# Advanced Complex Analysis-Part 2

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**Lecture 27**

**Introduction to the Arzela-Ascoli Theorem-Passing from abstract Compactness to verifiable Equicontinuity**

See you know I am trying to bring in the context of the context to explain why we need the Arzela-Ascoli Theorem, okay in our discussion, okay so you know the as I was telling you in the last lecture the main thing we are concerned about is compactness of families of functions, okay. So at the background is we have the need to get a proof of the Picard theorems, okay and for that you have to study compactness of families of Meromorphic functions, okay.

And we have to understand that properly, alright and therefore you must understand compactness from basic point of view first of all compactness in the plain topological sense what it means for metric spaces and then you know what it means for spaces of functions, okay and once you know all this then you your mind is now then properly in tune to understand the you know the the proof of the so called Montel's theorem okay which is a key theorem that is used for going for getting a proof of the Picard theorems, okay.

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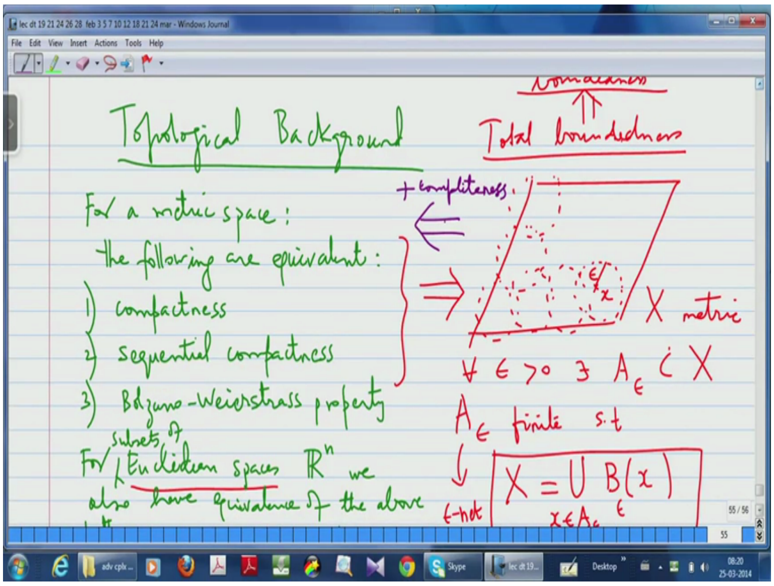


So so again you know so let me tell you very briefly let me again recall very briefly the topological background that I gave you last class. So you know if you start with if you start with the topological space you know there is a notion of compactness, okay namely every open cover has a finite sub cover and that is a very that is a very general notion, alright and it can be defined for any topological space because all you need for the definition is the idea of open set which is there in any topological space which is fundamental to any topological space.

Now but the point is that this this abstract definition of compactness is not useful all the time because it is a very it is a condition that you have to check for all covers for example if I want to check a topological space is compact I have to take an arbitrary cover of open sets and then I have to from that I have to show that I can pick out finitely many that is something that is not so easy today because it is you know why it is not easy to do is because a cover can be very abstract, it can be arbitrary and form something abstract it is very hard to extract something that is very specific, okay.

So so as usual as we do in mathematics normally you have a definition which is very abstract and the reason why you like that definition is because it has caught lot of power the abstraction gives it a lot of power, the abstract definition of compactness gives you lot of power in the sense that you know it tells you that whenever you have an open cover you can always get a finite sub cover.

So it is a very powerful thing but practically it is not so useful when you want to really use it in particular situations and just like same situation mathematics most of the time you have some definition involving a property which is very abstract and you make that definition because you know that property is very powerful it is a very powerful strong property which you can use.

But then how do you put it to use then the theory tries to give equivalent conditions so that you can which are easier to verify or which are more handier to verify, okay. So so in the same way if you look at compactness okay then there are conditions which help you to verify compactness in a more easy way by so what are equivalent conditions I was telling you yesterday that is you if you take a metric space then for a metric space compactness is a same as compactness is a same as sequential compactness and that is same as the Bolzano-Weierstrass property, okay.

