Advanced Complex Analysis-Part 2
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Lecture 28
Proof of the Arzela-Ascoli Theorem for Functions-Abstract Compactness Implies Equicontinuity

Let us continue with our discussion of this Arzela-Ascoli Theorem, okay. So you know the situation is that we are trying to understand what compactness means for functions, okay basically we want to understand compactness for families of Meromorphic functions defined on a domain in the complex plane, okay or external complex plane okay but then we want to get some idea about what compactness means from basic topology, okay.

So as I was telling you suppose you are having a topological space, you have the notion of compactness, okay and if the topological space is is a metric space then compactness is the same as it is equivalent to sequential compactness and it is also equivalent to the Bolzano-Weierstrass property, the sequential compactness is the property that every sequence has a convergent subsequence, okay the Bolzano-Weierstrass property is a property that every infinite subset has a limit point, okay and these three are equivalent, right.

And then there is also this this this implication that if you have a compact subsets it is always closed and bounded if you are in a metric space, okay but the converse is not true. So if you take R infinity you can easily look at the unit ball in R infinity the closed unit ball it will be closed and bounded but it will not be compact because you can write out your sequence which does not admit any convergent subsequence namely the sequence consisting of 1 at the ith place and 0 elsewhere, okay any two terms of this sequence differ by a distance of root 2, okay are separated by a distance of root 2 and therefore this sequence cannot have a convergent subsequence, okay because terms of the sequence do not come you cannot find a subsequence whose terms whose elements come closer and closer together, okay.

So but ofcourse in any in any metric space compactness implies closeness and boundedness the converse is not true the converse is true for eucledian spaces and in fact I even told you that if you take a Banach space the condition that it is finite dimension is equivalent, every closed subspace every closed bounded subspace is compact if you want to put that condition then the Banach space has to be finite dimensional it cannot happen if it is a infinite dimension Banach space, okay.

So now you see alright so the so there is a related notion that is this notion of total boundedness, okay which is implied by compactness, alright total boundedness is that is basically the property that you know you can cover the whole space by open balls of a fixed radius no matter how small the radius is the point is ofcourse by finitely many such open balls okay.

So given any positive radius epsilon I should be able to find just finitely many open balls of radius epsilon whose union is the whole space and I should I should be able to do this for every epsilon greater than 0 this is total boundedness is very strong form of boundedness this implies boundedness, okay and in fact space that is totally bounded is actually even it even has finite diameter, okay because the diameter of the space can be compared to the diameter of this finite collection of centres of these open disks which is called a net an epsilon net if the radius you are talking about is epsilon, okay.

And the point is that total boundedness itself though it is a strong form of boundedness total boundedness is not enough to give compactness, okay what you have to add to it is completeness. So if you have something that is complete and totally bounded a metric space that is complete and totally bounded then it is compact, okay and so that is the so that is another equivalent version of compactness.

So you know so we check to check compactness either you check the the abstract definition of compactness which is very hard you have to take an arbitrary open cover and you have to show that there is a finite sub cover that is very difficult in practise, okay and probably if you are given a particular open cover specific open cover may be you might be able to because of your knowledge of geometry and topology you might be able to extract a finite sub cover but you cannot do this in an arbitrary way so it is abstract, okay trying to get hold of a finite sub cover from an any given open cover is a very abstract thing it is not easy to check, okay.

So we end up checking sequential compactness, okay namely you take a sequence and show that there is a convergent subsequence if you check that then that is equivalent to compactness. The other thing you can check is that the space has the metric space has the Bolzano-Weierstrass property show that every infinite subset has a limit point, okay and ofcourse if you are in if you are working with subsets eucledian space you know what to do you will just check that the subset is that you want to say is compact is both closed and bounded so there is not much to do, alright.

But point is that and ofcourse now there is also this new there is also this extra condition that you know if you want to check a spaces compact you check that it is complete and it is totally bounded, okay and ofcourse completeness is the condition that every Cauchy sequence converges, okay that is you know whether that can be easily checked or not depends on the particular case you are looking at okay it is also not so easy, okay.

And the other thing is total boundedness and that is also very abstract, okay total boundedness is very abstract thing you have to say that you know you can find an epsilon net for every epsilon greater than 0 and that is so you have to produce an epsilon net for every epsilon greater than 0 and epsilon net is those finitely many points in the space centred at which if you take open balls of various epsilon these balls will cover the whole space, okay you have to you have to show the existence of those finitely many points that epsilon net and you should do this for every epsilon it is not an easy thing to do, alright.

