Advanced Complex Analysis-Part 2.
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Lecture-24.
Defining the Spherical Derivative of Meromorphic function.

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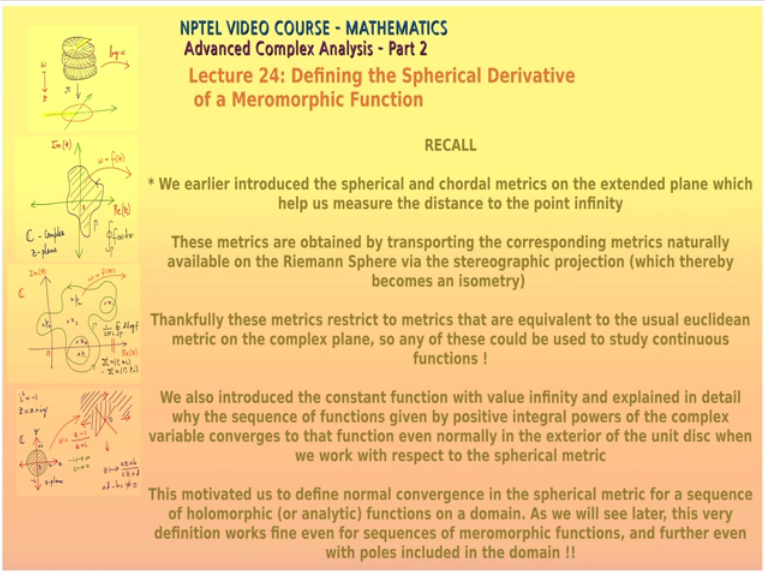

