.

So let me write that down this set this is an open cover cover of X, hence admits a finite sub cover and we are we label the finitely many elements the finitely many points of X which are centres of balls in that finite cover and we label the set of let me put set of as C sub n, okay. So C n is those finitely many points of X such that the open balls centred at those points, radius 1 by n covers X, okay and you do this for every n each C n is a finite set, okay.

Then union n equal to 1 to infinity, C n is a countable dense subset ofcourse it is countable because it is a countable union of finite collection of sets so it is countable, okay and why is it dense? It is dense because give me any point of X and give me any epsilon greater than 0, I can find an n large enough so that 1 by n is less than epsilon, okay and then if you that point has to lie in the open cover consisting of balls of radius 1 by n so it has to be within distance of 1 by n from one of the points of C n, okay and that means that the points of your set the union of all the C n's they come in infinitely close (())(20:32) close to every point of X that means that they are closure this all of X that is the reason why X is this subset is dense, okay.

Now what we will do is that since it is a countable subset we will again re label the points x 1, x 2, x 3 because it is countable, okay. So let us label the points of the set of this set as x 1, x 2 and so on, okay alright. So so the step 1 is to get this you can think of this as some you know test set of points where you want to test convergence, okay so this test set is just it is countable dense subset, okay.

And you know why you need the countability because you want to be able to extract a subsequence and extracting a subsequence means you need some algorithm for getting hold of a countable subset, okay so you need countability, right okay and now you see now what we will do is so this is step 1 we have got this this countable subset of points which is dense. Step 2 is to show that you can extract a subsequence from your original sequence of functions which converges at all of these points converges point wise at each of these points, okay.

Now that is done by a standard trick which is called a diagonal argument, okay so idea is very very simple the idea is see I have already a sequence of functions f 1, f 2, f 3 which I call as S now what you do and I have this sequence of functions on the one hand, on the other hand I have this sequence of points I have these sequence of points x 1, x 2, etc which is a countable dense subset of your metric space compact metric space, okay.

What I do is I test point by point I first take this point x 1 I substitute in these functions so I get f 1 of x 1, (f 2 of x 2) f 2 of x 1, f 3 of x 1 I get this I substitute the point in the function in the set of functions and what I am going to get I am going to get a bounded sequence of real numbers, why is it bounded? Because I know all these functions are uniformly bounded so bounded sequence of real numbers and you know a bounded sequence of real numbers has to admit a convergent subsequence.

So there is a sudden convergent subsequence of functions there is a subsequence of functions which is converges at x 1 I will extract that sequence, then what I will do I will repeat the process with x 2 for that subsequence of functions. So from that subsequence of functions I will get another subsequence of functions which will converge at x 2, okay and then I will repeat this process ad infinitum. So what will happen is at the nth stage I would have extracted nth at the nth level I would have extracted a subsequence of subsequence of subsequence at the nth level, okay which converges at x n.

And if I do this ad infinitum okay the point is that you take you write the functions on each stage in the form of a row, okay and you have a you write it as rows and then you get countably many rows and then you take all the functions in the diagonal the point with all the functions you are taking the sequence of functions in the diagonal is a ofcourse you will get a countable subsequence but the point is that this sequence will converge at all points of the dense subset because at some stage if you take all the functions in the diagonal the ith functions will converge at it will converge at x 1, x 2, x 3, etc upto x i okay and then what will happen is that if you if you keep letting i become larger and larger you will see that all the functions converge at all points of this countable dense subset, okay and that is the critical step.

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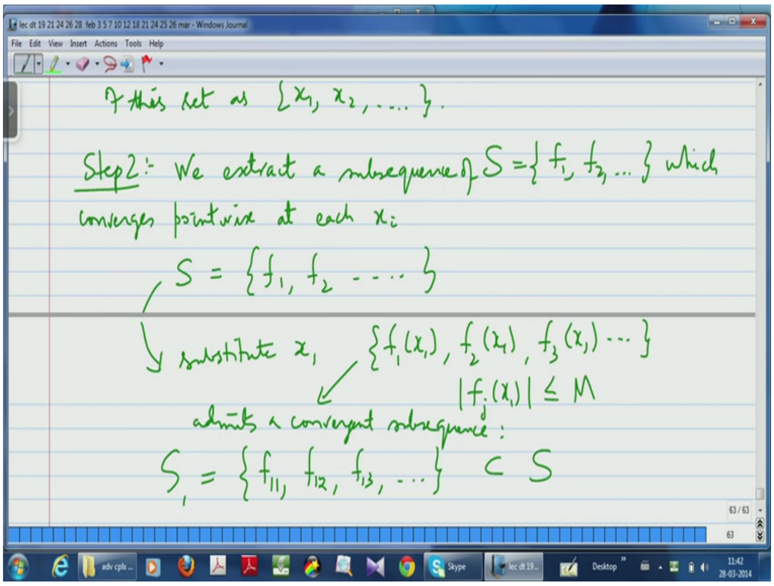