So so what is the sequential compactness it is the property that every sequence has a convergent subsequence alright and and what is a Bolzano-Weierstrass property? The

Bolzano-Weierstrass property is that every infinite subset has a limit point, okay and we are more used to looking at eucledian spaces, okay R n, n dimensional real space and you can also take C n, n dimensional complex space the n C n can be thought of as R 2n if you want, okay and these eucledian spaces they have the property that you know compactness there is equivalent to closeness and boundedness, okay and that is what we are whenever we are doing whenever we are working with eucledian spaces if you want to check compactness all you will do is you will check it your set is bounded, you will check it is closed, okay and that is it then you know it is compact, okay.

So it is as easy it is that easy and mind you just take for example the close close take a close disk on the complex plane take a close disk on the complex plane it is easier to say that it is a close set and that it that it is a bounded sets and therefore it is compact it is easier to say that then to say then to take an arbitrary open cover and try to pick out a finite sub cover, okay it is not easy if I give you an arbitrary open cover of a close disk, okay on the complex plane it is not easy to pick out finite sub cover, alright. Whereas, I can assert that this will be true because of compactness and why because I can check compactness by the equivalent condition that it is both closed and bounded which is easy for me to check, okay.

So but ofcourse you know it is very important that so this is see in all these issues we are in the context of metric spaces we are not in the more general context of topological spaces because the metric is involved in all these things for compact when you define compactness metric is not involved, okay but when you define sequential compactness, okay you are worried about convergence of a sequence, okay and that is easiest to define if you have a metric otherwise if you do not have a metric you have to worry about nets and things like that, okay.

And Bolzano-Weierstrass property, okay can well in fact it can also be defined more generally but these properties are all very easily defined when you have a distance function between two points which is a metric, okay. So you see if you take ofcourse you know if you take a so what I was saying was that you know in all these things we are working with a metric space, okay and and the point is that with a metric space things are better because I mean you can visualize a lot of things because there is a distance function, okay.

So if limit if suppose you have if you say limit x tends to x not in a metric space it makes sense because you are actually saying the distance between x and x not is tending to 0, okay you are letting a function to tend to 0 so it makes clear sense if you cannot make so easy sense of this if you did not have a metric, okay. So so what I want I tell you is that now you know so the point is that somehow if you are working with eucledian spaces then compactness is just the same as closeness and boundedness put together, okay.

But if you take an arbitrary topological space one way is always true if something is compact, okay then it will always be if you take a subspace of a topological space if it is compact, okay well I should say if you take an arbitrary metric space not not not eucledian spaces, okay you take an arbitrary metric space if you take a compact subset it will be closed and bounded, okay but the converse need not be true, okay eucledian spaces are very special for which closeness and boundedness implies compactness is equivalent to compactness.

So the standard example is R infinity okay you look at all the sequences of real numbers such that the sum of the if you write out the series which consist of the sum of the squares of the modulus I the so called square summable sequences, okay in in modulus, okay then so you know you take all the so you take all sequences x1, x2, etc xn of real numbers such that sigma mod x i square okay you take the modulus of the entries of the sequence and square it and you sum it mind you when you sum it you are getting a series that series should converge, okay.

The set of all these sequences they are called square summable sequences the set of all these sequences is called R infinity is called the infinite dimensional eucledian space, okay and in this R infinity it is a metric space, okay because it is very easy to define a metric using just extending the same distance formula only thing is that now you have infinitely many coordinates the usual distance formula is what you take difference of coordinates square them, sum them up and then take square root.

Now what you do is you do it for all the coordinates even though there are infinitely many coordinates this will work because I know that if you sum all the if you take the sum of the squares of all the coordinates it converges, okay. So what will happen is that you get this R infinity and in R infinity if you take the if you take the open when you take the closed ball radius 1 centre the origin, okay you are taking all those elements whose distance from the origin is less than or equal to 1, okay then what happens is that this is ofcourse closed and bounded it is a closed ball so it is close and its radius 1 so it is bounded but it is not compact, okay because if you take the sequence which consist of only 1's along the diagonal okay you take a sequence of sequences, okay which if you write down one below the other so that you know you get the only 1's along the diagonal and 0's elsewhere okay then that sequence you know it will be in this in this closed ball but it will never have a convergence subsequence and that is because the distance between any two members of sequence is always fixed at positive quantity it is a constant value in fact it is it is root 2 actually, okay.