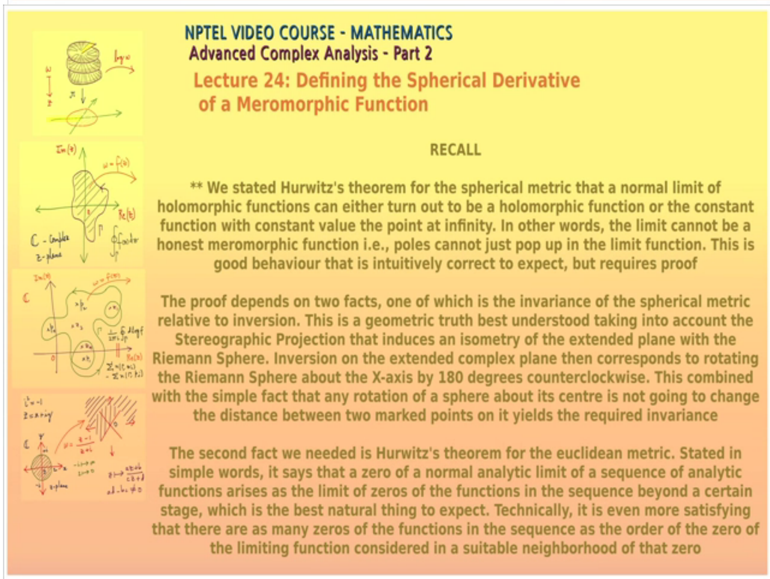

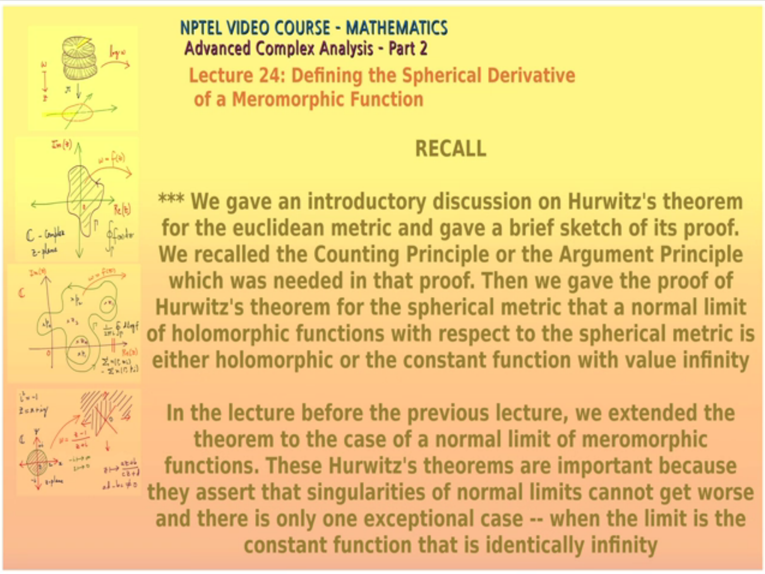

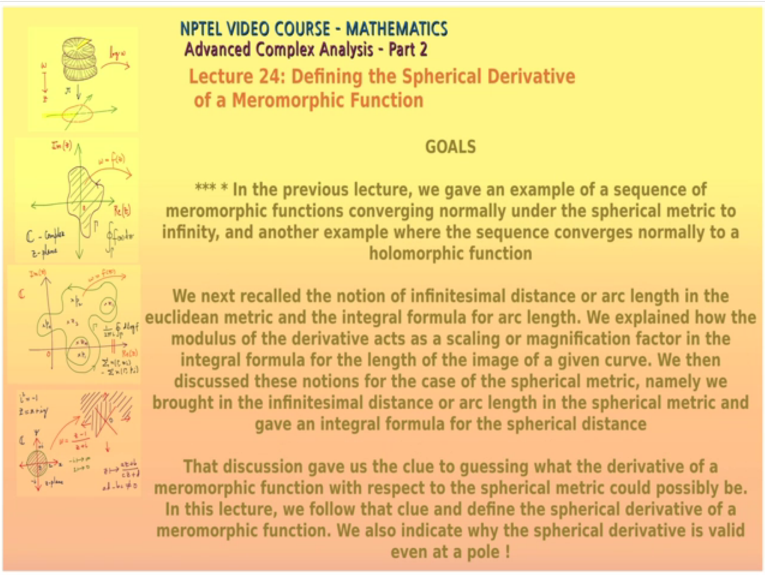

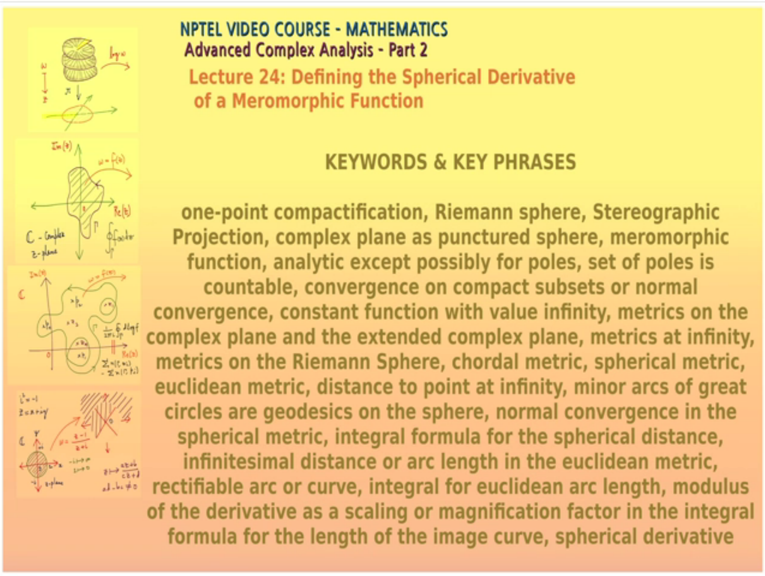

Alright, so what we are doing now is trying to understand what the spherical derivative of meromorphic function is, okay. So, well, you know the reason for all this is, this idea of spherical derivative is important to study the topology of families of meromorphic functions, okay. See the reason is that normally, you know there are, there is a relationship between, see basically we are interested in compactness, okay. And you know compactness is the same , is strongly related to sequential compactness. Okay, which is, given any sequence, you have, at least you are able to find the convergence of sequence, okay.