So these are the various versions of compactness that we have but you know we are interested in compactness of functions okay. Now for functions what is it that what is that is easy to you know verify normally what you normally easily verify about functions you verify continuity okay you verify differentiability, you verify boundedness of a function, okay try to find a constant which bounds the modulus of the function values, okay.

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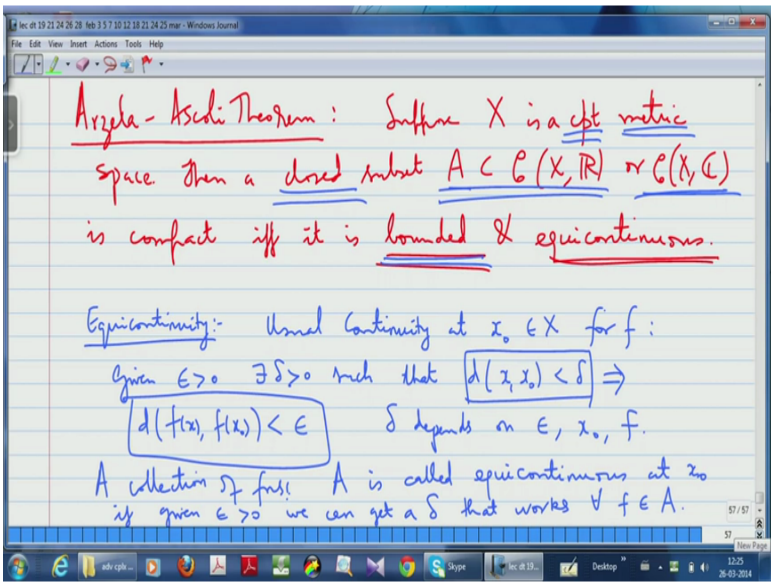

So these are the things that you verify, okay so you need something in the situation of functions if you are working with a space of functions then you need better conditions for compactness and that is where the Arzela-Ascoli Theorem steps in. So basically you know we are looking at a compact metric space X, okay so X is a compact metric space and so let me go back to what I wrote last lecture X is a compact metric space and let me let me use a different color now X is a compact metric space and you are looking at a I am looking at a I am looking at this C X, R which is the you know it is a Banach algebra it is a complete norm linear space of continuous functions on X with values in real numbers, okay.

And you can also take with values in the complex numbers you will get the other Banach algebra C X, C okay and the question is that you want to and ofcourse mind you since X is compact and a continuous function on a compact set A is uniformly continuous and always attains its bounds every function that we are worried about is already uniformly continuous and it is bounded, okay so normally when you write C X, R or C X, C you is you mean also not only just continuous functions but also continuous bounded functions, okay but here boundedness is automatic because X is compact alright.

So what I want I want I want to take a close subset of I want to look at a subset of functions I want to look at a family of functions or a collection of functions, okay which you think of as a subset of the space of functions C X, R or C X, C and I want to say that it is compact that is my aim I want to get a nice condition to say that a collection of functions is compact, okay. Now you know since we are in the metric space something that has to be compact a subset has to be compact means that it has to be both closed and bounded.

So certainly the subset to begin with should be closed, okay so we put this pre condition that it is already a closed set, if it is not a closed set it is not going to be compact because compact implies its closeness, okay and ofcourse it has to be bounded because I told you that in any metric space compact subset is always closed and bounded, okay. So if you start with the closed subset A the condition that it is compact is equivalent to being bounded and the here is the extra condition the extra condition is equicontinuity, okay so equicontinuity is something that that helps us to get hold of compactness, right.

So you know I want to see at this point I want at this point say the following thing I want to say that if you look at it from the view point of total boundedness, okay then you see to say that A is compact it is enough to say that A is totally bounded, okay because if A is totally bounded, alright then A is also complete not then I mean A is already complete because A is a close subset of complete metric space so it is complete.

And if you have total boundedness together with completeness you will get compactness because that is another characterization of compactness I told you okay. So if A is totally bounded then I know it is compact, okay so the only issue is with replacing this total boundedness with something else and I told you this total boundedness is not so easy to verify because you have to show the existence of an epsilon net for every epsilon that is not practical, okay.

