Advance Complex Analysis-Part 2
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Lecture 26
Topological Preliminaries - Translating Compactness into Boundedness

So what we are supposed to worry about is you know see we are worried about compactness of families of meromorphic functions, okay. So so basically you are trying to to topology on collection on meromorphic functions and see this is the technical background that is required to prove the Picard theorems and many others theorems in fact because the root is through so called Montel's theorem, okay.

So you know what I wanted to do is I want you I want to go back to some topology and tell you about compactness, okay so that you realize how whatever we are going to do is connected with all this we will have to bring into the discussion Arzela-Ascoli theorem and then Montel's theorem, okay and then we will you see so let me say the following thing you know what we have done so far is the following.

We have defined a spherical derivative, alright. So first of all so let me sum up what we have done so far, we have first we have tried to think of a Meromorphic function as a continuous function even at a pole, okay that is because we allowed the value infinity and so we are not only look at looking at complex valued functions we are looking at functions with values in the extended complex plane.

So we allow the value infinity the advantage of allowing the value infinity is that a Meromorphic function at a pole can be given the value infinity and it becomes a continuous map it becomes a continuous map when you consider it as a map into the extended complex plane which is identified with the Riemann sphere, okay you know it is a complete compact matrix space, alright.

Now so first we have to deal with the point at infinity, okay so we try to think of infinity as a isolated singularity when is infinity an essential singularity, when is infinity a removable singularity, when is infinity a pole, okay all these things we discussed behaviour at infinity, okay and then value of the function at infinity that also we have we worried about, okay. So you allow in principle you allow functions not only to take the value infinity but you also want to study functions at infinity, okay so the you see these are two different concepts in the in the in the in the co-domain of the function usually we are interested only complex functions but now you allow the value infinity the advantage is that you can make a Meromorphic function in a continuous map even at a pole, okay.

Then not only that in the domain normally the domain of the function is usually a domain in the complex plane but then you also want to study the function at infinity itself. So you want to put infinity also in the domain, okay so you have to define you have to understand the behaviour of the function at infinity, okay. So a function may have a pole at infinity, it may go to infinity at infinity which is the case for example if you take polynomials one constant polynomials they all have poles at infinity.

So you want to be able to work in this kind of generality that is the reason why we have to study the function behaviour at infinity thing of infinity as a isolated singularity and classify that kind of singularity and we also want infinity to be a value taken by the function. For example the value of a Meromorphic function at a pole, okay so we had to deal with infinity that was the first thing.

Then the second thing is is we were worried about this defining spherical derivative, okay we were concerned about defining spherical derivative and see the important thing about the spherical derivative is that the spherical derivative will not change you can first of all you can define it for a Meromorphic function, okay. So it is a derivative that will work even at a pole. See if you take a Meromorphic function by which by definition is a function which is which has only pole singularities, okay of course it may be completely holomorphic, may be completely analytic but we are interested in in the situations we are going to really encounter those in which they are actually poles, okay.

So if you look at Meromorphic functions (())(5:20) Meromorphic functions you take a pole at the pole it is not differentiable because after all you know at the pole the function goes to infinity and it is not differentiable because it is a singular point it is not a removable singularity it is a pole, the function is not differentiable in the usual sense of the term, okay and the function value is also not defined in the usual sense of the term but what we do is we define the function value at the pole to be infinity that is an extra definition we make and then since you cannot (def) you cannot differentiate the function at a pole.

So what you do is you do this (())(5:55) of differentiating not with respect to the usual metric which is Euclidean metric but you try to differentiate with respect to the spherical metric so you introduce what is called the spherical derivative, okay so that gives you a derivative of a function which will work even at a pole you see that is the advantage, okay.

If I take a Meromorphic function at a pole I cannot differentiate it but if I take the spherical derivative the spherical derivative will exist and I have told you that spherical derivative we calculated it last time I think it was 2 2 divided by the modulus of the residue at the simple pole if it is a simple pole and it is 0 if it is if it is not a simple pole if it is a pole of higher order.