So you can see that if the distance between two members of the sequence is a constant such a sequence cannot have a convergent subsequence because the convergent subsequence means it has to become Caushy and distance between terms should come closer become very small, okay so it cannot have a Caushy subsequence. So what it means that you are able to get a sequence which does not have a convergent subsequence so it means it is not sequentially compact but if it is not sequentially compact it cannot be compact because there is this theorem which says that for a metric space compactness is same as sequential compactness, okay so R infinity is an example of metric space where you closeness and boundedness does not imply compactness okay.

So you are in this closed and bounded implies compact works for us for most of the time because we are worried about only usually we are worried about subsets of eucledian spaces, okay if you are working with R n or C n, n dimensional real or complex space then you are in good shape okay. Now but then you know the but you take an arbitrary metric space compactness will always imply closeness and boundedness, okay a compact subset will always be closed and bounded, okay that will always be true compactness is always stronger, compactness will give you closeness, compactness will give you boundedness but if you want to get back compactness from closeness and boundedness that will not work in an arbitrary metric space as we saw it does not work in R infinity but it will work in R n or C n, okay.

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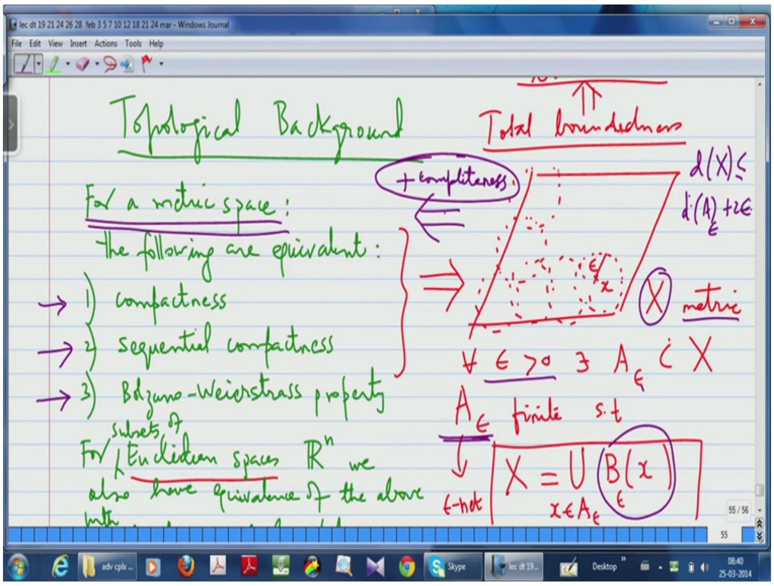


Now the truth is compactness is not only that strong in fact it is far more stronger it gives you something called total boundedness and what is this total boundedness? This total boundedness is as I was explaining to you know last time by this diagram you can have a look at this diagram so you see I have this space X which is a metric space mind you and the total boundedness is that you know you give me any positive number epsilon, okay you think of it is a very small radius, okay then what you can do is you can find finitely many points points of a subset A sub epsilon called an epsilon net such that you know every point of X is within an epsilon distance from one of these points atleast one of these points.

So in other words if you take the if you take the open balls centred at these points these finitely many points which are elements of A epsilon so called epsilon net with radius epsilon, okay then the union of all these balls will cover the whole space. So you know it is of mind you it is you are covering the space by only finitely many open balls of given radius epsilon and this will and this should work for every epsilon that is the that is even if you make epsilon smaller you will still find another finite set of points at which centred at which if you take these epsilon balls and you take the union it will still cover the whole space, okay it is a very strong property.

And this will this will ofcourse fall through if it is compact why because you know the truth is that give me if the space is compact give me any epsilon and you take the collection of all balls centred at various points all possible points with radius epsilon you take all these open balls that is an open cover and because it is compact it has to have a finite sub cover so I found an epsilon net for every epsilon I can find an epsilon net just simply because of the very definition of compactness so compactness will give you total boundedness.