And of course if you are worrying about euclidean spaces, then of course compactness is the same as close and founded but then you know for saying things like bounded you need a metric and so on and so forth. But you know if you are working with spaces of holomorphic functions or analytics functions, the point is that you know, you will have to work only with normal convergence, you will not get uniform convergence, okay. You will get uniform convergence only on compact subsets. And then its, had it been only uniform convergence, then you could have taken the soup norm, okay and you could have used to define a metric.

But the point is that you do not have uniform convergence, you have only uniform convergence only restricted to compact subsets, that is called normal convergence. So it is not so easy to think of a metric, all right. But then you still want to think of compactness and compactness is kind of related to sequential compactness. And so then you know this is also connected with uniform bounded less, it is connected with each equicontinuity, okay, and it is also connected with, for example if you want boundedness of the derivatives, okay, so these are bunch of interrelated results, okay.

In the, on the topological side this is the so-called arzela ascoli theorem, okay. And on the holomorphic side or on the complex analytic side, it is the so-called montel theorem, okay, which we need to prove, okay. And, so you see that somehow we want to do it not only for analytic function, we want to do it for meromorphic functions. Because you see if we have to worry about meromorphic functions because that is the, these are the functions that you need to study families of such functions to get to the proof of the picard’s theorem which is what our primary aim is, okay.

Since you are worried about meromorphic functions, the problem is that you know they are not always different, they are not differentiable everywhere, i mean they are not analytic everywhere. If you go to a pole, at the pole of course the function goes to infinity, so you cannot, you cannot differentiate the function at the pole, it is not differentiable because it is a singular point basically, okay. So, for your usual derivative will not work, your usual derivative will not work at a pole. So what we will do? The method is that you introduce a spherical derivative because spherical derivative is something that will work even at own, that is the whole point, okay.

So i want to tell you in general while we are getting so worried about, why we are making so much noise about the spherical derivative, because see that is what we need, that is a thing that will work even for meromorphic functions, it will work even at poles, okay. Whereas ordinary derivative you cannot think of at the pole, because it is a, the moment you say the pole is a singular point and singular point derivative does not exist, then you know you are a lot of trouble. That is the reason we introduced spherical derivative. So i was, so let me continue with what i was telling you last time.

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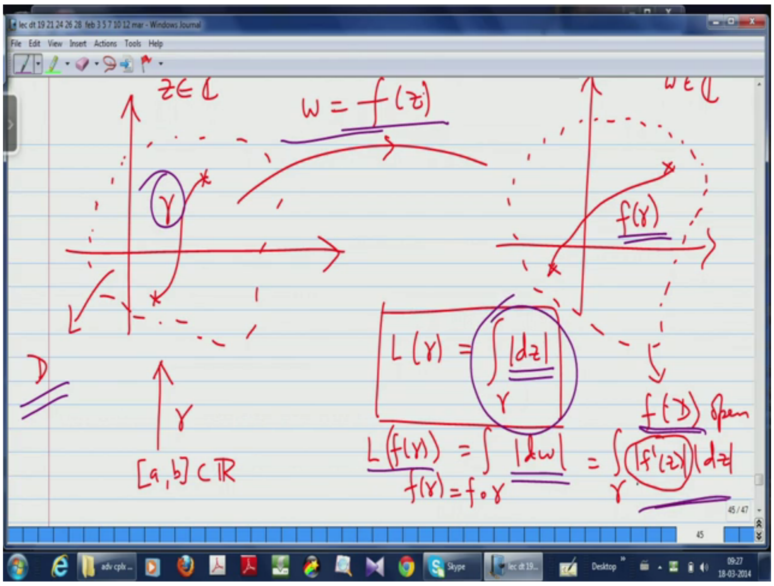