So even the spherical derivative make sense and on top of all this one more beautiful thing about the spherical derivative is that the spherical derivative will not change if you change the function by its reciprocal that is if you take the Meromorphic function f and calculate the spherical derivative we will get the same thing if you took 1 by f, okay mind you which is also Meromorphic with only the only thing is that the poles and zeros will get interchanged when you move from f to 1 by f but for 1 by f also if we calculate the spherical derivative you will again get the same thing as a spherical derivative of f.

So what it tells you is it if you are studying the spherical derivative you can actually apply the thrift analytic functions and stop worrying about even poles because at a pole of f I can simply if I am working with the spherical derivative in the neighbourhood of a pole of f it is a same as a spherical derivative in a neighbourhood of that point for 1 by f but for 1 by f that point is 0, okay and therefore it is analytic 1 by f becomes analytic at that point that is the advantage.

So working with a spherical derivative allows you to reduce to analytic functions, okay you do not have to even worry about poles that is an advantage and the other thing is it gives you a derivative of that works even at poles, okay. Now so this is the this is this is what we have done so far. Now why did we do all this we did do the idea is that there are two concepts on the one hand we are worried about compactness of a family of Meromorphic functions that is our main aim we want to do topology on a collection of Meromorphic functions on the space of Meromorphic functions, we want to what kind of topology of course topology means there are many things right there is connectedness connectedness, compactness so on and so forth but we are interested in compactness, okay.

And so that is on that is on the one end on the other end what we have is this spherical derivative that is that is what we have which is close to a derivative in the case of a in the case of a Meromorphic function, okay. So now I need to I need to tell you people how I need to tell you people how to connect these two things, okay so we need to do some topology. So I will give you some topological background.

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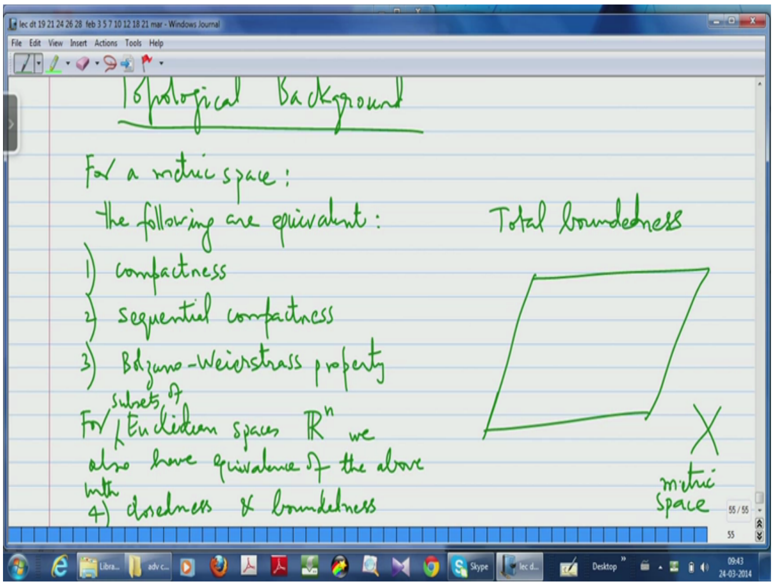

So topological background so this is very very important because only then you will understand what is going on, okay in the broad sense what are we trying to do, okay. So if you want to get an idea of that this is very very important. So so what we will do is we start with let us say let us say you are working with a metric space suppose you are working with a metric space, okay mind you the topology I am worried about the topological property that I am worried about is compactness, okay.

So we will try to do try to understand everything connected with compactness, right so start with a metric space which is a simplest kind of topological space that you can think off which naturally occurs, okay then what do you have? The following are equivalent is a compactness, number 2 is sequential compactness and number 3 is the so called Bolzano-Weierstrass property, okay.