But total boundedness and ofcourse total boundedness implies boundedness ofcourse because you can check as I was telling you last class that the the diameter of the topological space can be compared is always less than or equal to the diameter of the epsilon net plus 2 times epsilon. So you will get this so I am little short of space but let me write it here d of A is d of X not d of A d of X d of X is less than or equal to d of A plus 2 epsilon d of A epsilon plus 2 epsilon you will get this for every epsilon you will get this and d of A epsilon mind you the finite quantity because it is the diameter of a finite set diameter is supposed to be supremum of the distances between two points on the set, okay it is trying to measure how large the set is using the metric, okay.

So and we put (())(17:04) because sometimes the set may be unbounded and in that case your diameter may be infinite, okay. So in any case what this tells you is that if a space is compact if the metric space is compact and you know this tells you that it is bounded because it has finite diameter alright and and ofcourse I told you it will also be closed if a subspace of a metric space is compact it will be bounded and closed ofcourse, okay and you also get an epsilon net for every epsilon which means it is totally bounded alright and total boundedness a very strong condition it is a very strong form of boundedness.

Now the point is that from total boundedness how do you get back compactness, can you get back compactness? And the theorem is that you can get back compactness with this with this extra condition, okay. Suppose your space is totally bounded and complete then it is compact, okay so it is a you know it is rather powerful theorem so you know compactness so you know to check that something is compact it is enough to check that your metric space if you have a subspace of a metric space suppose you want to check its compact all you have to check is well now you have one more condition you check it is complete that means you have checked that every Cauchy sequence converges and you check that it is totally bounded, okay so this is this is another thing which helps.

And you know as far as general topological spaces are concerned things are really bad, in fact you can have suppose you have a general topological space which is not a metric space suppose it is not a metric space, okay then you can have a horrible situation like this the whole space will compact you can have a whole space which is compact not a metric space, topological space whole space is compact you can have a (sub) you can have a subspace which is also compact but it is not closed such a horrible thing can happen this can happen in an arbitrary topological space it cannot happen in a metric space, in a metric space a compact subspace is always closed and bounded, okay that is what you have to remember, okay.

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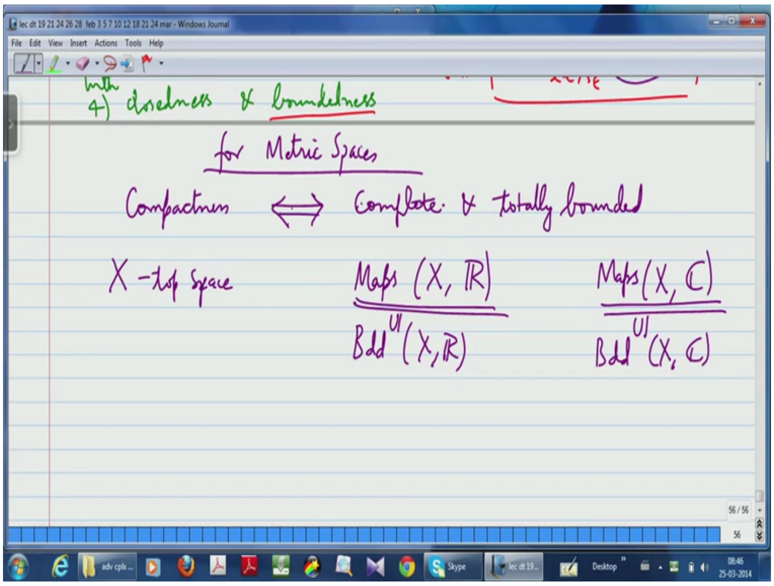


So so well now the point is that so what you get is you know so let me continue with this discussion so you have compactness on the one hand and then ofcourse we are working with metric spaces you have compactness on one hand and that is you can get that from complete complete and totally bounded and in fact this way also this is correct because you see if a space is compact it is ofcourse totally bounded I have told you if a metric space is compact it is totally bounded I told you it has finite diameter in fact and finite diameter which is comparable to diameter of any epsilon net, okay which exist because of total boundedness and compactness also implies completeness because you see if it is compact it is sequentially compact, okay.

