Advanced Complex Analysis-Part 2.
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Lecture-25.
Well-definedness of the Spherical derivative of meromorphic function at a pole and Inversion-Invariance of the Spherical derivative.

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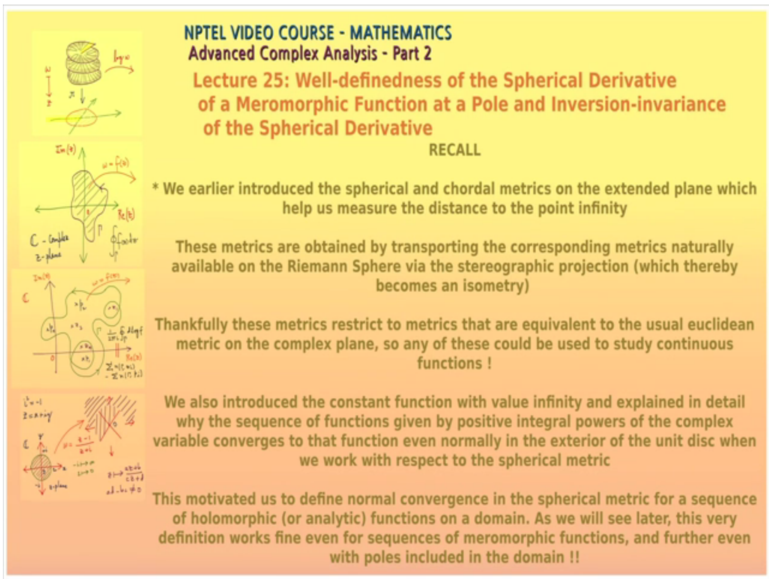
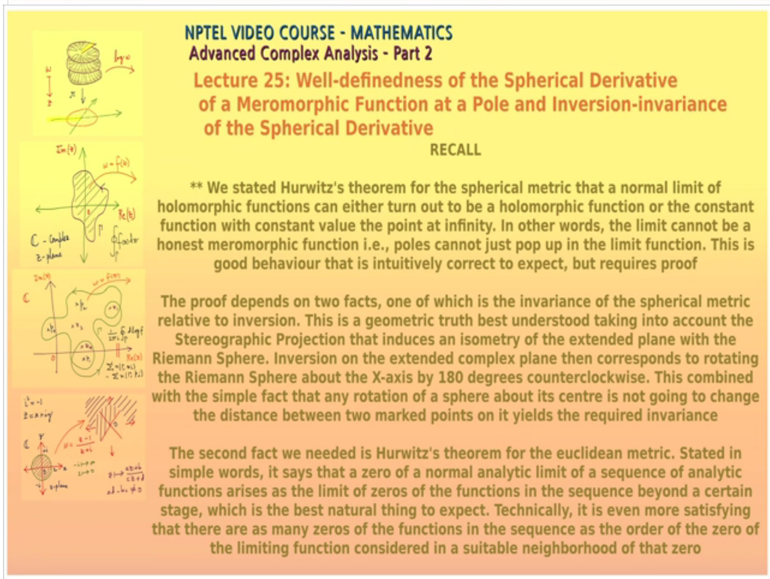
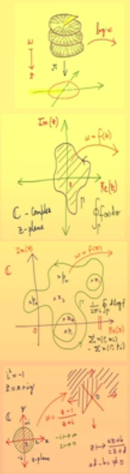






NPTEL VIDEO COURSE - MATHEMATICS

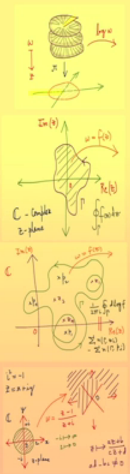
Advanced Complex Analysis - Part 2

Lecture 25: Well-definedness of the Spherical Derivative of a Meromorphic Function at a Pole and Inversion-invariance of the Spherical Derivative

RECALL

*** We gave an introductory discussion on Hurwitz's theorem for the euclidean metric and gave a brief sketch of its proof. We recalled the Counting Principle or the Argument Principle which was needed in that proof. Then we gave the proof of Hurwitz's theorem for the spherical metric that a normal limit of holomorphic functions with respect to the spherical metric is either holomorphic or the constant function with value infinity

We finally extended the theorem to the case of a normal limit of meromorphic functions. These Hurwitz's theorems are important because they assert that singularities of normal limits cannot get worse and there is only one exceptional case -- when the limit is the constant function that is identically infinity



NPTEL VIDEO COURSE - MATHEMATICS

Advanced Complex Analysis - Part 2

Lecture 25: Well-definedness of the Spherical Derivative of a Meromorphic Function at a Pole and Inversion-invariance of the Spherical Derivative

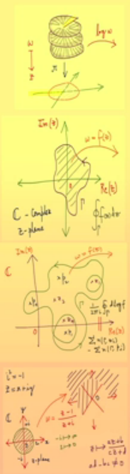
GOALS

*** * In the lecture before the previous lecture, we gave an example of a sequence of meromorphic functions converging normally under the spherical metric to infinity, and another example where the sequence converges normally to a holomorphic function

We next recalled the notion of infinitesimal distance or arc length in the euclidean metric and the integral formula for arc length. We explained how the modulus of the derivative acts as a scaling or magnification factor in the integral formula for the length of the image of a given curve. We then discussed these notions for the case of the spherical metric, namely we brought in the infinitesimal distance or arc length in the spherical metric and gave an integral formula for the spherical distance

That discussion gave us the clue to guessing what the derivative of a meromorphic function with respect to the spherical metric could possibly be. In the previous lecture, we followed that clue and defined the spherical derivative of a meromorphic function. We also indicated why the spherical derivative is valid even at a pole ! In the present lecture we expand on this and explain in full detail why the spherical derivative of a meromorphic function is well-defined even at a pole and why the spherical derivative is continous throughout the domain

We also explain how the invariance of the spherical metric with respect to inversion induces the invariance of the spherical derivative with respect to the inversion of the meromorphic function being spherically differentiated