So so we have these three these three properties are equivalent, okay so I am just trying to recall what is equivalent to compactness, okay just it helps to translate a property in different ways to find out equivalent properties so that you can work with them, okay. So compactness is so this is a this is something that you should have done in the first course in topology, what is compactness? Compactness is that every open cover admits a finite sub cover, okay that is when you are are given a collection of open sets whose union is a full space then it is enough to pick only finitely many among those collections among in that collection whose union is also the whole space you can extract a finite sub cover from every open cover that is compactness, okay.

It is a very it is you see it is defined only in terms of open sets and it is a very general thing so it works for any topological space compactness space can be defined for any topological space because for any topological space open sets make sense, okay defining the collection of open sets is exactly the what giving a topology is, okay so compactness make sense for any topological space but it is a very abstract notion at least for metric spaces where the topology is induced by a metric, okay that means that you know your open sets are defined to be unions of open balls and open balls are they are the analog of open balls in euclidean space you take points of the space and then you take all points which whose distance from the given fixed end point is less than some positive number which you call the radius of the open ball, okay.

And of course you say strictly less than because if you put less than or equal to then you are also include the boundary and it will not remain an open set it will become a close set, okay so you put strictly less than the distance should be strictly less than some positivities, okay and if I said is called open if it is union of such open balls and this is how you and you know this involves the notion of distance that is why the metric in the space is used.

So the metric space the metric induces a topology so when we say metric space and you think of it as a topological space you always mean the topology induced by the metric, okay the open sets are precisely those which are given by union of open balls and open balls are defined by the metric alright. Now for such a metric space compactness which is a very abstract thing is connected with what is is equivalent to sequential compactness, what is sequential compactness it has to do with sequences what it says is that you give me any sequence of points in the space I can always find a convergent subsequence that is what sequential compactness is, okay.

If you give me a sequence in the space the sequence itself may not converge but at the worst you can pick out a subsequence which converges, okay that is sequential compactness and that is equivalent to compactness is what this basic result says. And then there is a third property which is called the Bolzano-Weierstrass property what is Bolzano-Weierstrass property? It is just a property which is satisfied by the euclidean spaces which you namely which you would have come across namely the fact that you take any infinite subsets it has an accumulation point or a limit point, okay given any infinite subset all there is a cluster point there is a point where there is a point at the space such that if you take any open neighbourhood of that point and delete that point there is a point of your infinite subset there, okay.

So points of your infinite subset come closer and closer and closer to atleast one point of the space and that point is a limit point of that set, okay. Now that every infinite subset has a limit point is Bolzano-Weierstrass property and that is also equivalent the space having this property is also is also compact okay so all these three are three different avatars of compactness, okay alright sequential compactness and then Bolzano-Weierstrass property, okay.

And well if you are looking at euclidean spaces okay that is R n n dimensional real spaces finite dimensional real spaces then what happens is that this is also equivalent to if you look at a subset of euclidean space, compactness is equivalent to closeness and boundedness put together, okay and that is what we most of the time when you are working in R n n dimensional real space we keep using that all the time. Whenever you want to say something is compact you say it is you just verify that it is closed end bounded.

For example if you take the close disk in the complex plane that is close disk in the complex plane is compact because it is disk of finite radius so it is bounded and it is closed so it is both closed and bounded so it is compact so we keep using this all the time, okay.

So let me write that down for for euclidean spaces R to the n we also we also have equivalence of the above with with 4 so this is for if fact I should say for subsets of for subsets of euclidean spaces. So the subset should be closed and boundedness, okay. So if something is closed and bounded is compact and conversely, okay. So mind you you know you know my bag what is the background of our trying to understand all this the background of our trying to understand all this is you want to do this for functions for space of functions you want to do this for space of functions.

For a space of functions if you take a space of functions it will be a subset of all functions of the given type. So for example if you take a space of continuous functions, real valued functions it will be a subset of space of all continuous if you want continuous bounded real valued functions, okay or you might be looking at a space of analytic functions or you might be looking at a space of Meromorphic functions that is the that is the background in which that is the generality in which you want to do all this and you want to make sense of compactness for such a set of functions.