So whereas total boundedness being difficult to verify we have boundedness is easier to verify, okay so you from the total boundedness you remove the total and you just put boundedness which is an easy thing for verify for functions, okay and you because you made the condition weaker from total boundedness to boundedness you have to put something else to make it strong enough to get compactness to give compactness and that strong enough thing is the equicontinuity, okay.

So I will tell you what this what this equicontinuity is, okay so so basically the idea is very very simple the idea is see if you want function a function to be continuous at a point then what you do is you say that that is usual you know you have the usual you know you have the usual definition of the epsilon delta definition of continuity at a point and what is that give me epsilon greater than 0, I can find a delta such that whenever the whenever the point is within a delta of the given particular point then the function value is within an epsilon of the image of the function value at that point, okay.

Now so the pole issue here is that if you change the point, okay so you are looking at continuity at a point for a given epsilon you have to verify it, okay now the delta that you get will depend on the epsilon ofcourse because if you make epsilon smaller you might expect that the delta must become smaller, okay and so the delta depends on the epsilon the delta also depends on the point and the delta is particular to that function, okay.

So the delta depends on three things, okay delta depends on three things if you are so you know so let me write this down so what is this equicontinuity? So you see what is usual continuity usual continuity at let us say x not for f what is this? Given epsilon greater than 0 there exist a delta greater than 0 such that whenever the distance between x and x not is less than delta then we have that the distance between f x and f x not is less than epsilon this is the usual good old definition of continuity epsilon delta definition of continuity.

And you see the point here is that ofcourse this this the first d that I used in d of x, comma x not is the metric on the space on which the function is defined whose points are x and x not, x not is a fixed point, x is a variable point, okay. So basically this d x, comma x not less than delta refers to you know basically it refers to all the points in a delta open ball centred at x not in your space, okay and what you are saying is that whenever a point whenever you take a point delta open ball centred at x not then its image is in an epsilon open ball centred at f x not that is what you are saying, okay.

So the image of this delta open ball under f goes into this epsilon open ball centred at f x centred at f x not, okay that is what usual continuity is but the point is that this epsilon you see given an epsilon okay delta depends on what? See depends on ofcourse epsilon it depends on x not and you know it depends on f also ofcourse we are looking at a single function so we forget the function because if you are dealing with only one function there is no confusion but if you are looking at a family of functions then you will have different functions f okay and as you change the function for the same epsilon and even for the same point okay if you have a family of functions which are continuous at a point x not, okay then for that family of functions even if you take a single epsilon the same delta will not work the delta will change with the function, okay and ofcourse delta will change with the point.

Now equicontinuity is the fact that you are is the condition that you know you are able to get a delta that works for all functions in one stroke that is equicontinuity, okay. So collection collection of functions of functions so well A so let me call it as A is called equicontinuous at x not you know if given epsilon greater than 0 we can get a delta that works for all functions in that collection that is equicontinuity you get a common delta that is equicontinuity, alright.

And well and ofcourse you say that a family is equicontinuous on a subset of the space if it is equicontinuous at every point of the space, okay. So you make a point wise differentiation and then you say that this definition holds if it holds on a holds for a subset of points if it holds for every point in that subset, okay. So this is just like saying that function is continuous on a set if it is continuous at every point of the set it is like that.

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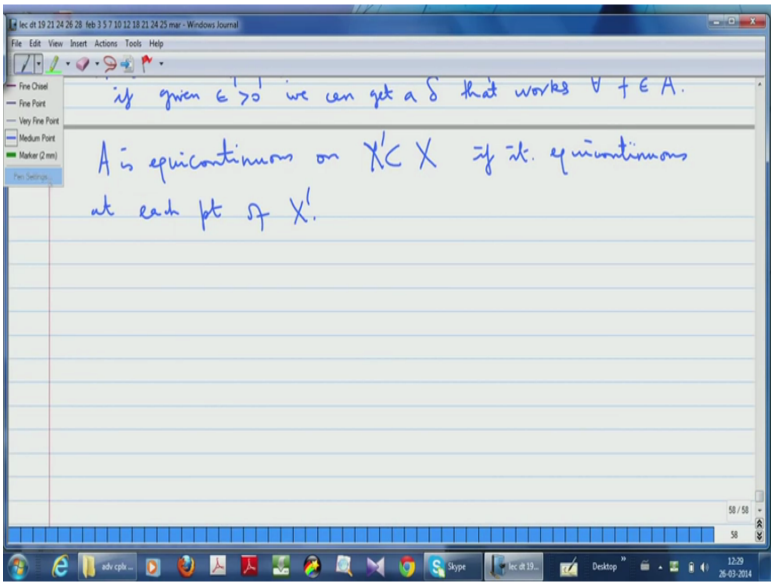