I was trying to tell you that the spherical derivative, if f is a, if small f is a meromorphic function, if you can see here, so let me, so let me use a different colour for the moment. So if you have this f which is a meromorphic function on the domain d in the plane, then there is a spherical derivative, okay. F hash of z, and in fact this is spherical derivative in absolute value, mind you i have put 2 times mod f dash of z by 1+ mod fz the whole square, so it is nonnegative real valued function, okay. Normally it derivative should be, derivative of a complex valued functions would again be a complex early functions but this is not exactly a derivative, this is complex valued, it is actually positive real, nonnegative real value.

So you should think of it as absolute value of the derivative, okay. So when we say that spherical derivative, i mean absolute, in absolute value, okay. And why is it that we are interested in this absolute value, because it is a scaling factor. You see, so you see what is happening, see the point is that, you know as i told you, as i was telling you last time, see if you look at this, if you take this function w equal to f of z, to the transformation from the z plane to that the blue plane, okay. Then you know, you assume it is, it is an analytic function, you can assume it is not constant.

So then what happens is that you know if it is nonconstant analytic function, then the image of any open set is open. So if i start with this open set d here, which is supposed to be for example in this diagram the interior of this dotted boundary, okay, then the image of that which is f of d is an open set. And if i take a curve gamma inside d, the image of gamma will be f of gamma and this f of r gamma is now going to be a curve in the image which is f of d which is open. And you know if i calculate the length of gamma, it is given by the formula, namely integrating mod dz, okay because mod dz is the infinitesimal version of the euclidean distance which is more z1 minus z2, okay.

So you take, if you take 2 points z1 and z2 on the complex plane, then the distance between them is given by mod z1 - z2. If these points are very close to each other, you can call one point as z1, you can call the next point as z1 plus delta z, then mod z1 minus z2 will become more than delta z and you replace delta by d to get the infinitesimal version, so you get mod dz. So mod dz is just the infinitesimal version, it is called, you may also call it the element of arc length which you have to use to integrate. And if you integrate the curve over the arc length, you will get the length of the curve, okay. And of course it is very important that the curves needs to be rectifiable, okay, it should be a curve which is finite length, okay.

And that is why we always put a condition that we work only with contours and contours are you know they are continuous images of closed bounded intervals which are in fact piecewise smooth and, in fact piecewise continuously differentiable, okay. And for such, for such curves, such contours, the length will always be a finite quantity and, so, so you have this. So the point is that you integrate mod dz, you will get the length of this gamma, all right. Now on the other hand what happens if you, if the, by the same philosophy if you integrate mod dw here over its image which is f of gamma, you should get the length of f of gamma. Which is the, f of gamma is the image of gamma under f, it is the image curve.

But you know, but here the variable of integration w is f of z, so if you integrate mod dw , i mean if you substitute for w f of z, then mod d fz will become mod f dash of z into dz, okay. And you see the difference between this formula here and this formula here, is that there is this x factor of mod f dash of z, okay. So what it means is that if you simply integrate over mod dz, you will get the length of the source curves, if integrate, if you multiply it by the modulus of the derivative, you get the length of the target class, they may serve. So the point is that the extra factor you have to put in the integrand to get the length of the image curve is the derivative, the modulus of the derivative, okay.