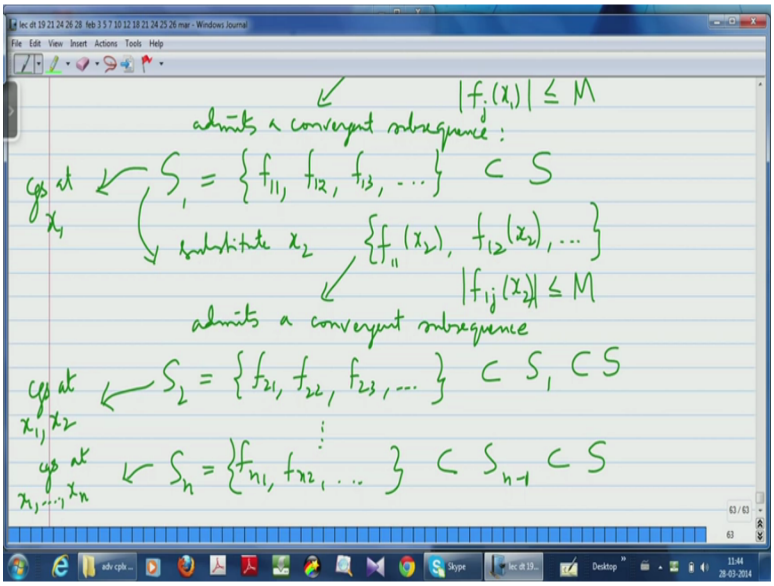

So let me write that down, so step 2 we extract a countable countable means a subsequence S subsequence of S which converges point wise at each x i, okay that is it converges point wise on that countable dense subset, how do you do that? So let me do it figuratively so you have S is equal to f 1, f 2 and so on, okay and what you do is well substitute x 1 what you will get is you will get f 1 x 1, f 2 x 1, f 3 x 1 and so on, okay and note that mod f j x 1 is less than or equal to M because this M is uniform bound for all functions of A and after all the sequence S has been taken from A, okay.

So but this is a this bounded sequence of real numbers okay and you know if we were working with complex functions you would have got a bounded sequence of complex numbers any bounded sequence of real numbers are complex numbers does admit a convergent subsequence, okay. So let me write this here this admits a convergent subsequence so what you will get is you will write you see you get a convergent subsequence of real numbers, okay or complex numbers whatever it is, okay depending on other we are now looking at real valued functions if it work on complex valued functions you will get a convergent subsequence of complex numbers.

But then forget now the numbers and look at the functions that have come up, okay forget the x 1 and look at the f so what you let us call that as f 11 because I want to use this I am relabeling that so f 12, f 13 and so on, okay mind you this is a subsequence of S okay and I am relabeling in principle I should write as f if you want well you know I could use different methods of relabeling but let me use this okay.

So thee f i j's f 1 j's are just these the first subsequence I picked out from this set, okay. Now what I do is I repeat the same process for this S 1with x 2 okay and and I proceed so let me write that once more. So though this substitute x 2 then what you will get is you will get this f 11 x 2, f 12 x 2 and so on and again you will have mod f 1j x 2 is certainly less than or equal to M because it is a uniform bound it works for all functions and for all points.

Again this admits the convergent subsequence this admits a convergent subsequence, okay and again now what you do is relabel the subsequences f 21, f 22 so I will get S 2 which is now f 21, f 22, f 23 and so on. Now this is a subsequence of S 1 which intern is subsequence of S, okay but the point is that this S 1 this converges at x 1 when you plug in x 1 this converges what about this? This converges both at x 2 and x 1 because it is already a subsequence of S 1. So this converges at x 1, comma x 2, okay and then now you proceed by induction what you get is you will get S n which will consist of functions f n1, f n2, etc this is a subsequence of S n minus 1 which is which intern is subsequence of S and this will converge at x 1 dot dot dot upto x n, okay and you can do this for every n, alright.