NPTEL VIDEO COURSE - MATHEMATICS

Advanced Complex Analysis - Part 2

Lecture 25: Well-definedness of the Spherical Derivative of a Meromorphic Function at a Pole and Inversion-invariance of the Spherical Derivative

KEYWORDS & KEY PHRASES

one-point compactification, Riemann sphere, Stereographic Projection, complex plane as punctured sphere, meromorphic function, analytic except possibly for poles, set of poles is countable, convergence on compact subsets or normal convergence, constant function with value infinity, metrics on the complex plane and the extended complex plane, metrics at infinity, metrics on the Riemann Sphere, chordal metric, spherical metric, euclidean metric, distance to point at infinity, minor arcs of great circles are geodesics on the sphere, normal convergence in the spherical metric, integral formula for the spherical distance, infinitesimal distance or arc length in the euclidean metric, rectifiable arc or curve, integral for euclidean arc length, modulus of the derivative as a scaling or magnification factor in the integral formula for the length of the image curve, spherical derivative at a pole, invariance of the spherical metric under inversion implies invariance of the spherical derivative under inversion of the function being spherically differentiated

All right, so we continue with our discussion of the spherical derivative, okay. So there are few things i wanted to point out with regards to the spherical derivative, okay. So let me, so let me just recall.

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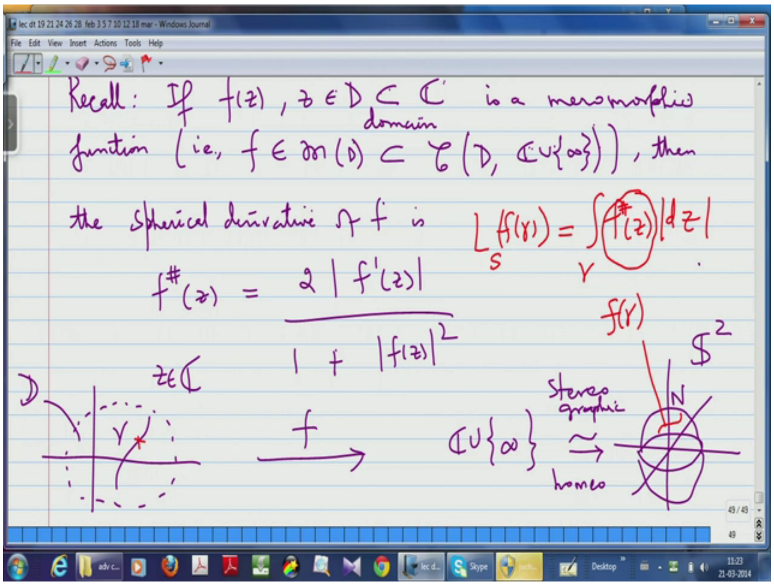

If f of z z in a domain d, z varying over capital d which is a domain in c is a meromorphic function, function, that is f belongs to m of d and mind you the set of meromorphic functions on d is consider now as a subset of, you know in fact continuous maps from d to the extended complex plane which is c union infinity, okay. So the script c denotes continuous maps and the point is that you make the meromorphic function continuous even at the poles by defining the function value at the pole to be infinity, okay. So you consider it like this and then the spherical derivative of f is f hash of z, it is defined to be 2 times more f dash of z divided by 1+ mod of fz also. This is the definition of the spherical derivative.

And mind you why, why did we need the spherical derivative is because of the following reason. So suppose the, you have this is the complex plane with the variable z and you have this, suppose this is your this, the area inside this dotted region is your domain d and suppose you had an arc gamma inside d, you take the image of this arc gamma under f in the external complex plane, okay. So it means that you know you are allowing also the value infinity, so for example you know gamma may pass through your pole f. F is a meromorphic function, so f is meromorphic on d means f is holomorphic, that is analytic on d, except for a subset of isolated points of d where f has poles.

But at the poles also, the value of f has defined to be infinity. So your gamma, your curve gamma pass through the poles and that is the technical thing that i want to explain to you about. Now you identify this external complex plane via the stereo graphic projection with the riemann sphere which i will briefly draw like this so this is riemann sphere, which is s2, okay. And this, this isomorphism is actually a homeomorphism given by the sphere graphic projection, this is the stereo graphic projection with the, with the point infinity going to the north pole which is this point here, all right.

And the fact is that the image of gamma will, see gamma will give you, you know, if you take the image of gamma, what will happen is that you will get some curve here on the riemann sphere, okay. So it is a curve in the external complex plane but you know, you are thinking of the external complex plane as the riemann sphere when you think of, you may imagine that the image of gamma is a curve on the riemann sphere itself, okay. And what is that curves, since this, this is just s of gamma, okay, this is f of gamma and what is the big deal about the spherical derivative, the big deal about the spherical derivative is that you can get the spherical length of f of gamma, okay.

You can calculate the length of that image curve, okay, and i have put subscript s for spherical lens because it is the length you are computing the arc length on the sphere, okay. And how do you get it? You get it in the following way, you simply integrate over gamma with the variable, see normally if you now integrate over modern dz, if you integrate over mod dz simply on the plane over a curve gamma, you simply get the arc length of the curve, okay. That is what integrating over mod dz means because mod dz is infinitesimal arc length on the euclidean plane, on c common complex plane, on the complex plane thought of as r2, okay.

It is usual arc length, but you know if you put, if instead of doing this, suppose i put, if i add the magnification factor given by the, suppose i add the magnification factor given by the spherical derivative. So that means i put f hash of z here and do this. Then what you will get is , i will get actually the length of the image curve on the riemann sphere, okay. And so this is where the spherical derivative is used, okay, the spherical derivative will give you, it is, so without this if i do not, see if i remove this spherical derivative factor, okay, i will get simply integral of us gamma mod dz and that is just length of gamma.

But if you put the spherical derivative there, okay, then i will not get the length of gamma but i will get the length of the image of gamma under f. And mind you gamma can pass through, it can pass through a pole, the only thing is it means that this image curve will pass through the north pole, that is all, it is not going to create any problems. Because if it passes through a pole, the function value there is infinity at infinity corresponds to the north pole on the riemann sphere under the stereo graphic projection, okay. So the point is important that the spherical derivative is that it gives you the spherical length, okay.