So every every sequence has a convergence subsequence. So in particular if a sequence is a Cauchy sequence if then it will have a convergent subsequence but if a Cauchy sequence has a convergent subsequence then the Cauchy sequence itself must be convergent, okay so so compactness is equivalent to complete and totally bounded, okay. Now the problem is that you know this is this is what we have in general topology if we are only worried about spaces okay not just topological spaces but we are worried about more specific spaces namely metric spaces, okay but our context is different our context is we are worried about functions, okay.

If fact our application is we want to study Meromorphic functions on a domain in the external complex maybe alright that is the context where we have to go to. So somehow you know these results which are for spaces you have to translate them they are not still good enough for our use you have to translate them into results for spaces of functions, okay because we want to apply everything to spaces of functions.

So what spaces of functions will you think of? So you know the you know if you take so let me recall again if X is a topological space, okay then you know you can take this you can take the set of all maps from X to R okay or you can take the set of all maps from X to C so you can take the set of all real valued functions on X or you can take the set of all complex valued functions on X this is these are algebras, okay.

So you see you take the set of all maps from the space to the real line these are just maps theoretic maps I am not assuming anything continuity nothing, okay. So just maps from X to the real line or just you take the maps on X to the complex plane, okay. So just real valued functions or complex valued functions, okay now that is a these are algebras, algebras means that they are so for example we will take the set of all real valued functions that is a vector space it is a real vector space because you can add two functions and you can multiply a function by a constant real constant it is a real vector space.

And if if you think about it it is also an algebra it is a commutative ring namely there is a multiplication in it which is commutative and that is just multiplication of real valued functions if you multiply two real valued functions point wise you get again another real valued function so this is a ring it is a commutative ring which has a vector space structure and the vector space structure is compatible with the ring structure multiplication distributes over addition in the right sense, okay and so on and therefore this is a nice ring it is a ring plus a vector space such a thing is called an algebra.

So I have mentioned this before in one of the earlier lectures, so this is set of all maps from X to R is a real algebra set of all maps from X to C is a complex algebra, okay and the point is that if you are looking at among these maps if you look at only bounded maps okay you look at maps whose images are bounded, okay in the image I can talk about boundedness because the image is going to lie either in R or in C if you take a if you take a real valued function the image is going to be a subset of R, if you take a complex valued function the image is going to be a subset of C, okay.

So the images make sense and subsets of R and C therefore boundedness of the image make sense therefore see what I can do is I can look at so let me write this if I take the bounded maps from X to R or if I take the bounded maps from X to C what will happen is that this is this is a subset of this, okay in all these upto this point you know I do not even need X to be a topological space X could have even be a non-empty set I am even the topology on X I am not going to use I have not used okay because I will worry about the topology on X if for example I am looking at something connected with maps which are connected with the topology namely continuous maps, okay but I am just looking at maps.

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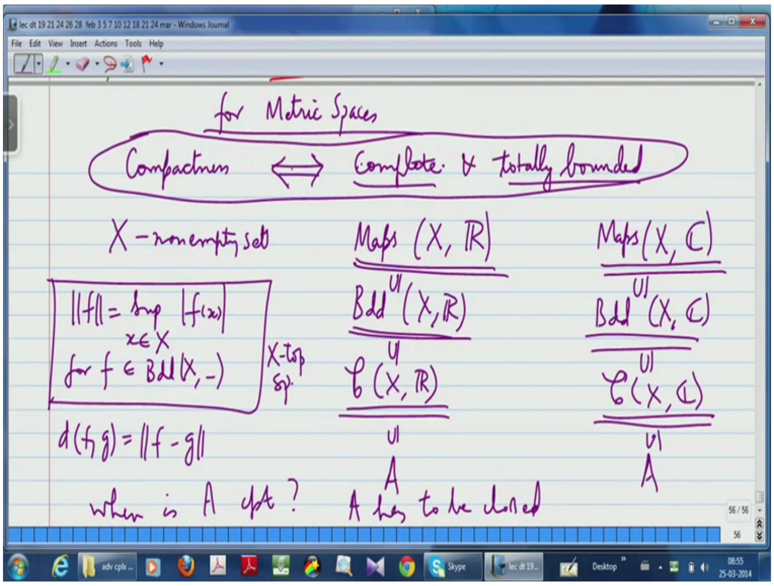


So X could need not even be a you know topological space in fact I can just say let me I can just rub this and say X is just a non-empty set. So I get these two algebras and I get the subsets of bounded maps, okay. Now these bounded maps also if you if you check they also form sub algebras okay because the sum of two bounded maps is bounded, product of two bounded maps is bounded and so on.