Now what you do is you write down all these f ij's as an array and pick the diagonal sequence that is the sequence that we are looking for that is the sequence that will converge at all points at all these points of this countable dense subset in that sequence that is going to be a Cauchy sequence as we will see, okay.

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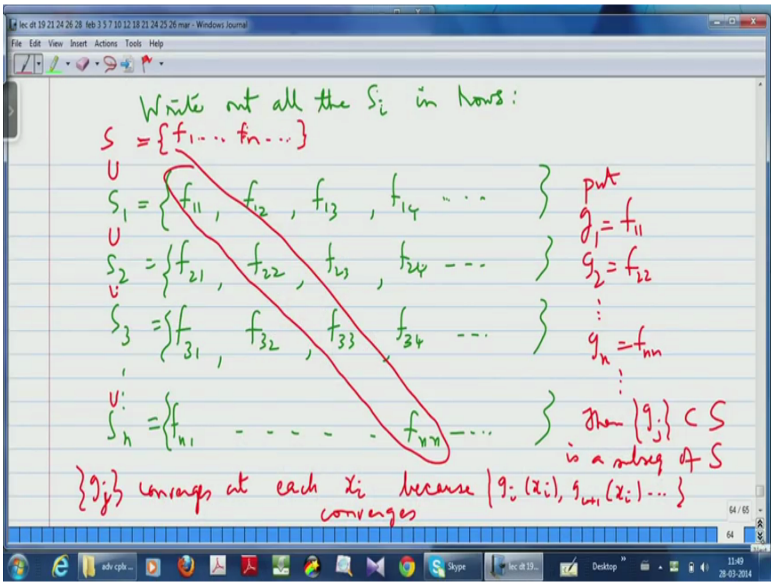

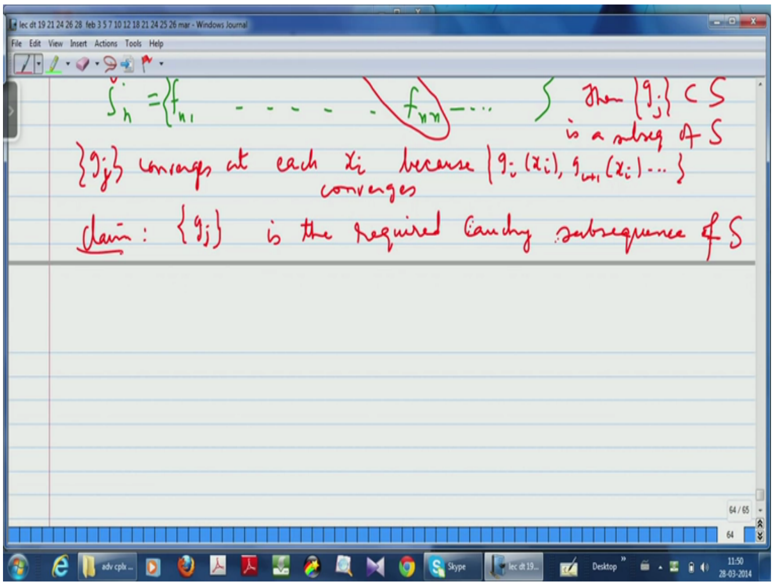

So write all the functions all the S i in rows so here is S 1 this is f 11 f 12 f 13 f 14 and so on, there is S 2 that is f 21 f 22 f 23 f 24 and so on and it goes on like this and if you write S n I am going to get f n1 going to get f nn, okay and ofcourse and well that is how it is and there are commas in between if you want but any way, okay. Now what we are worried about is the diagonal subsequence so so we are worried about this guy pick this put g 1 is equal to f 11, g 2 is equal to f 22 and so on g n equal to f nn and so on, okay.

Then the claim is that ofcourse this g 1, g 2 that is a subsequence of the original sequence S mind you this S 2 is contained in S 1, S 3 is contained in S 2, S n is contained in S n minus 1 and all this is contained in S which is the original sequence which consisted of you know f 1, etc f n and so on, okay. So what you get is you get this sequence g of g j’s these g j's are all subsequence of the original S but the beautiful thing with this g j is that these g j's will all converge point wise at every point of x, okay. So in fact at every point of the countable dense subset, okay.