But there are, there are a few technical things about this , there are a few technical things about the spherical derivative which i just indicated towards the end of my last lecture and i want to be more you know elaborate about that. So you see, so i want to draw your attention to the, to this formula which is a formula for the spherical derivative, okay. This is the formula for the spherical derivative, there is something that is a little troublesome about this formula.

See when i have defined and you go before that, let me also tell you that here in this formula for length, spherical length of f of gamma, you know if i replace the spherical derivative, if i instead of putting f hash of z, suppose i put mod f dash of z, suppose i put modulus of the derivative of f, then what i will get is actually the, and i assume that f is you know the holomorphic function. Then i will get the image, the length of the image of gamma under f. Then f will map only into the complex plane, if f is differentiable, okay, everywhere on gamma, okay, then it is a meromorphic.

So the image of gamma will lie in the plane itself, it is not going to go to infinity, okay because there are no poles, okay. And because f is, if you assume f to be an analytic function, okay. And then if you integrate over gamma mod f dash of z into mod dz, what you will get is the length of the image under f but this will be the euclidean length, it will be just length on the plane. But if you integrate over gamma, mod dz with the coefficient f hash of z which is a spherical derivative and in addition you allow also f to be meromorphic, you will actually get the length of the image curve on the riemann sphere, part of the extended plane, that is what you understand.

Okay, fine, you know there is a problem with this at 1st sight with this definition of f hash because you see there is this f dash, okay. F dash is the derivative of f at the point z but the problem is that if z is a pole, then you are in trouble. At the pole the function is certainly not differentiable, it is a single point, it is a pucca similar point, it is an honest singular point, it is not a removable singularity, okay, the function is not differentiable. Alright. So you are in trouble, so when i wrote this definition last time, you know i was only, you know i was trying to heuristically tell you things but now i am going to tell you things more seriously, so let us worry about, let us worry about this, this, this situation.

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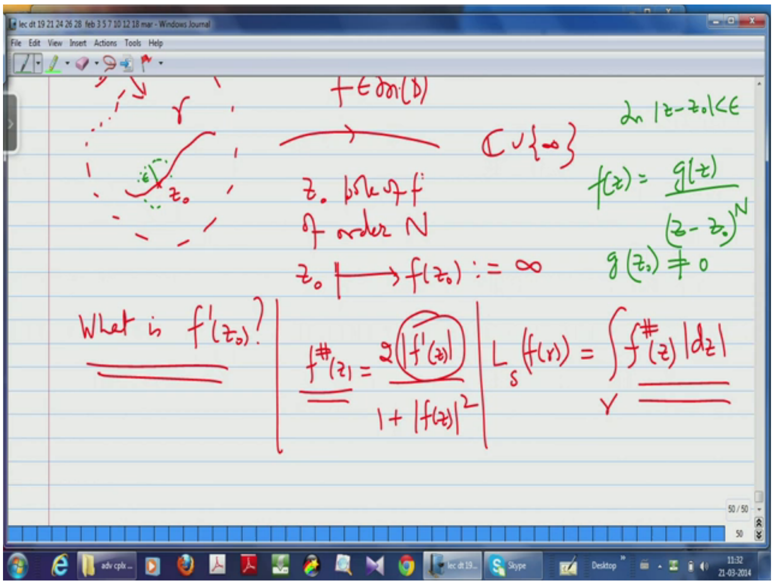

So here is my, here is my domain d which is interior of this dotted line, it is an open connected site, okay. This is inside the complex plane and suppose i have a point z0 and of course i have this map f, f is a meromorphic function of d and you know of course f is taking values in c union infinity and z0 is a pole of f, of order let say, of order n, okay. And of course z0 will go to f of z0, which is by definition infinity, okay, this is our definition. Now what about the spherical derivative? Okay.

See, what is, so the question is what this f dash of z. So this is an, this is an issue, you see because what is f dash of z. If we have to worry about this, the reason is because, see suppose i have a gamma, suppose i have a path gamma passing through a pole, okay, then my formula for the length of gamma, the spherical length of, the image of gamma under f on the riemann sphere which is identified in the extended plane, what is the formula, it is l spherical of f of gamma, you know that is what i have shown in the previous slide , the spherical length of f of gamma is integral, you integrate over gamma.

So i put, if i put just mod dz, i will get just the length of gamma but i have put the magnification factor given by the spherical derivative of f with respect to z. And what is a spherical derivative of f with respect to z? It is, well it is f hash of z, you can write it, it is 2 times mod f dash of z divided by 1+ more fz the whole square, this is what it is. It was there in the previous slide also. Okay, now the point is, if i put z equal to z0, z0, see now gamma passes through gamma passes through z0, okay.

So when i calculate this integral on the right side gamma i have, when you do, when you do and integration the variable of integration will lie on the region of integration. In this case region of integration is the path gamma. So z will pass through z0, it will bury, at some point z will become z0. But when z become z0, there is this integrand which is f dash, f hash of z, the spherical derivative, that is in trouble. Because you know f hash of z depends on f dash of z0 in the numerator. F hash of z0 will be, will involve f dash of z0 but f dash of z0 does not make sense.

Why, because z0 is a pole, i cannot differentiate at a pole, i just cannot find derivative at a pole. So what is it, what is the big deal? So there is, so you see this formula as we have written it last time has this issue that has to be fixed. And the reason is because, the fact is that, as i was telling you last time, even at z0, this f dash of z0 is not defined but this spherical derivative is defined as it finite quantity. That is the beauty, that is the reason why this integral works, okay, that is what i want to explain to you. So you see, so let me, so let me say that, so you see, let us assume that z0 is a pole of order n.

So then what happens is that you know you will get a small disc surrounding z0, so let me use a different colour, see i will get a small disc surrounding z0, okay, i can find a small disc surrounding z0 where z0 is the only pole, okay. Because you know the poles of an analytic function are isolated in any case. And in fact our meromorphic functions are supposed to be having only pole singularities, okay. These are the only singularity that are allowed, okay. So i can find a small disc surrounding z0 where z0 is the only pole and well, you know, if you, if you call this, if you call the radius of this disc as say epsilon, okay, then mod z minus z0 less than epsilon, which is interior of that small disc.