So these will give you sub algebras and the beautiful thing is that the boundedness allows you to define a norm the so called supremum norm, okay. So what you can do is that you can define norm f to be supremum over x small x in capital X mod f x for for f for f a bounded map, okay I can define this supremum norm alright and the point is that this is a norm once it is a norm it is a norm on a vector space so it becomes a norm linear space and once you have a norm linear space the norm induces a distance function alright so you get a metric space, you get the metric induced by a norm and once you have a once you have this metric you have a topology induced by the metric.

So these becomes very nice topological spaces in fact they become Banach spaces, okay they become Banach algebras they become complete norm linear spaces and the completeness is just because of the completeness of the target it is because of completeness of real line completeness of complex numbers, okay. So what you get is that you get these two even for a non-empty set you get this two beautiful Banach spaces Banach algebras alright one is a real Banach algebra the other is a complex Banach algebra depending on whether you are considering real valued functions or complex valued functions.

Now what you can do is, now you put one more so so let me write that the distance between f and g is is just norm of f minus g I can I can define this is the metric defined by the norm, alright and now we are in a nice metric space norm okay and now what you can now what happens is that you know if X is topological space then you can go one step down and then look at the set of all continuous functions from continuous bounded functions from X to R or you can look at the set of all continuous bounded functions from X to C, okay.

Then what happens is that this you can check that the set of all continuous functions that subset is a closed subset. The reason is because if you take a sequence of if you take a sequence of continuous functions if it converges to a limit function the convergence here will correspond to uniform convergence because of the sup norm and you know a continuous a uniform limit of continuous functions is continuous therefore what it means is that if you take a sequence of continuous bounded continuous functions, okay and if it if it is a Cauchy sequence okay then you know it will converge in the whole space because the whole space is ofcourse complete mind you it is a Banach algebra the whole space is complete but the limit will also be continuous because it is a uniform limit and why it is a uniform limit is because of the sup norm if you check, okay.

Therefore the space of all continuous real valued functions bounded real valued functions C X R or the space of all continuous bounded complex valued functions C X C that is a close it is a closed subset and you know a closed subset of a complete space is again complete therefore this C X R and C X C are very beautiful Banach algebras also they are Banach sub algebras closed sub algebras, okay.

And now the point is suppose you are so now you know you are coming to now slowly you know we have come into discussing spaces of functions, okay now you know I would like to do topology on a subset of of this so I would like to look at a space or a family or a subset or a collection of let us say continuous real valued functions and on that set considering as a set

A subset A of C X R okay I wanted two topology on that and what kind of topology I am interested in compactness.

So the question is suppose I start with an A here or here the question is when is A compact this is my question you see because my aim is what my aim is to study compactness of spaces of functions you know and mind you let me again keep reminding you so that you do not get lost our final aim is to study compactness of a family of Meromorphic functions, okay so I am trying to do it in the simplest case in the case of just say topological the topological case of this continuous functions continuous real valued bounded functions, okay.

Now when is A compact? If you see by whatever I told you because A is anyway a subset of a metric space, alright since A is subset of a metric space to check A is compact I can check ofcourse I can check the usual definition of compactness that every open cover has a finite sub cover that is highly in practical, okay then I can check sequential compactness I can check that you know every every sequence in A has a convergence subsequence, right that is another thing that I can check.

The third thing is I can check that A has a Bolzano-Weierstrass property because these are all the characterizations of compactness on a metric space okay and you know of all the three one possible thing that I can check for A is that it is sequentially compact, okay I can so I can check that if you give me any sequence in A you give me a sequence of functions in A if I can check that there is a convergent subsequence then also I will get compactness, okay.

So that is something that is I have but mind you if I have to check that I the sequence in A converges to it has a convergence subsequence mind you the convergence is now uniform convergence because you are under the sup norm I have to check uniform convergence, okay that is what it means, alright. So that is one thing that I can do, what other things can I do? I just now told you that for metric spaces as I have written above compactness is a same as completeness and total boundedness.