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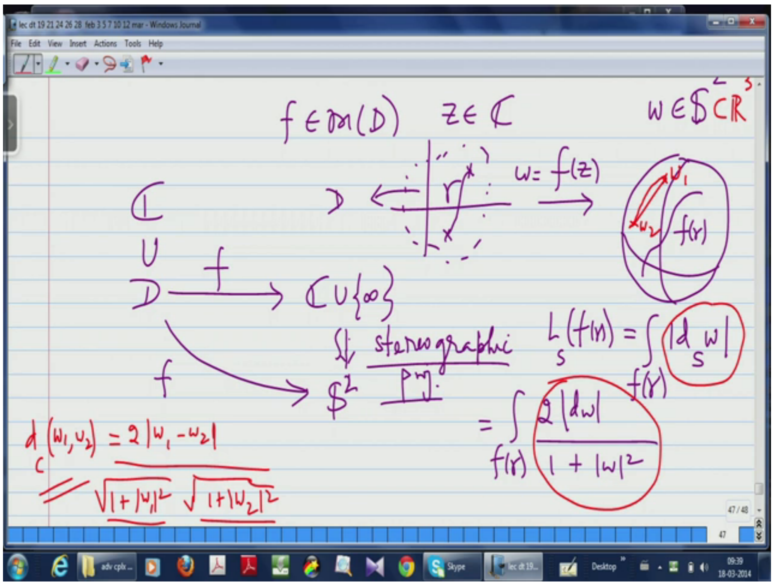

Now in the same way what has happened, going to happen if the following thing. If you take f to be, if you take the function f to be meromorphic function on d, okay, now what you are doing this, say you have the complex plane, and you have this domain d inside it. And you have this function f and this is the, i am now thinking of this function as a function into cu infinity. Mind you it is a meromorphic function, so i am thinking of it as a continuous function into the extended complex plane because i define the value at a pole to be infinity, okay, and with that it is continuous.

And how to actually think of this? So you see that is this, it that is this, this is identified via the stereo graphic projection to the riemann sphere, okay. Which, i think i should put this as 2, s2, 2 sphere, okay, centred at the origin in the 3 space, real 3 space radius 1 unit, okay and with the north pole being identified with the point at infinity, okay. Now you see what you do is you essentially think of this, look at this function. Okay, if you look at this function, think of the function as a map from the domain in the complex plane onto the sphere.

So geometrically whenever you are thinking of the riemann, whenever you are thinking of the extended plane, think of the riemann sphere dramatically, that is how you should think of, okay. So this function , think of function f as going into the riemann sphere. So if you want you know, in abuse of rotation i will still call this as f, in fact it is f followed by the stereo graphic projection. So in principle i should give it some other name but i will still call it as f because i am thinking of this as an identification, i am thinking of the stereo graphic projection as an identification, okay.

I am identifying the extended plane for all practical purposes with my riemann sphere. Now what happens, now what happens is that on the one hand you have the complex plane, and you have the domain of the complex plane, okay. So this is my domain in the complex plane and the variable here is n and 10 on the other hand i have this sphere, i have the riemann sphere, okay. And what is happening is that you know if i now take, if i now take a small, if i take the, if i take a curve gamma here, okay in the complex plane, in my domain and take its image under f, okay.

Then what will happen is that the image of this curve will also give me a curve here, on the riemann sphere. Now, so this will become the curve f of gamma and how will i guess the length of f of gamma? Okay, now think about what i just told you sometime ago, to get the length of image curve, you have to integrate over, you have to integrate over the original curve with the original, the original infinitesimal element of arc length and then you have to scale it by the modulus of the derivative, okay. But now you see what i am doing this i am actually trying to get the , i am trying to get the length of the image curve it is, the length i am getting is actually the spherical length.

See it is the length of the riemann sphere and the length of the riemann sphere corresponds to the length on the extended complex plane even by the spherical metric. See the only difference, the only point for you to remember is, in my target the metric is not euclidean metric, it is a spherical metric. I have to use a spherical metric, i have to use the element of the spherical metric and what i have to integrate. So what is the length of f of gamma under the medical metrci? So i put l sub s just to indicate that this is length of spherical metric.