So A is equicontinuous at or let me say on on let me say subset Y in X if it is let me just explain if it is equicontinuous at each point of X prime, okay. So so this equicontinuity is basically if you want to you know think about it in a very simple way equicontinuity is given an epsilon you find it delta that works for all functions that is the point, okay a single delta will work an epsilon for every function in your collection then that collection is called equicontinuity an equicontinuous collection or equicontinuous family of functions, okay.

And what the Arzela-Ascoli Theorem says is that well the Arzela-Ascoli Theorem says that if you put equicontinuity together with boundedness then that is good enough to say that your collection is compact, okay so compactness of a family of functions is equivalent to that family be equicontinuous and boundedness and and you this is a much much nicer condition then total boundedness is okay equicontinuity is it is a kind of continuity that you have to check it is just continuity you have to check but make sure that you get one delta for all the functions for a given epsilon, okay that is like checking continuity and that should be far more easier checking boundedness is not a it should not be a big problem, okay.

So these are all things that we normally check for functions but given a family of functions you never it is not common that you check total boundedness total boundedness is you have to find finitely given an epsilon you have to find finitely many functions so that the distance of any function is within an epsilon to one of these finitely many functions you have to find this epsilon net that is not easy that is not common, okay that is not practical.

So Arzela-Ascoli Theorem helps you by saying that well you want to check family of functions is compact do not do much you just check that it is bounded, check that it is equicontinuous and you are done, okay. So I just wanted to you know indicate something about the about this this equivalence of compactness with boundedness in equicontinuity because it is a because it is a very very fundamental thing and you need to know how things work, okay.

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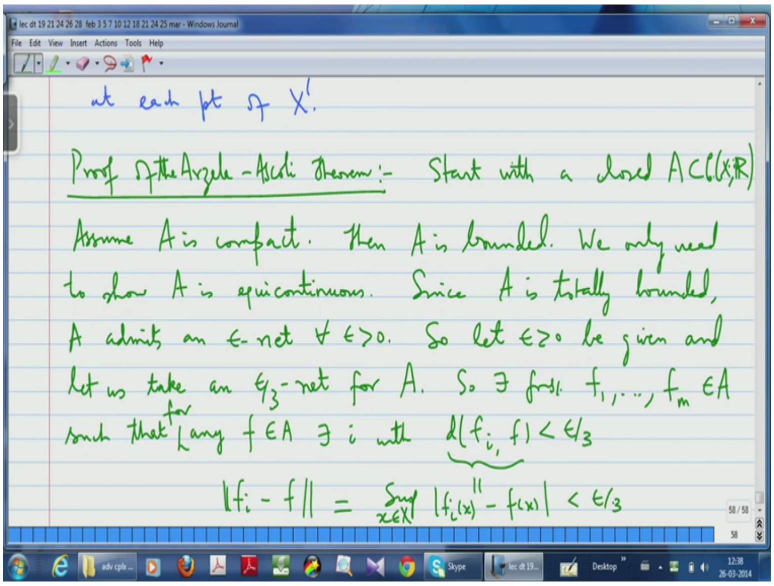

So so let me so let me do the following thing I will I will again change colour and use a different colour so I will say proof of Arzela-Ascoli Theorem. So so let me go through the proof because I wanted to understand the ideas involved, okay you will also get familiar with all these notions of total boundedness, epsilon net and how equicontinuity is actually used, okay. So so start with a closed start with a closed A inside C X, R start with the closed set what do you have to show? You have to show that A is compact if and only if it is bounded and equicontinuous, okay.

So assume A is compact assume A is compact alright now what do you have to show so this is one way of the theorem you have to show that if it is compact it is bounded and is equicontinuous, okay. I told you that in a metric space subset is compact implies it is both closed and bounded, okay so it is bounded, okay compactness always implies closeness and boundedness in a metric space, right.