You see you can write f of z, you can write f of that as you know g of z divided by z minus z0 to the power of n where g of , where g of z0 is not 0, okay. So you can, you can write it like this. And of course g analytic in mod z minus z0 less than epsilon, okay. So, okay, i will give a little more space, let me rewrite that. Let me just write g analytic. I can do this because this is how the function looks near a pole of order capital n, okay. Now, now watch carefully. Let us calculate the derivative of f, not at z0 because at z0 you cannot calculate the derivative.

But in the deleted neighbourhood of z0, let us calculate f of z. And you also when i, when i write this in mod z minus z0 less than epsilon, of course z should not be z0. I mean i cannot literally plug-in z equal to z0 because z equal to z0 is a for land z minus z0, denominator vanishes, i cannot write that. Of course we have agreed to put it as z equal to z0 and equate it to infinity, that is when you consider f to be a function with values in c union infinity but nevertheless if you think of it and the usual function, then you do not plug-in z equal to z0 because you do not divide by 0, okay.

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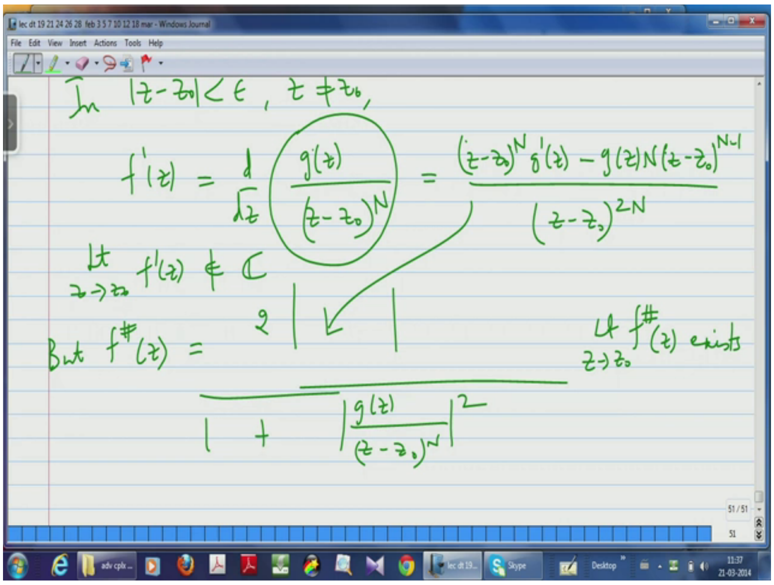

And that is the situation you must be, if you want, if you want really differentiate this, okay. So you see, so now let us calculate, let us do this calculation. See what is, so let me write this, in mod z minus z0 less than epsilon, z not equal to z0, what is f dash of z. F dash of z is just d by dz of f of z which is now g of z by z minus z0 to the power of n, okay. And you can calculate this by if you want quotient rule from basic calculus. What will be, i will get z minus z0 to the power of 2n and i will get, what will i get here, z minus z0 to the power of ng dash of z plus, minus g of z ,n z minus z0 to the power of n minus1, okay, this is what i get, if i do this computation.

And you see now what you do is, see this is all right for z not equal to z0, all right. And you will certainly have a problem if you let z tend to z0, okay. Limit, if you actually see in the usual sense if i let z tends to z0, then what will happen is that in the numerator the 1^{st} term will go, all right, the 2^{nd} term, well it will go, provided n is greater than 1, okay. If n equal to1, i will get gz0, okay but the problem will be the denominator. As z tends to z0, i will end up with, essentially i will, what will happen is that because f has a pole of order capital n at z0, its derivative will have a pole of capital, order capital n +1 at z0, okay.

That is what could happen, so it is going to only get worse, limit z tend to z0, f dash of z will not exist. And if, in fact the worst-case you want to make it exists is you can define it to be infinity by thinking of f dash also is a meromorphic function but now with values in c union infinity, you can do that. But in any case it is not a finite, it is not, it is not a proper limit in the usual sense, okay, you have to include the value infinity. But then, so i will say limit z tends to z0, f dash of z does not, it is not a complex number, okay. It is, if you, if you include c union infinity, then you can call it as infinity, that is that, but that is not the case.

We do not want to include the value infinity when we are talking about derivative. But, you see, what look at what on the other hand you look at what is f hash of z. Look at the spherical derivative, if you look at the spherical derivative, what i will guess, i will get, c will get 2 times modulus of f dash of z, okay, so i will get 2 times modulus of this whole quantity, okay, divided by 1+ mod f the whole square and mod f the whole square will be, mod f is the modulus of this quantity, so it will be 1+ modulus of g of z by z minus z0 to the power of n the whole square, this is what i will get.

Okay, this is what i will get and now you take limit z tends to z0. Now you take limit z tends to z0, of f hash of z you take, you calculate this limit. What will happen if you see the, in the denominator you have 1+ mod gz the whole squared divided by mod z minus z0 to the 2n, okay. So denominator will go through infinity, the denominator will go to infinity faster than the numerator, so this, so the whole quantity will be bounded as z tends to z0, that is the whole point. So you see if you calculate it, okay, if you calculate it, what will happen, so, so let, so this exists. So if you write it out, you know i am going to get, so i will, to simplify things i will multiply both numerator and denominator by z minus, mod of z minus z0 to 2n, which is what is the common denominator.