So I can check that A is complete and totally bounded but you know mind you I told you a compact subspace of a metric space is always closed and bounded okay the converse may not be true okay as I have told you in the case of R infinity okay. So if you expect A to be compact A should atleast be closed that is a necessary condition okay so that is always that A has to be closed this this you cannot avoid so you must have closeness alright.

And what this means is that therefore if you check that there is a if you so if you want to check sequential compactness in A it is enough to check that every sequence as a Cauchy sequence that is enough because we will check it has a Cauchy sequence then it means it has a convergent subsequence and the limit will also be in A because it is closed, right. So A has to be closed you cannot avoid that and what does this thing that I have written on top tell you to check that it is compact I have to ofcourse it has to be closed I have to check that it is complete and totally bounded, okay.

And the point is completeness is automatic, why because A is already a closed subspace of a complete metric space so it is automatically complete. So the only thing I have to check is totally boundedness total boundedness.

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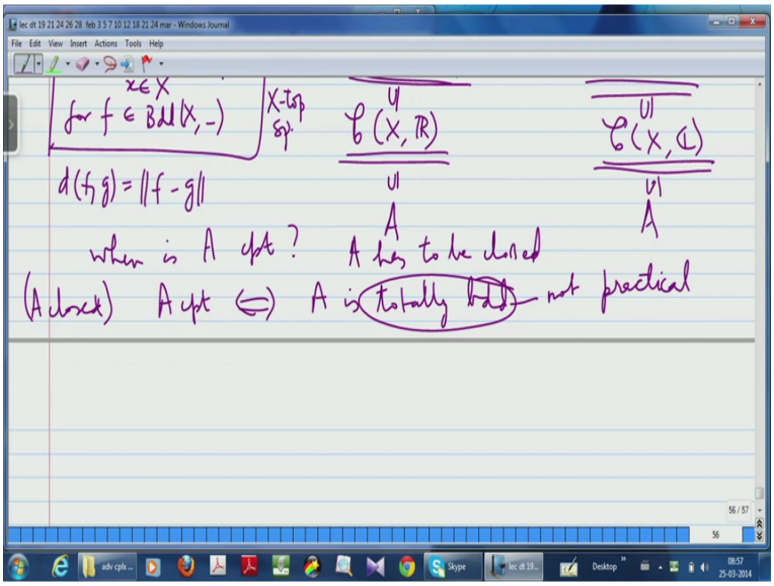

So so A is compact so let me write it here A compact if and only if A is totally bounded okay this is what we get this is what we get mind you ofcourse A is closed that is already already assume A is closed because if it is not closed you cannot expect compactness okay. So we have ended up at this point we have ended up at this point where you are saying so finally what does all this translate? Give me a bunch of functions real valued continuous bounded functions on a space alright, how do I check that as topologically it is compact how do I check it is compact?

So this tells me check it is totally bounded, okay. Now what does that mean? It means that you have to find an epsilon net for every epsilon what does that mean it means given an epsilon positive I have to find finitely many functions from this family such that the distance of any other function from atleast one of these functions these finitely many functions is less than epsilon that is what epsilon net means, okay.

So I have to pick given an epsilon greater than 0 I have to find finitely many functions from A such that the epsilon open open balls at those centred at those finitely many functions with radius epsilon that covers A that is what an epsilon net for A means okay and you see this is also very abstract you see it is very very abstract it is if you if I start with an abstract family of functions usually families of functions are abstract because you know they will depend on some property I might have functions which have some differentiable property or some some they may be defined by some abstract property and from an abstract collection it is not so easy to pick out finitely many it may not be so easy to do it.

So this total boundedness checking this total boundedness for a family of function does not work, okay so what comes to help us here is something that can really be checked for families of functions and that is the Arzela-Ascoli Theorem.

So let me write that down so so this is this so let me write this this is not practical this is not practical by that I mean this is not easily verifiable in practise okay you cannot demonstrate it in practise so easily, okay so what comes to our help is so called Arzela-Ascoli Theorem. So what is that so let me write that down.