So then so let me write it here then g j is a subsequence of S, g j converges at each x i and why is that true because if you give me an x i then from g i onwards okay all g i, g i plus 1 etc they will all converge at x i okay alright because g i x i g i plus 1 x i converges, okay. See for a sequence to converge at a point it is enough if beyond a certain stage all the members of the sequence converges.

So if I want to check the g i if I want to check g i when I substitute when I take the sequence g j and substitute x i, if I want to say g j of x i converges it is enough to show that g j of x i converges beyond a certain j and what is that j? that j is from i onwards okay that is what I am saying okay so this is the so you see this is the diagonal trick which is very cleverly used, okay so we have this now the claim is that g j converges in fact I should say I should let me say g j is the required Cauchy subsequence of S, okay so this is this is claim this is the claim, okay.

So finally we are able to extract a Cauchy subsequence and that is good enough, right. Now how do you check it is Cauchy, how to check it is uniformly Cauchy okay because it is Cauchy in the space of functions means that is it uniformly Cauchy with respect to points of x, okay so you have to bring in points of x somehow alright and the point is that we will have to use what we have not used so far we have already used the closeness of A because we have reduced from finding a convergent subsequence to a Cauchy subsequence, okay and we have used the closeness to used the fact that A is already complete, okay close subspace of complete subspace is complete, we have not used the equivicontinuity we have to use the equivicontinuity, okay.

And what you will have to do is that you will have to use the equivicontinuity of this family A in particular it also applies to the functions in the subsequence g j because after all it is they are all functions from the same family A, okay this collection A, we have to use the equivicontinuity and but then the thing is that we will also have to use the see we have to we will have to use the compactness of A again.

See this is because first of all what we have proved is that we have got a because of the diagonal argument we have got an hold of a subsequence that converges at all points of a countable dense subset, okay but form that you want to conclude that it converges at all points, okay first of all you have to conclude that it converges at all points, you have to conclude that using the information that it converges only on points of a countable dense subset, okay and for that how do you reach an arbitrary point from from this given dense point there you will have to use compactness, okay and there you will have to use equicontinuity.

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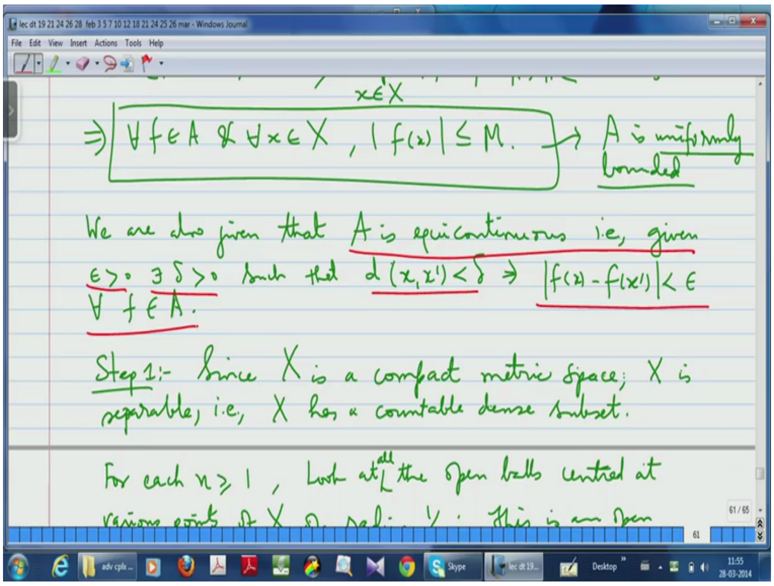