So usually we use various sometimes we say family of functions if you want to specify an index set, or sometimes we say sequence of functions if you want to think of sequence of elements which each is a function or you take a subset of the space of all functions okay so you refer to it in different ways but then basically you are looking at a subset of functions and you want to study compactness for that, okay.

Now now you see the question is ofcourse that you know how do you how do you go from this to something else. So there is a very important there is a very very important property and that is called total boundedness, okay there is something called total boundedness, okay. Now what is this total boundedness? It is a very very strong form of boundedness it is a very very strong form of boundedness.

So what is this total boundedness so I will try to explain to you so basically what happens is that you know you have some space x okay and let us assume that x is x is a say metric space. Suppose x is a metric space there is something so the idea of total boundedness is like is to you know fill out the whole space by finitely many open disks of a fixed radius, okay no matter how small that radius may be that is the idea.

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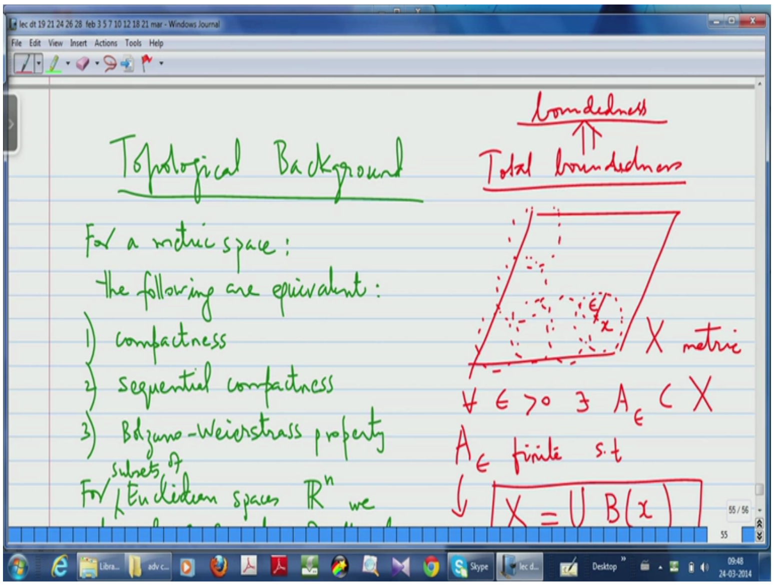

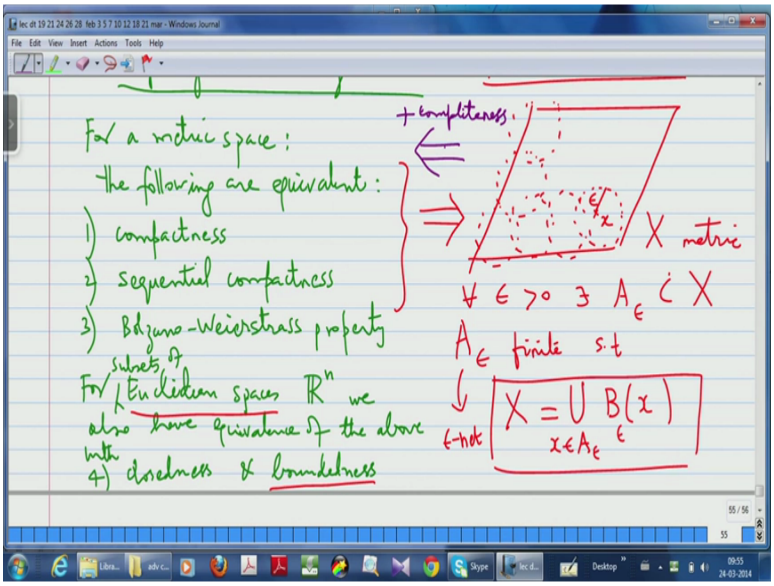

So total boundedness so here is my space x it is a metric space, okay and then for every epsilon positive no matter how small it is there exist a subset A epsilon subset of x A epsilon and this is the point is this is a finite set so it is only a finite collection of points A epsilon finite, okay such that you see the union if you take the union of all the if I take the union of all the open balls centred at points x i of A epsilon and take radius epsilon and I do this for i equal to i so you know in fact let me not put a subscript let me get rid of the subscript and just put x belongs to A epsilon. So when I say x belongs to A epsilon there are only finitely many such x because A epsilon is finite.