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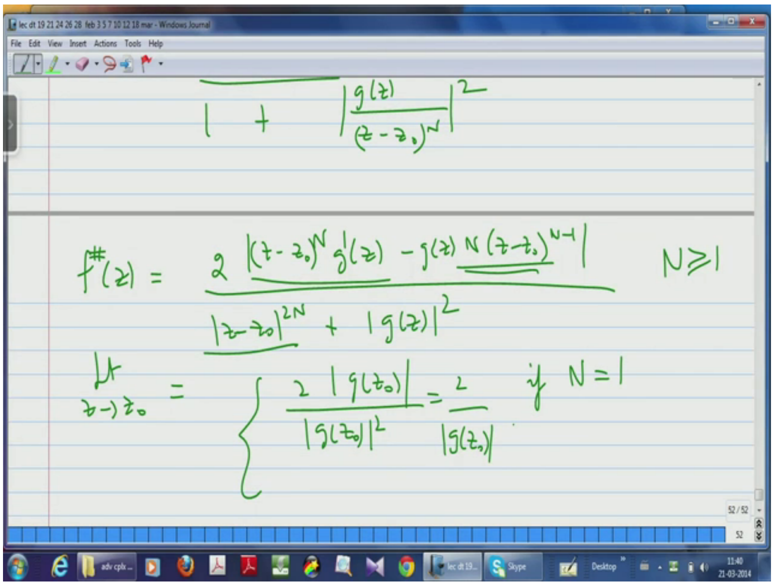

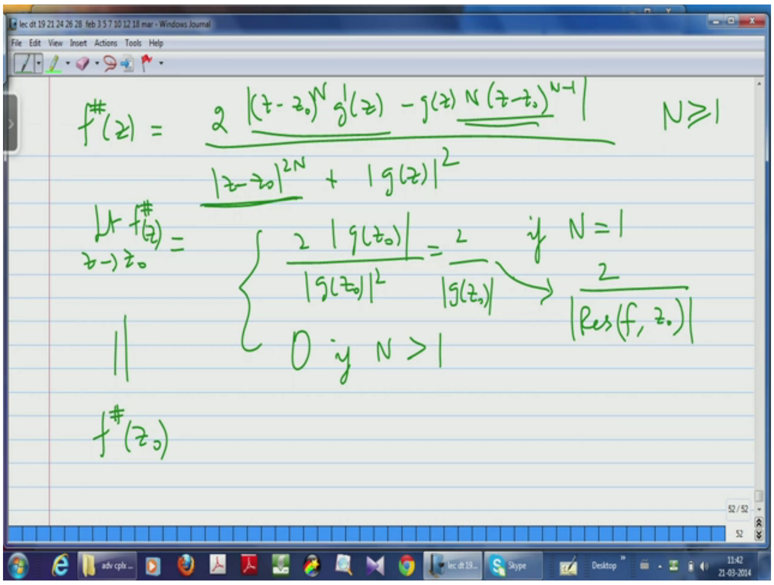

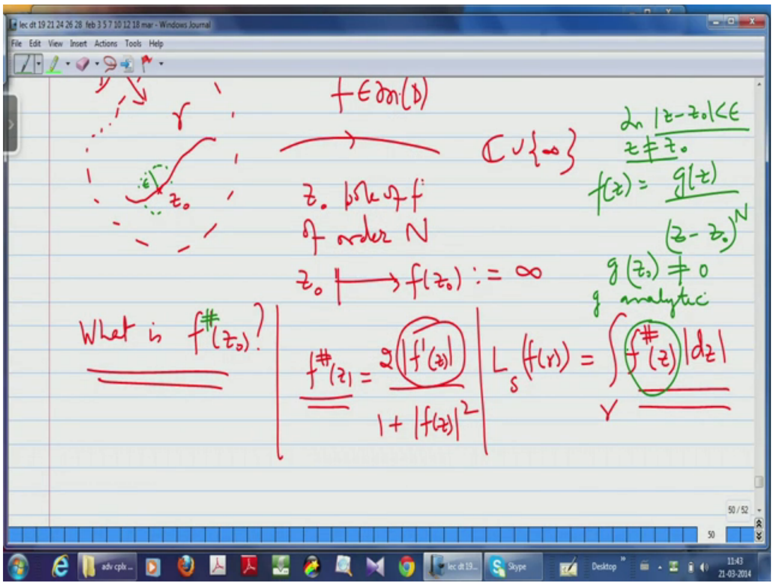

So what i will get is that i will get, well let me write it here, f dash of, sorry f hash of z is going to be 2 times, i will get the numerator of this, which is more z minus z0 to the n, so i will get this, mod z minus z0 to the power n gz minus gz, oops, i think that must have been, that is g dash of z minus g of zn into z minus z0 to the n minus1 mod divided by, okay, i have multiplied by, multiplied by this modulus of this quantity, mod z minus z0 to the 2n, okay. So that is gone, so in the denominator i will get more z minus z0 to the 2n plus mod gz the whole square.

This is what i will get if i multiply it by mod z minus z0 to the 2n, okay. And mind you the spherical derivative is an absolute derivative, so it is only absolute value, so it is a nonnegative real valued by the way. Now you, now you do, if you take limit at z tends to z0, what is going to happen. You see as z tends to z0, this term will vanish because z minus z0 power n is there and of course this n is of course greater than or equal to1. It is the order of the pole, so this whole of order 1 or higher, okay. So this is going to vanish and this fellow here, what will happen here, depends on whether n equal to1 or n is greater than 1, okay.

See, if, if n is equal to1, what is going to happen, if n is equal to1 then this term does not exist, okay. And let z tend to z0, i will get, i will get 2 times mod gz 0, g is anyway mind you in analytic, discontinuous, so z tends to z0 gz, g z0 and modulus also a continuous function, so i can push the limit inside the variable, okay, inside the argument of the function. And then that is what i said in the numerator, so this is, this is if n is 1, okay, this is if n is 1. And in the denominator what i am going to get, this term is going to vanish as z tends to z0, i am going to simply get mod g, again i will get mod gz0 the whole squared, i will get divided by mod gz 0 the whole square, which is just able to by mod g z0, this is what i will get.

And mind you gz0 is not 0 because g z0 is , g is the, you know if you want, g is analytic function divided by z minus z0 power n which is equal to f in the neighbourhood of f. In fact you know g z0 is, if you check very carefully, g z0 is, if the coefficient of the, of 1 by z minus z0 power n, if you write down the lagrange expansion, okay and that is not supposed to be 0, okay, because g, f has a pole of capital n, right. So this is what you will get.