What is this, this is just i have to integrate over f of gamma, okay, i have to integrate over f of gamma, have to integrate over modulus of dz sub s, i will keep putting this sub s just emphasise that i have to integrate over an element of the spherical arc length. But what is, what is the element of spherical arc length? You see, in fact i think i should not put, so it is important that my variable, let me call this variable as w. Okay and i should be careful inadvertently to make mistakes like this, see i am integrating over f of gamma, so my variable should be in f of gamma, variable of integration and that has to be not z, z is in the source.

Z is on gamma, that is w is what is on f of gamma, so you know this not have been z, i should correct this, this should be dw sub s. It is an element of arc length, spherical arc length with respect to the variable w on the sphere all right, where w is f of z. So this transformation is given by w equal to f of z, okay. And the only funny thing is that this w is now on the sphere, mind you w can take the value of the north pole, patches corresponding to w equal to infinity, that is also allowed now, okay, because we have allowed values in c union infinity extended plane, all right.

Now you see but what is this, what is this, what is this element of spherical arc length? I told you that last time that the element of spherical arc length is actually 2 times mod dw, the usual euclidean element of arc length divided by 1+ mod w the whole square. This is what the element of spherical arc length is. And the reason why you got this, is if you want as an aside, let me write that down. I will use a different colour, so you see, so what happens was that if you take the spherical distance between 2 points w1 and w2, okay, then that turned out to be 2 times more than w1 minus w2 by square root of mod 1+ w1 the whole square into square root of 1+ mod w2 the whole square.

This is the spherical distance between 2 points w1 and w2 on the on the extended plane or the riemann sphere, okay. This is a spherical distance, it is actually the, in fact i should not even say, this is a spherical distance, sorry, this is actually they, this is the cordal distance as such. Yah, this is still not the infinitesimal version, sorry. So this is the d sub c, this is the cordal distance. So you take 2 points w1 here and w2 and you join them by this cord, okay. It is a line segment, it is a line segment in 3 space, joining those 2 points on the riemann sphere.

But you see, minute those 2 points are actually points from the extended plane, i am simply identifying the extended plane with the riemann sphere. So i am still writing w1, w-2, where actually i mean the stereo graphic projection of, i mean the stereo graphic projection of w1 and the stereo graphic projection of w2. So if see this is the, this is the cord, this is the cordal distance, this is the cordal metric, and what is this metric? This metric is a metric in r3, minute this, this here, this riemann sphere is sitting inside r 3, this is inside r3, okay.

And in r3 i am simply measuring the distance, and i asked you to check that this is the, this is an exercise for you to check that, it is an exercise for you to check that this is the distance formula, okay. I asked you to do that, you should do it if you have not done it so far. Now you see, this is the, this is a cordal arc length. If i want the spherical, what is the spherical arc length? Spherical arc length is this, this arc length and what is that arc length? I take the great circle, that is only one big circle on the rim, on the sphere which passes through these 2 points.

And that circle, with these 2 points, 2 points on a circle determine a minor arc and a major arc, okay. And you take the length of the minor arc, that is the definition of spherical distance. So, so if i want the, if i want the spherical length, okay, and i want the infinitesimal spherical length, that is, that is what this quantity is. This dw sub s if the infinitesimal element of spherical length and for that what i will have to do is i will have to bring w1 and w-2 very very close, okay and as i bring w1 and w-2 very very close, the cordal distance will approximate the, it will come close to the spherical distance, okay.

So what i do is you know in this calculation, and this formula what you do is you put w2 is equal to w1 plus delta w, okay and then, and then you write it in such a way that you only allow, you only worry about dw and do not worry about delta w whole square, delta w whole cube, higher-order terms because you think of them as being very small and negligible. And then you change the delta w to dw, so what will happen is that will, this thing, this formula as w1 tends to w-2, okay, this formula will give you this formula.

So you see in the numerator instead of w-2 if you put w1 plus delta w, the numerator will become 2 delta w and this, both of these quantities will become equal to root of 1+ w square. So you will get the square of that which is 1+ mod w the whole square, okay. So each of these quantities will become square root of, so the 1st, the 2nd term will become square root of 1+ w1 plus delta w mod the whole square, okay. And as w tends to 0, you will get this quantity. So this is the, this is how you get this infinitesimal element of spherical arc length.