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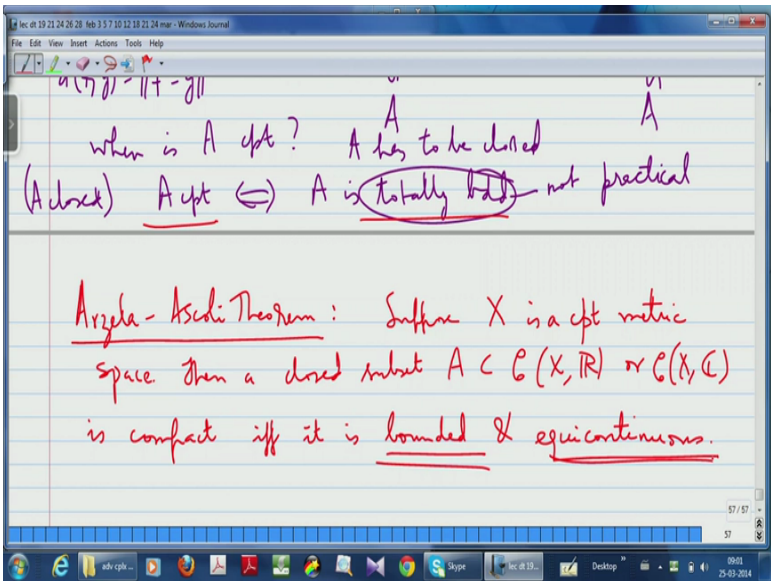


So here is the Arzela-Ascoli Theorem and what is this theorem and what does this theorem say it says that suppose X is compact X is a compact metric space you assume you are working on a compact metric space okay the advantage of working on a compact metric space is that automatically all continuous functions have bounded okay any continuous function on a compact any continuous real valued function on a compact sets is you know it is bounded it attains it is bounded uniformly continuous you know all these things.

So if you are working with a compact space then you do not have to restrict to bounded continuous functions every continuous every continuous function is automatically bounded, okay so you work with a compact metric space okay then a closed subset A of C X, R or C X, C is compact if and only if it is bounded and equicontinous, okay. So this is this is (())(37:46) this is a much more easier thing to verifying principle.

So what you do see what we had above is that to check A is compact we have to check it is totally bounded and totally bounded is not practical but checking its bounded that is more practical see checking the checking a family of functions or a collection of functions is bounded is a very easy thing because you have to check that there is a bound for all there is a common bound for all the functions and mind you it is a uniform it is it will be a uniform or common bound because you are the metric you are working with is induced by the norm and the norm is a sup norm.

So when you say it is bounded you mean bounded in the sup norm and that means it is uniformly bounded. So that means you must find a single positive constant such that mod f x is always less than or equal to that positive constant for all f in A and all X in X you must find uniform bound that is why this is sometimes called a uniform boundedness principle, okay Arzela-Ascoli Theorem is sometimes called the uniform boundedness principle.

So what you do instead of checking set of functions is totally bounded what you do you just check that there is uniform bound find the bound for all the functions in your family which will work at once for all the functions in a family that is one thing that you have to check, okay that is much more practically easy easier and the other thing is you have to check this so called equicontinuity and what is this equicontinuity? The equicontinuity is a very very simple thing what it says is that no matter what points you chose okay the moment you decrease the distance between points then the distance between the function values will decrease no matter what function you chose in your family it will work for all.

So you know usual definition of continuity is given an epsilon you find delta, okay now that epsilon given an epsilon the delta will depend on the point at which you are checking continuity and it will also depend on the epsilon but what you want is you want given an epsilon, you want a delta which works for any two points which works for any point and for any function at the same time in your family that is equicontinuity, given an epsilon you find delta which works for every function in your family and for all points in one go that is equicontinuity.

And this is also something it is a property of continuous functions so it can be checked unlike total boundedness where you have to pick up some finitely many functions explicitly which is not so easy, okay that is why the Arzela-Ascoli Theorem is a very useful a tool to check compactness of a family of functions, okay and this is what you get from topology. Now what we will do in the next lecture is that I will tell you we will try to understand how you can translate this to our situation where we are working with analytic functions and Meromorphic functions you have to translate this, okay so I will stop here.