So and mind you that is true only for metric spaces, if you go to arbitrary topological spaces things can become very bad you can have a compact subspace of a compact topological space which is not closed, okay it can behave terribly you can have a whole space which is compact, you can have a subspace which is compact but the subspace is not closed in the whole space that this can happen for an arbitrary topological space, it cannot happen for a metric space. For a metric space compactness always forces closeness and boundedness so house of spaces also house of spaces if you want ya ya.

So you see so assume A is compact then A is A is bounded bounded, okay because compactness will imply closeness and boundedness it is all A is already closed, okay but if you want boundedness that that is implied by compactness. We only have to show that A is equicontinuous, okay we only need to show A is equicontinuous, okay that is what you have to show. So you see so incidentally let me make a remark here to in this case we are anyway going to the way we prove the Arzela-Ascoli Theorem is by using the other version of compact characterization of compactness which is compactness is equivalent to A being totally bounded, okay this total boundedness will come into the picture, okay atleast in the proof, okay that is the key tool that helps in the proof of Arzela-Ascoli Theorem.

So if you think of since if you think of that given that A is compact I know it is totally bounded and I told you totally bounded also is a strong form of boundedness so it implies boundedness totally bounded is the condition that there is an epsilon net for every epsilon, okay and you know that diameter of the space can be compared to the diameter of any epsilon net, okay to within an epsilon net to within an epsilon, okay or two epsilon, okay.

So therefore A is always bounded in fact it has finite diameter alright. We only need to show that A is equicontinuous and for the equicontinuity also you use the fact that it is totally bounded, okay.

So since A is totally bounded well A is A admits an epsilon net for every epsilon greater than 0, okay so here I am using the fact that compactness implies total boundedness, right and compactness implies total boundedness is a very very it is a very simple thing there is nothing complicated about it, okay basically what you are saying is that you should cover the space by finitely many open balls of the fixed radius, okay that radius being epsilon and that is very that is very easy to see because if you give if you take all open balls of radius epsilon that is the cover for the space you allow every point of the space to be a centre and you take the collection of all open balls of radius epsilon that is obviously an open cover for the space and if you know it is compact it has to admit a finite cover and that finite cover if you take the centres that will give you the epsilon net that you want, okay. So it is very that compactness implies total boundedness is absolutely easy to see, okay. So there is a there is an epsilon net, okay.

So so let epsilon greater than 0 be given and let us take an so here is a you know this is the this is the kind of thing you fiddle with to get epsilon finding your answer so you know usually the idea is that you use a triangle inequality you know finally all these epsilon arguments they finally end up with a triangle inequality if you are breaking a distance into two pieces then you try to and you want that distance to be less than epsilon you know you try to make each piece less than epsilon by 2, if you are breaking it into 3 pieces then you know you make each piece less than epsilon by 3. So what I am going to do is I am not going to take an epsilon net I am going to take an epsilon by 3 net, okay.

So so let epsilon greater than 0 be given and let us take an epsilon by 3 net for A, okay. So what does this mean? This means that so there exist functions f 1, etc upto f m which are in A such that well the any function in A such that for any function in A there exist an i such that i with the distance between f i and f less than epsilon by 3, okay so this is what epsilon net means basically you cover all points of the space by looking at open balls of radius epsilon by 3, centred at the points which correspond to the epsilon by 3 net, right this is this is this is what you get.

Now I wanted to you know at this point you know I want you to remember what is this distance here? See this distance is the distance in this distance is the distance in the in the Banach algebra C X, R okay it is mind you all these what is A? A is a subset of C X, R and C X, R are you know functions which are continuous functions on X real valued functions, okay and so what is the and ofcourse X is compact X is compact so these all these functions are bounded, okay continuous functions on a compact set is bounded and in fact it is uniformly continuous (())(30:44) it bounds, right.

So so what is the distance the distance is the supremum now, so what is this this quantity is supremum as X belongs to capital X as X varies over capital X of you know mod of f i of x minus f of x this is what it is, okay. In fact it is actually it is precisely norm of f i minus f that is what the distance is the distance between f i and f is just norm of f i minus f and the norm is the sup norm so you calculate f i minus f at each point and then you take the mod of that and then you take the supremum, okay and this should be less than epsilon by 3 this is what you have, okay this is what you have.