And for each such x i so you know so here is one x here and then I have this this ball centred at x this open ball centred at x and radius epsilon, okay and I do this for all the points of A epsilon I take the open ball centred at each of the points of A epsilon with radius epsilon, okay and if I take the union that should be equal to x that is the that is the requirement. So I can cover x by finitely many such open balls and the beautiful thing is that the all these balls have the same radius epsilon, okay and there are only finitely many of them they cover all of x, okay.

And this must happen for every positive epsilon this should happen for every epsilon (()) (22:10) if it happens for a particular epsilon such a collection of points finitely many points see epsilon is called an epsilon net, okay so this is called an epsilon net so this is called an epsilon net and this is the net condition, okay. Now this is this this see you are saying that no matter how small an epsilon you take I can make sure I can find I can make sure to find only finitely many points in x such that every other point of x is at a distance less than epsilon from atleast one of these balls that is what you are saying, right so let me repeat it what is this epsilon net condition? Given an epsilon no matter how small, okay you are able to find finitely many points that they will constitute their elements of the set A epsilon such that given any point of x it is distance from atleast one of these points is less than epsilon that way you cover every point of x, okay.

It is a very very strong point and you know the point is that this is this is a very strong form of boundedness because this implies boundedness because you see why does this imply boundedness if you see you know so so let me say it inverts so let me put this here this implies rather let me write it above I will put it here this implies boundedness and why is that true?

See in fact what it will tell you is that you know it will tell you that diameter of x is comparable to the diameter of any of these A epsilons that is what it will tell you. See what is the diameter of a space a metric space the diameter is supremum of the lens between two of its points and you allow those two points to just vary so it is like trying to draw the longest line segment through that space if you want to think of it and measure the length of that ofcourse this longest may not exist so it might become infinite so your space may have infinite diameter. So that is the reason instead of taking maximum you take supremum.

So basically what you do is you take supremum of the distances between two points of your space and you allow the points to vary, okay. If that has a finite value that is called the diameter of your space and the point is if your space is totally bounded then its diameter can be compared to any epsilon net. So for example you know if you take an epsilon net such as A epsilon, okay and you measure the distance between two points of the space, what you can do is that each of these points is within an epsilon from one of the points in the net and the distance between two points and the net cannot exceed the diameter of A epsilon mind you A epsilon is only a finite set so it has a finite diameter the finite subset always has a finite diameter because you are just going to take supremum of the finitely many distances between pairs of points in that set and that is only finitely many pairs, okay.

So the diameter of any finite subset is of course finite, alright and and you know if you look at the diameter of A epsilon okay that will be an upper bond for the distance between any two points of A epsilon, okay. Now if you take any two points of the space for each point I can find a point of A epsilon which is to within an epsilon so what this comparison will tell you by triangle inequality is that the diameter of the space cannot exceed the diameter of A epsilon plus 2 times epsilon if you write it out, alright if you use a triangle inequality the diameter of the space cannot exceed the diameter of A epsilon plus two epsilon for every epsilon greater than 0 that will tell you that the space has finite diameter, okay.

So you can see it is a very very strong condition and ofcourse if the diameter is finite it means the space is bounded if the diameter of the space is finite that means if the diameter of the space is say lambda positive number lambda then the space is ofcourse bounded because you take any point in that space and take disk take an open ball of radius greater than lambda the whole space will be contained in that so it becomes bounded. So totally bound is very strong it implies boundedness, alright.