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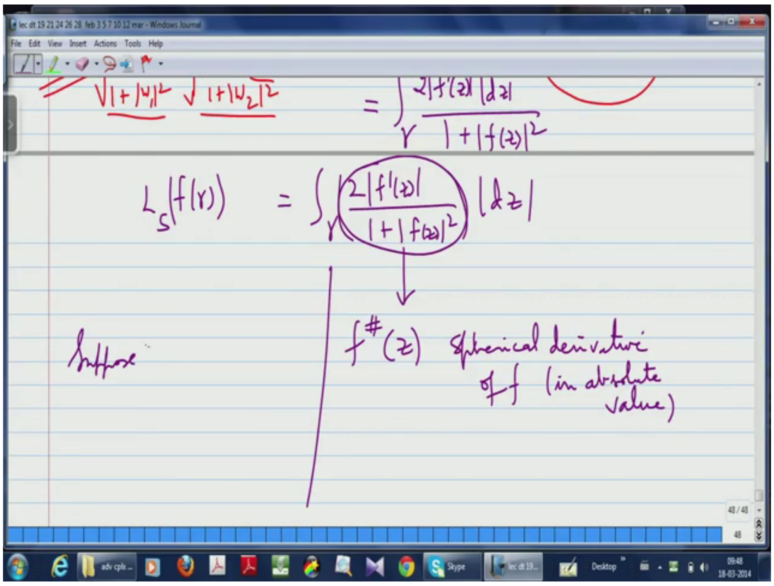

And integrating over that, integrating that over the curve f of omega should give you the length of f of omega, spherical length of f of omega, which is what we are interested in, okay. But then in this you plug-in what w is. Your w is fz, so plug fz inside that. If you plug fz inside that, what will get, you will end up, well, let me go back to the other colour that i was using. So what will i get it, i will get, well, i will get integral, since i have changed, i made a change of variable w equal to f of z, now the variable becomes z and z now varies over gamma, so i will put gamma here, okay.

And if i now calculate this mod dfz will become mod f dash of z into dz, so what i will get is able to mod f dash of z mod dz by 1+ mod f of z the whole square. And what is this, if you look at it carefully, this is just 2 times, sorry this is just integral over gamma, this is just integral over gamma 2 mod f dash of z divided by 1+ mod fz the whole square, this whole thing multiplied by mod dz. So what are you getting, you are getting the length of the image curve in the spherical metric of gamma under f is this formula. And go, you go back to what i was telling you some time ago, if you want the length of the image curve, you have to multiply by the modulus of the derivative, okay.

If you simply integrate, you will get the length of the source curve. But if you multiply by the modulus of derivative, you will get the length of the image curve. So you see what this tells you, this tells you that if you want the length, spherical length of the image curve f gamma under f, you will have to multiply by this, this quantity here. And that quantity therefore has to be the absolute value of the spherical derivative. Therefore the spherical derivative, so this is what we call as, so this is what we call as f hash z, this is called the spherical derivative of f, okay.

And mind you this is, this is i should write it in bracket, it is an absolute value, this is an absolute value, all right. Because it is, so you must remember, go back to your, your 1^{st} course in complex analysis, when you take f dash of z, okay. If you take a point z0, if you take f dash of z0 where suppose z0 is a point where function is analytic, so the derivative exists, you take f dash of z. What is the, what is the geometric significance of f dash of z0? See the modulus of f dash of z0 is the scaling fraction, it is locally the factor by which an image is scaled.

If you take some, if you take a small square containing z0, a very small square containing z0 and integrate its image under f, you will get something that looks like a square, okay, okay and this area will be, you know its length will be scaled by mod f dash. So the modulus with derivative is a scaling factor, the argument of derivative is a rotating factor, is a factor of rotation, okay. The argument of z0 is that the angle by which the tangent rotates, okay. If you have a source point and you have a curve passing through the source point z0 and you have tangent at that point.