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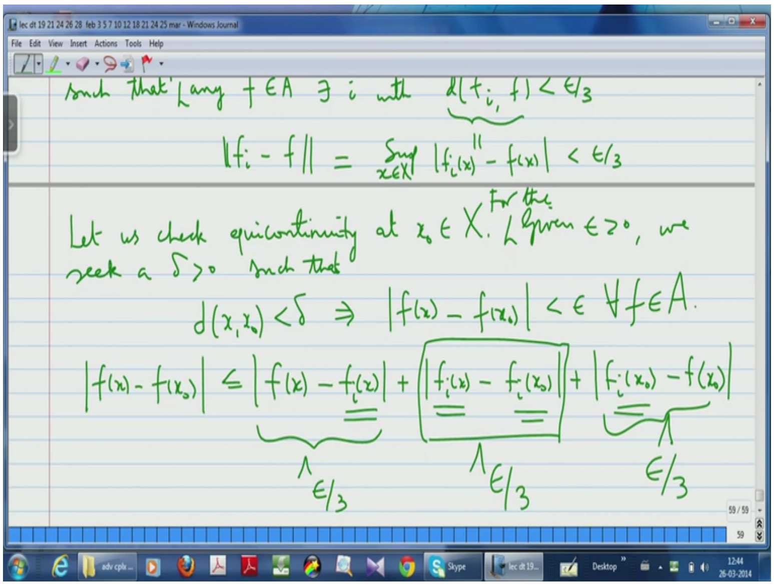

Now you see now we are kind of in more or less in a very good shape because see now you can get equicontinuity very easily. So let us check equicontinuity at a point, okay. So let let us check let us check equicontinuity at x not in X let us check equicontinuity there okay then what is what is it that is going to happen well what do you have to check for equicontinuity you have to find a delta such that whenever mod x minus whenever the distance between x minus x and x not is less than delta, okay the the distance between f x and f x not is less than epsilon and this should work for all f in your family A in your collection A, okay.

So given for the given for the given epsilon greater than 0, we seek a delta greater than 0 such that mod I keep saying mod because you know at the back of at the back of once when one keeps thinks of eucledian space but it is not mod you should replace it with d so the distance between x and x not less than delta implies that mod f x the the distance between f x and f x not which is mod f x minus f x not is less than epsilon and this should work for all f in A this is what you want, alright.

So you see how do you do this the idea is very very simple that is this one the one hand you have an X which is a variable point you have this point x not and you have a arbitrary f alright and you will have to now connect that arbitrary f in A with these particular f i's which are the elements of the epsilon net, okay. So you do that by a you know basically by using a triangle inequality to break down some distance into three pieces, okay.

So what you do is that well you write this mod f x minus f x not you write this as you know I have already told you that given this f there is an f i with the property that the norm of f i minus f is less than epsilon by 3 that is already there so use that f i. So you write this as modulus of you know well f of x minus f i of x plus mod f i of x minus f i of x not plus modulus of so I should put less than or equal to modulus of f i of x not minus (f i) f of x not this is what I will have to do, okay. So I introduced this this f i cleverly this f i values at x and f i values at x not to break this.

And then what happens is that you see the you know f for any x in x okay for any x in x the modulus of f i x f x is always less than epsilon by 3. So the point is that this fellow here this is less than or this is less than epsilon by 3, okay and so is this fellow here this is also epsilon by 3, okay I have to be worried only about the central term this is the only term I have to worry about with that this is I cannot apply that epsilon by 3, bound to that because the points are different the function is the same it is the same f i okay the same f i but the points are x and x not that is in the first term the point is x, the third term the point is x not, the term in the middle has two points with the same function, okay.

What do I do that what do I do here? So here what is do is basically I I use the fact that all these f i's are all uniformly continuous in fact they are all continuous they are continuous on a compact set so they are uniformly continuous. So because you are uniformly continuous I can make sure that you know if I if I chose a delta sufficiently small whenever the distance between x and x not is less than delta I can make the central term less than epsilon by 3 I can do this.

In fact I can simply do it because of continuity of f i at x not but the point is that I can do this for all f i at the same time because there are only finitely many f i's, okay there are only finitely many f i's. See if I want f i x minus f i x not to be less than epsilon by 3 I can ofcourse chose delta such that I can find a delta for which whenever the distance between x and x not is not less than delta then mod f i x minus f i x not is less than epsilon by 3 I can do this because of continuity of f i at x not, okay.