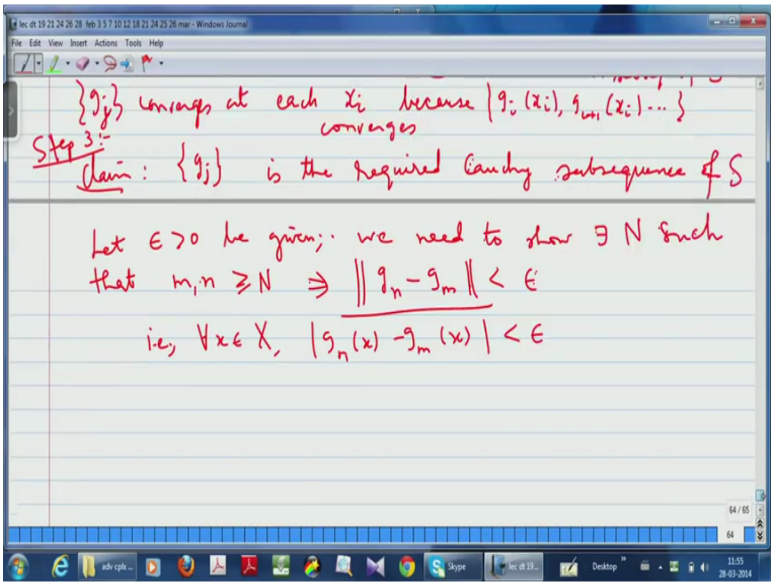

So what we will do is that so let me write out so this is the final step more or less so this is well let me put this as a so what was I what I had called it as step 2 may be. So this is step 3 actually I have to show that this is the required Cauchy subsequence. So what you will do is that so let epsilon greater than 0 be given, okay. What do we have to show? We need to show there exist an N such that m, comma n greater than or equal to N implies mod g j sorry g m so this is the Cauchy condition so I should say distance so let me write down so norm g n minus g m can be made less than epsilon this is the Cauchy condition mind you this is the norm g n minus g m is the distance between g n and g m in the metric induced by the norm in the space of functions, okay.

I have to show that this can be made less than epsilon for if the subscript of the sequence goes beyond a certain stage, okay. Let points of the sequence come to within an epsilon beyond certain stage that is the Cauchy condition, okay but ofcourse this norm is the sup norm what this means is that that is for all x belonging to X you must show that mod g n of x minus g m of x can be made less than epsilon, okay and you see the important thing is that here you see this this is because the norm is a sup norm, right. So we will have to show this.

So how do you do it? Mind you the x is an arbitrary point of X, so you have to tackle an arbitrary point of X so what you do is so at some points you know you have to reach an arbitrary point of X from one of the points in the countable set, okay but then this is but there are too many points in the countable set there are countable meaning but you have to reduce to finitely meaning so this is where you again use compactness of X, okay.

So what you do is you look at a cover of X by open balls of radius delta, okay where you see where delta is this delta that we are already we already have for the equivicontinuity. So where is it you see here so there is this delta here if you go back A is equivicontinuous so given epsilon greater than 0 there is a delta such that whenever the distance between x and x prime is less than delta then mod f x minus mod f x prime is less than epsilon, okay we have to use that delta.

But then you see since we are going to interpolate with points from finite subset of points the distance in the triangle inequality will be broken in three terms so we will take this delta corresponding to epsilon by 3, okay.

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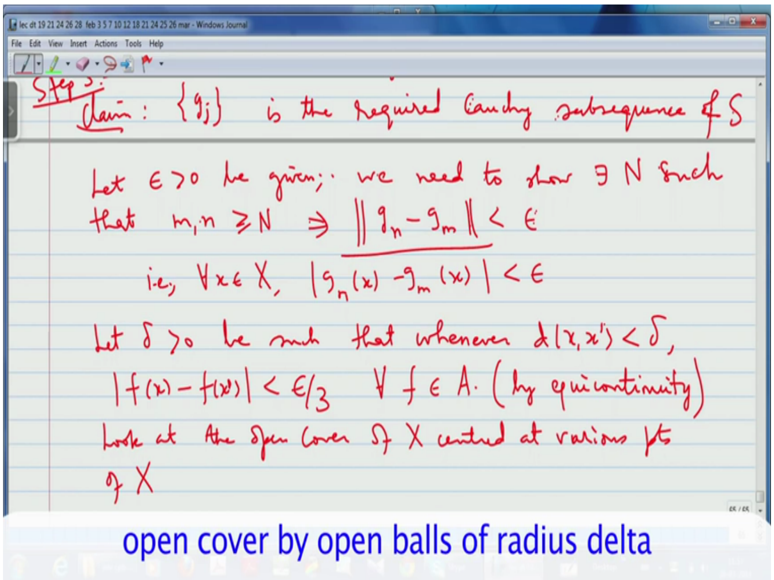