And in fact actually for euclidean spaces you see boundedness is same as total boundedness, okay and in fact more more generally if you take a Banach space you take a complete norm linear space you take a Banach space even for a Banach space you see the fact that every subset that is bounded is also totally bounded is a very strong condition it will happen if and only if the Banach space is finite dimensional, okay it cannot happen in infinite dimensional Banach space, okay.

So if you go to infinite dimensional spaces okay which is like non-euclidean kind of spaces then you are in trouble, okay there is a difference between total boundedness and boundedness, okay but total boundedness A priory is a very very strong condition, right. So for example you know if you take R infinity infinite sequences of you know infinite sequences of real numbers.

Then what will happen is that if you take the unit ball there that is ofcourse you know bounded but it is not totally bounded because if you take the diagonal sequence which consist of 0 everywhere 1 in the ith place for i equal to 1, 2, 3, 4 is called the diagonal sequence, okay then that sequence will never have a convergence subsequence because distance between any two points of that sequence is finite quantity.

So it is a finite positive quantity it is a constant, okay and therefore you cannot have a convergence subsequence because if there is a convergence subsequence then distance between points should come closer and closer but this does not happen all distance between any two points in that sequence is equal to some fixed positive quantity, okay. So if you take R infinity the unit ball is bounded but this is certainly not totally bounded, okay.

And what I am trying to say here is basically a theorem in fact what I am trying to say is that you know if you have compactness which I have written on the left side in its various avatars in its various avatars I have written compactness sequential compactness. See all these things they all implied total boundedness, okay compactness or sequential compactness or Bolzano-Weierstrass property they are imply total boundedness ofcourse what I wrote below is that you know they they all imply for euclidean spaces they all imply closeness and boundedness, okay but it is not just compactness in general gives you very strong thing it gives you total boundedness.

Now the question is how do you come back from total boundedness how to you come back to compactness, okay and the answer to that is theorem if you want to come back this side what you need to do is you will have to put the condition that your space is complete, okay. So with completeness so let me let me try to use a different color so that you understand the implication that is involved with completeness. So if I go like this plus completeness. If I take a metric space that is totally bounded and I add completeness to it, okay completeness is the condition that every Cauchy sequence convergences, okay I will put this completeness condition then you will get compactness, okay this is a so you know in so what I am trying to tell you is see we are trying to move from compactness which is a very abstract thing to something that is related to boundedness, okay.

And why we are doing this is because when you are studying functions or spaces of function it is easier to verify something is bounded if you want to say a function is bounded that is easy to verify, okay where is if I want to say that a collection of functions is compact it is very very abstract, okay. So boundedness is something that for functions it is easy to verify under under many situations.

So that is why we are trying to move from compactness to boundedness and this is the route compactness implies boundedness for example in euclidean space, okay and if fact it is equivalent to closeness and boundedness but if you forget euclidean spaces, compactness gives you total boundedness which is a very strong form of boundedness but from total boundedness if you want to come back to get compactness you need to complete this.

So the translation so far is we so basic topology teaches us that you can translate from compactness to completeness plus total boundedness, okay. Now what I need to do is that I will have to now translate all this to functions spaces of functions, okay and that is where what we will come across is the so called Arzela Ascoli theorem and then so what we will do is there we will try to see how to decide a certain collection of functions is compact, okay.

So you will you can expect that you know you will say the condition that will be (())(32:31) total boundedness and completeness but completeness you will get if the collection is already a close subset because a close subset of a complete space is complete. So if you are working for example with the Banach space of real valued functions or complex valued bounded continuous functions, okay then any subset any close subset that any close subset will automatically be complete.

So the only thing that is required for it to be compact by what I just said is that it should be totally bounded, okay but then from total boundedness you want to even remove the totalness and come down to boundedness that is where you have to bring in the Arzela Ascoli theorem, okay. So I will explain that in the next lecture.