And you see, and mind you in the case that n equal to1, g z0 is actually the coefficient of 1 by z minus z0 power n which is 1 by z minus z0. But you know what is the efficient of 1 by z minus z0 called, it is called the residue. So actually this is 2 divided by your residue of f at z0, that is what it is. This is nothing but 2 divided by modulus of residue of f at z0, this is what happens if you get, if f is a simple pole, capital n equal to1, all right. And the point is that, now if n is greater than 1, everything is gone, because you see is n is greater than 1, there is no following the denominator, i will get mod g the whole squared, this term is anyways going to vanish.

And the numerator will also go now, numerator has z minus z0 term common, so it is going to go. So i will get 0 if n is greater than 1. So here is the, so here is the, so of course this is on the, i forgot to write f hash of z. So here is a nice thing, f hash of z0, you can now call, see you can define f hash of z0 by continuity to be equal to limit z tends to z0 f hash of z, okay. If you think of f hash as a continuous function, okay, if you want to think of the spherical derivative as a continuous function, then it is natural define f hash at z0 to be the limit as z tends to z0 as f hash of z, okay.

And you see, this, what this does is that it makes the spherical derivative continuous even at z0 and mind you z0 is a pole. So what this tells you is that the spherical derivative f hash of z is continuous at all poles. So it is continuous throughout domain and therefore because it is continuous at all, throughout the domain, this formula is valid, okay. What i really meant here is, what is f hash of z0, okay. Of course f dash of z0 does not make sense, so the question is what is f hash of z0, all right. So , so now you know f hash makes sense even at poles, so this integral is well-defined, there is no issue.

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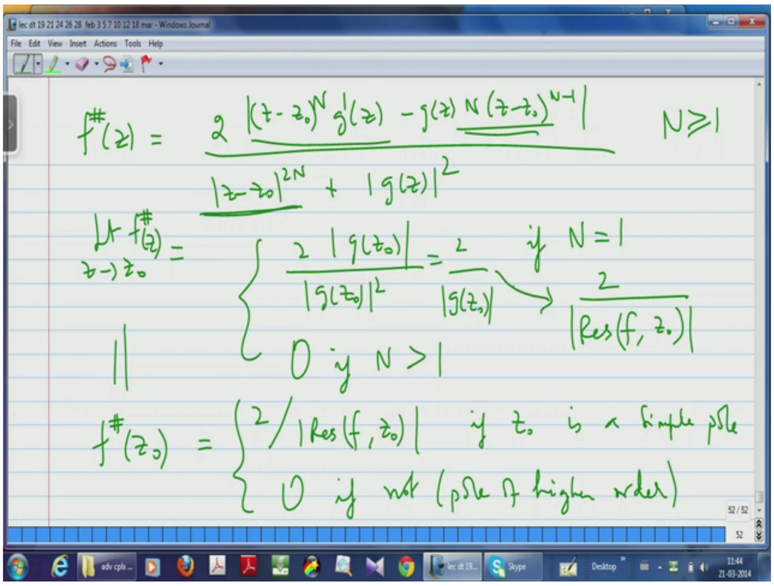

You can blindly integrate f hash, you can, you cannot blindly integrate f dash because f dash will not exist at a pole. You can integrate f dash, mod f dash only where so long as you are an apart which is not going through any poles. And if it is going through a pole, you cannot integrate f dash but you can integrate f hash always, even if you are passing through a pole, that is a big deal, that is a big deal. So that is the reason why this formula works. And what this calculation we did just tells you is that the spherical derivative is actually 2 divided by modulus of the residue of f at simple pole z0 if z0 is a simple pole and it is 0 if not, this is not means, i mean pole of higher-order, all right.

So the moral the story is that you know, you are in, you are in good shape. F hash spherical derivative is a very nice thing, okay. And therefore when you, whenever you want to find the arc length, you can integrate mod dz over, multiplied with, with the, you know integrand as f hash and that is pretty important. Now, and you know again, i will tell you why we are doing all this, we are doing all this because you know somehow the kind of analysis that is required to prove picard's theorem is involves montel's theorem, okay.

And this, i will tell you roughly the idea is that you know there are, there are these, there is a, there is a very close relationship as i told you between compactness and sequential compactness and equi-continuity and normal, normal convergence, okay and bounded mass of the derivatives, okay. So this is the, this is a bunch of results same analysis which is usually covered by the arzela ascoli theorem, okay. And that is a, there is a, the montel's theorem is something that comes out of that, okay. And why we are doing all this is because you know you, basically you know the idea is that you want to look at the space of meromorphic functions on a domain, okay.

So you have some domain, all right, this is a domain in extended plane, it could include infinity also, okay, on that domain you are looking as meromorphic functions, all right. And you are looking at, you want to think of them at least as continuous functions, so you are allowing the value infinity at a pole. So you are looking at that pace of meromorphic functions and you see the convergence that you are worried about is normal convergence, okay, it is not uniform convergence everywhere, it is only uniform convergence restricted to compact sets, which for normal convergence.

And with this convergence idea you want to study apology of this space of functions. And explicitly what kind of topology you want to study compactness, you want to study compactness, okay. You want to study compactness and you know if you have studied, for example in euclidean space, compactness is as good as sequential compactness which is the same as saying that you know every sequence, if you have an infinite sequence, there is always a convergence of sequence, all right. And so you have compactness is somehow strong related to sequential compactness, okay.

So basically you want, basically given, given a sequence you always want a convergence of sequence, all right. And now you want this also to happen for meromorphic functions, that is the, that is the central idea, the central idea is give me a bunch of, give me a sequence of meromorphic functions on a domain and now you try to find conditions, topological conditions, that will tell you, topological, of course includes analytics conditions. That will tell you that i always will be able to find from this sequence i find a sub sequence which converges.

But mind you now it is not just convergence, it is normal convergence, because in the context of complex analysis, in the context of holomorphic functions, uniform convergence will not work. You will get only uniform convergence on restricted subsets, namely only on compact subsets, that is called normal convergence, okay. So you have to worry about this and in order to do all this, see i need to, see the boundedness of the derivatives for example if something that is strongly related to all this. So i want to be able to work with derivatives. But the problem is that functions i am trying to work with are all what, they are meromorphic functions.