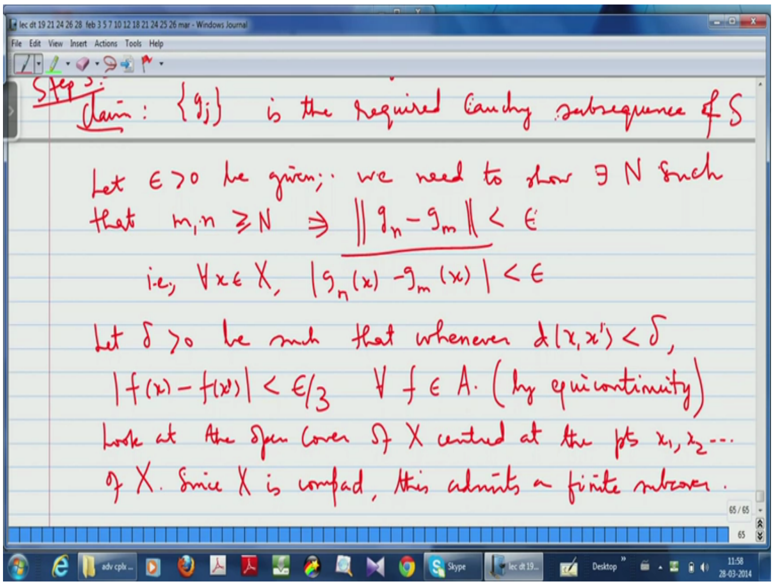

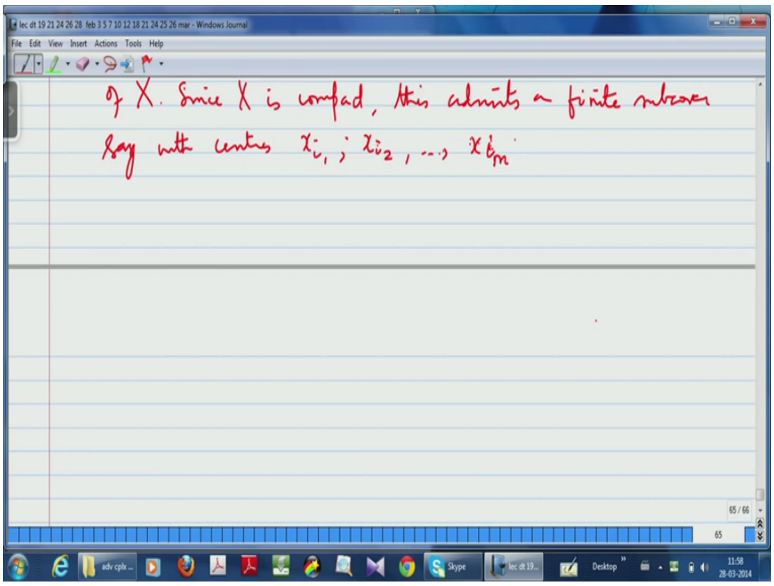

So what we will do is so let me write that down let delta greater than 0 be such that whenever the distance between x and x prime is less than delta, the distance between f x and f x prime is less than epsilon by 3 for all f in A this is where we are using the equivicontinuity, okay by equivicontinuity.

So you started with an epsilon but you consider epsilon by 3 and then for that epsilon by 3 you pick a delta now use this delta to form an look at all look at this open cover which is formed by open balls of radius delta that will have a finite sub cover because (A) because x is compact, okay and take those points those finitely many points, okay and use those points to interpolate between f x and f x prime using the g j's and you will get what you want, okay so that is the idea.

So look at the open cover of X centred at various points of X, in fact in fact but I would like those points to be one of the points in the countable collection so instead of taking all points of X I leave and take the points from this countable dense subset even that will form a open cover because the subset is dense, okay. So look at the open cover of X centred at at points so not all points at the points x 1, x 2, etc of X, okay.

So I could have taken all points of X but I am purposely taking the points of the countable dense subset, okay that will also form an open cover because their closure is all of X, okay. Since X is compact this admits a finite sub cover so say with centres x i1 so you will get a subsequence of their x i's so you will get which I am calling as x i1, x i2, etc x im, okay my i looks like j, okay.

Mind you these x i's are finitely among those points which form a countable dense subset, okay and now what you do is that you see now the trick is that you know I will I will have to interpolate mod f x minus f x prime with these x ij’s and the g j's okay and get what I want alright.