But then this delta may depend on the f i, okay and if I change the i to other f i's okay then the delta will change but there are only finitely many of these f i's so I could have taken the minimum of them and that would work for all f i's all the elements of the epsilon net in one go alright and mind you you could even forget the point x not that is because all these functions that we are studying they are uniformly continuous, okay so there is also you can you can not only do this you can not only do this for all the f i's because they are finitely many, you can also do it irrespective of the point x not because of uniform continuity that is because all the f i's all the functions we are considering are continuous functions on a compact space and when you have continuous functions on a compact space you get uniform continuity.

Uniform continuity is that whenever the distance between source points is less than a delta then the distance between the image points is less than epsilon it does not matter what the source points are the only condition is they have to be within a delta you can find such an delta that is uniform continuity, okay.

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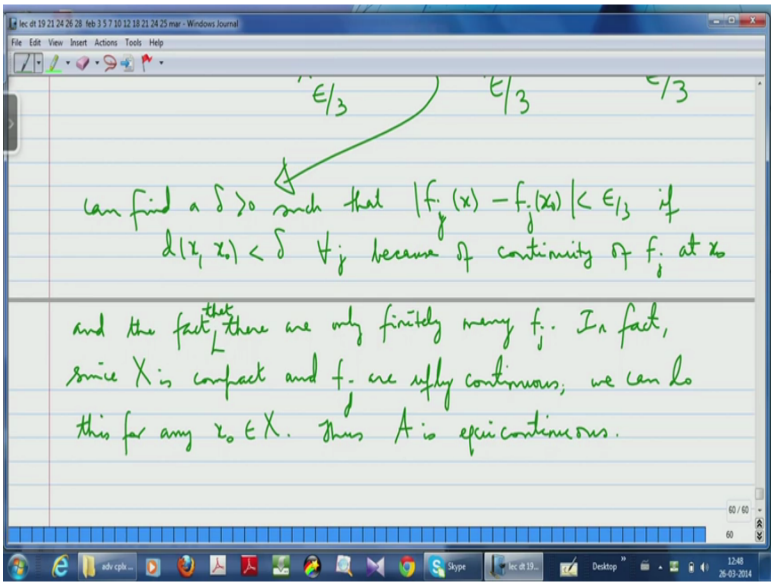

So so let me so let me write here can find a delta greater than 0 such that mod f i of x minus so let me write f j of x minus f j of x not is less than epsilon by 3 if d of x, comma x not is less than delta (because) for all j so let me write for all j because of continuity of f j at x not and and the fact that there are only finitely many only finitely many f j, okay I can do this, right. Not only that I can do more in fact because all of these are uniformly continuous I can do this irrespective of x not okay I can even forget x not, okay.

In fact since X is compact and f j are uniformly continuous so I am abbreviating it to ufly for uniformly uniformly continuous we can do this for any x not, okay. So you can find given an epsilon you can find a delta which neither depends on the f nor does it depend on the x x not, okay so you have verified equicontinuity in a very in uniform way, okay and that is how so you have got a equicontinuity that is it, you have found a delta that works for every f, okay. So thus A is equicontinuous, okay.

So so the moral of the story is we are able to see one way, why if you have a compact space of functions then these functions must form an equicontinuous family, okay. So equicontinuity is something that is is very very important and mind you I will have to prove the other way of the theorem but then before I do that which I will do in the next lecture what I will now say is I will tell you why this is called why Arzela-Ascoli Theorem is sometimes called uniform boundedness principle, it is because of the following reason you see the condition is that a close subset of functions is compact if and only if it is bounded and equicontinuous, okay.

Now this bounded is actually bounded with respect to the metric on the space of functions and that is that is given by the sup norm, okay and boundedness under the sup norm, okay is actually uniform boundedness it means see what does boundedness normally mean? It means that you are able to find positive constant such that the modulus of the function values at all points is bounded above by this positive constant, okay.

But ofcourse this positive constant would change if you change the function, so if it take different functions in a family each function may individually be bounded but you may not be able to find the common bound for all the functions in the family, okay and that is what uniform boundedness does. It gives you a common bound for all functions in your family or collection or subset, okay.

So boundedness in C X, R actually means uniform boundedness and Arzela-Ascoli Theorem is just that if you have a uniformly bounded family which is equicontinuous then it is compact and conversely that is what Arzela-Ascoli Theorem is and that is why it is called the uniform boundedness principle, okay. So I will continue in the next lecture and try to tell you the other way of the the other way of the proof, okay.