And meromorphic functions are not differentiable, at the poles they are not differentiable. So what do i will do at the poles with meromorphic functions? What i do is, the clever thing is i do not look at the ordinary derivative, i look at the spherical derivative. Spherical derivative makes sense even at pole, that is where spherical derivative comes in. Okay, that is what you offer understands all this is required for me to do analysis on a space of meromorphic functions. And that is a kind of, see that is the kind of you know analysis you have to do to prove the picard's theorem, okay.

Fine, so okay so what i want to do next is i want to tell you , well that this other fact that i was telling you last time that you know the spherical derivative has another important advantage. The spherical derivative allows you to, you know forget meromorphicity. It is a very clever trick, you see on the, you introduce a spherical derivative because you want to look at derivative of a function which is meromorphic at the pole for example, okay. That is why you introduce a spherical derivative but i, the, it is beautiful, once you introduce this notion, you can in most cases you can even forget the pole.

Which means you can reduce everything to just studying analytics functions. So that is the beauty, the reason is, the spherical derivative of meromorphic functions is the same as the spherical derivative of its reciprocal, okay. And what is the advantage of passing the reciprocal, passing to the reciprocal, the advantage is that a pole becomes is zero, okay. And a zero is a very nice thing, the function is differentiable there, all right, so that is the advantage. So that is a, that is an added advantage you get free, okay. And why is this true, this is true because the spherical length, that is invariant under inversion, okay.

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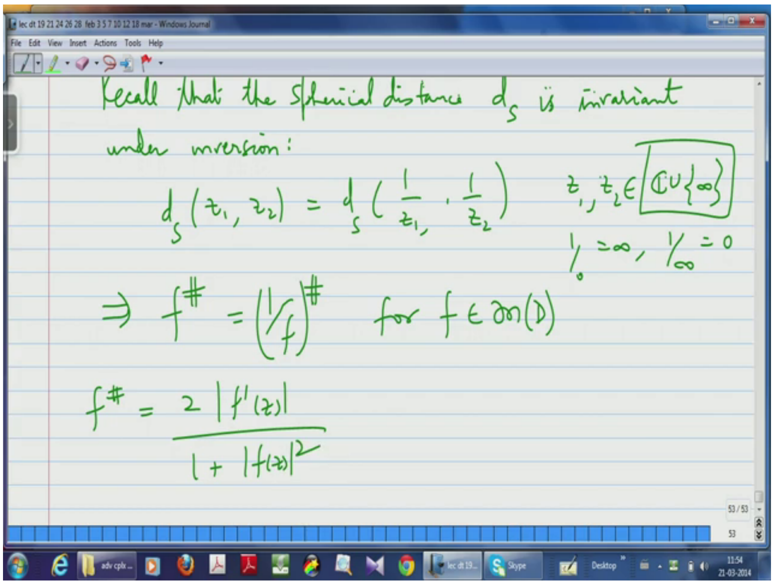

So that is what i want to tell you about next. So you see, so recall that the spherical distance d sub s is invariant under inversion, so you have the spherical distance between z1 and z2 is the same as the spherical distance between 1 by z1 and 1 by z2 where you know z1 and z2 are now taken to be in the external complex plane. And in the external complex plane mind you 1 by 0 is defined to be infinity and 1 by infinity is defined to be 0, this is the condition. So you know the spherical distance is the, is actually the spherical distance on the riemann sphere, okay which means the distance between 2 points on the riemann sphere is given by the length of the minor arc of the biggest, bigger circle that passes through those 2 points and which lies on the riemann sphere, okay.

And that length, how do you get it, and that length you get it by, for example, using basic analytic geometry, it is after all length of the arc of a circle and you can always derive's formula, okay. And that length you transport it via the stereo graphic projection to the extended complex plane. So the advantage of that is that i can measure for example the distance between the point in the complex plane and the point that infinity. Okay, i do, which i cannot do with the usual euclidean distance because you, euclidean distance becomes unbounded as the point, as one of the points is fixed and the other goes to infinity, all right.

So and i told you in an earlier lecture that you the mapping z going to 1 over z which is the inversion mapping, that is a map of c union infinity onto itself, it is a, it is a, in fact it is a homeomorphism and the fact is that under that homeomorphism , see what it does is that she has just maps the extended complex plane back to the extended complex plane, if exchanges 0 and infinity. Infinity goes to 0 and 0 goes to infinity, okay. But on the other hand since it is, it is itself homeomorphism of the extended complex plane, it will also induce itself homeomorphism of the riemann sphere because after all the riemann sphere is homeomorphism to the extended complex plane.

See whenever in mathematics whenever one object has an isomorphism and you take another isomorphic object, then an isomorphism on the 1st object will automatically induce an isomorphism on the 2nd object which is transported by this isomorphism between them. So the inversion will also induce an, a self isomorphism, a self isomorphism, a self homeomorphism of the, of the riemann sphere and what is it? It is nothing, i have asked you to check this, you should do it, i hope you have done it. So it is just rotation of the riemann sphere about the x axis about 180 degrees, that is all, that is what it is.

And you know if you if you take 2 points on a sphere, and you take the spherical distance between them, that arc distance, now if you rotate the sphere, that is not going to change, okay. So it is invariant under that rotation, all right. And therefore the net effect is that the spherical distance is invariant under inversion, okay. Now this implies that the spherical derivative of f is the same as the spherical derivative of 1 by f, okay. And so why is this, this is for f in meromorphic, f is a meromorphic function on d.

And why is this true? Because you see, i will tell you if you want you can try to do direct calculations, you can do a direct calculation. So you know the what is f hash f hash is just 2 mod f dash z by 1+ mod fz the whole square, this is what it is, all right. Now this is okay where f dash exists, this formula is correct where f dash exists. Let us assume that, for, for simplicity let us assume that f dash is not 0 at the point. Suppose derivative does not vanishes at the point, 1 by f also makes sense at that point, the usual derivative of 1 by f also makes sense at that point.

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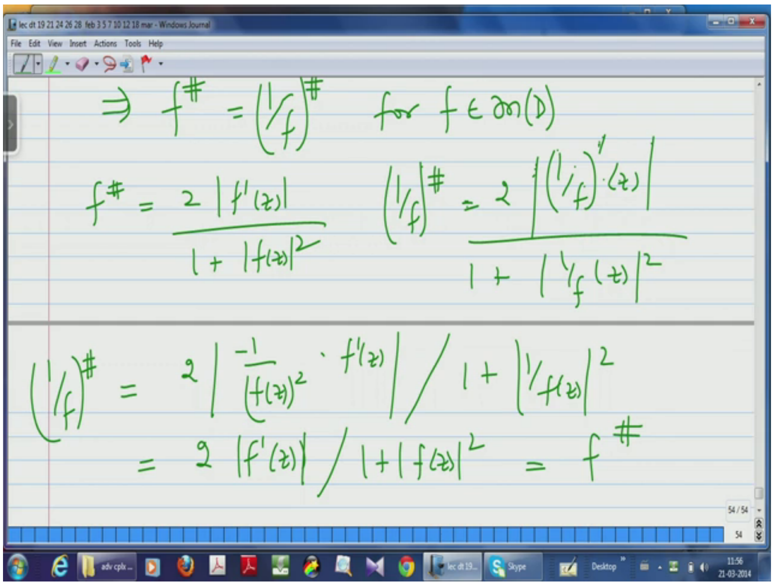

So if you calculate, if you calculate, it will, what it will be, it will be 2 times, you know i will get modulus of 1 by f dash of z divided by 1+ mod 1 by f of z the whole squared. Now if you calculate this what will i get, i will get is well you know 2 times, if you take the derivative of 1 by f, i am going to get minus1 by f squared into f dash, this is what i will get, okay, if i use a chain rule. And then i have to divide by 1+ 1 by f of z the whole square mod, okay. And now if my simplified, you see, i will again end up with 2 times mod f dash of z divided by 1+ mod z the whole square.

If i simply multiply numerator and denominator by mod fz the whole square, i will end up with this which is the same as f hash, okay. So this is a very heuristic, i mean it is a very simple calculation, the only thing, the only problem with this calculation is that you know, so i am, i am cancelling out mod f dash, i am cancelling out more f. And to cancel out mod f, f should not vanish, okay, otherwise i cannot cancel mod f in the numerator and denominator. So this is okay if f is, f is, f of z is, it is okay at the point where f is not 0, okay, so let me write that, valid if f is nonzero, that is one thing in the 2^{nd} thing is that f dash should exist, okay.

Derivative should exist, otherwise i cannot write, so f dash exists. So what i am saying is that the spherical derivative of f and the spherical derivative of 1 by f, they are the same, you can verify it at all points of f which is different from the zeros and poles, okay. Now what you do is, you check it at a pole using the same calculation that we did last time, all right and you will see the mobile last time i mean just some time ago, we did this calculation to calculate the f hash z0 at a pole z0, that is by basically writing out f locally at the point z0 which is a pole in this form, f is equal to g by z minus z0 to the n, all right.

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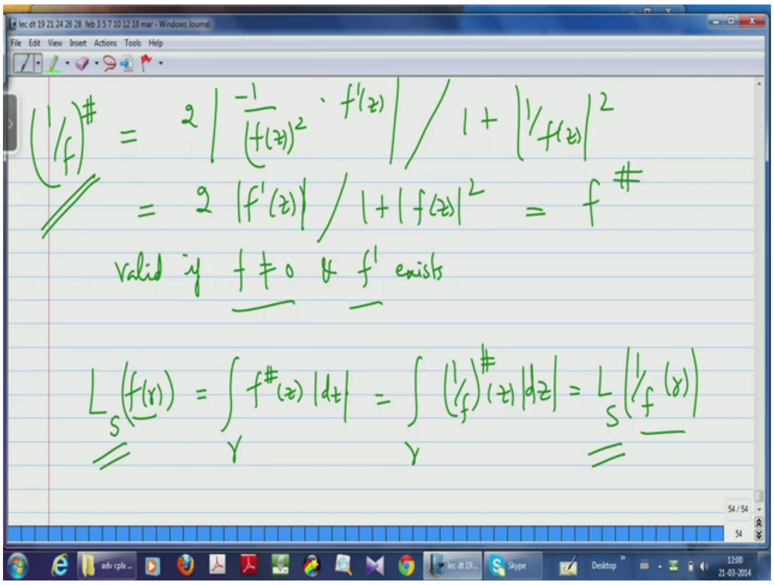

And you use the same calculation, if you use the same calculation, what will happen is that you can see that when f is 0 at a point, that is at the point where f has a zero or at a point where f has a pole, the same calculation is correct. What you do is you calculate in the deleted neighbourhood and then you let limit z tends to z0. You do it both at the pole and at t0 and you will see that the limit will exist and whether you calculate the limit for 1 by f or whether you calculate the limit for f, you will get the same thing. Okay, so that, that i leave it to you as an exercise, okay.

So one of the, one of the important things is that, see one of the important things is that you know if you take the spherical length of f of gamma, okay, this is going to be by definition integral over gamma f hash of z mod dz. And since f hash is 1 by f hash, this is also integral over gamma, 1 by f hash of z mod dz and this is by definition spherical length of 1 by f of gamma. And f of gamma and 1 by f of gamma, they only differ by an inversion and under the inversion, the spherical length should not change, so it is correct, okay.

So this way also you see that the, you know, you can, when you calculate the spherical derivative, whether you calculate for f or whether you calculate for 1 by f, there is no difference, okay. So what is the advantage, suppose you are proving something, involving spherical derivative and suppose you have to deal with a point which is a pole, okay. Suppose i have to deal with a function f at a point which is a pole and suppose i am working with the spherical derivative, without loss of generality i can replace f by 1 by f. Because by replacing f by 1 by f, my spherical derivative does not change but my pole for f becomes is 0 for 1 by f and 1 by f becomes analytic.

So i am dealing with a nice analytic function, okay. So that is the advantage of having the spherical derivative, okay. So now what we will do is in the forthcoming classes, we will use all this, all this background that we have developed so far do you know in a series of lemmas and propositions and finally theorems, we will prove the Picard's theorems, okay. And on the way we get the very important Montel's theorem, okay. So i will stop here.