Now you take the image curve which will pass through f of z0, the image point and you take the tangent there, the difference in the angles that the tangent makes with the x axis is precisely the argument of f dash of z0, okay. So the geometric meaning of f dash of z0 is that the modulus of f dash of z0 gives locally at z0 the magnifications factor and the argument of f dash of z0 gives locally the friction factor. This is house geometrically the map f behaves locally. And you know if the derivative f, f dash is nonzero, then you know it is conformal, you would have studied this in a 1^{st} course, conformal means you know it will preserve angle between the curves.

So for example if you take something like, something like a square, its image will be something like a distorted square, all right. If you take something like a circle, its image will be something like a distorted circle, you can expect it to be like for example something like an ellipse or something like that. And this is of course, if you consider it sufficiently small and at a point where the derivative does not vanish, okay. And this is why it has got so many applications to engineering because of conformality.

So you see why i am trying to tell you all this is that the modulus of the derivative is the magnification factor. And therefore multiplying by, multiplying the infinitesimal arc length by the modulus of the derivative always give you the length of the arc length of the image curve and that is what is happening here, you see this is a multiplication factor, this multiplication factor is therefore called the spherical derivative. Now i want to tell you a few things, few very very important things in this integral.

See the 1st and foremost, the amazing thing about this is that, you know i told you f is a meromorphic function, okay. F is a meromorphic function, so you see, f could have poles, okay, f could have poles, of course they are isolated but f can have poles. And the beautiful thing is your curve gamma, see your curve gamma can pass through those poles, okay. Now that is the amazing thing, we normally when you do integration, you never try, the integrand is always supposed to be continuous, okay.

When you integrate analytic function or for example whenever you do cauchy's theorem or you know you do argument principle, in all these, in all these things when you want to apply, you always make sure that the contour does not pass through any singular points. It cannot pass through poles, it cannot, there are cases when you are doing the logarithmic integral in the, in the argument principle, you assume that the contour does not pass through any poles and also through any zeros, okay. You do not allow the contrary which passes through 0 because you are integrating the logarithmic derivative exists f dash by f.

And if there is a zero, then denominator f will have a zero, then you cannot integrate. So in all these things that you have been, that we have been doing so far, we always make sure that the contour on which we are integrating does not pass through any zeros or poles. It should never pass through poles of course but also not true zeros if you are trying to apply the argument principle. But now mind you we are not, we are dealing with meromorphic functions, they have poles. And my point is, now the contour gamma can pass through as many poles as you want, it will not create any problem.

It will not create any problem for this integral because you see that is a matter of calculations that you have to understand. I will you roughly, suppose your contour gamma passes through some pole, my new there can be only finitely many such poles on the contour. Because you see the set of poles is anyways an isolated said, by definition of poles it is isolated, it is an isolated scenario. So set of poles is an isolated set and if you take the set of poles lying on gamma, it is an isolated said, it is an isolated subset of a compact set.

Mind you gamma is a , gamma is compact, any contour is compact, it is closed unbounded because it is actually continuous this image of an interval, closed unbounded interval, so it is closed unbounded, it is compact. And you know any isolated subset of a compact set is finite because if it were infinite, it will have a limit point, okay. And that limit point will not be isolated, okay. Therefore what will happen is that you have, okay if gamma passes through poles of f, mind you f is a meromorphic, it could have poles, if gamma passes through poles of f, it can pass through only finitely many poles because gamma is compact.

And what happens at the poles? See nothing happens to the integrand at the poles, it is bounded, that is a beautiful thing, that is why this integral is valid even if gamma passes through a pole or several poles. That is because you just imagine, suppose f has a pole at z0, okay, then in fact you know we can write this down. Suppose, so let me write this down.

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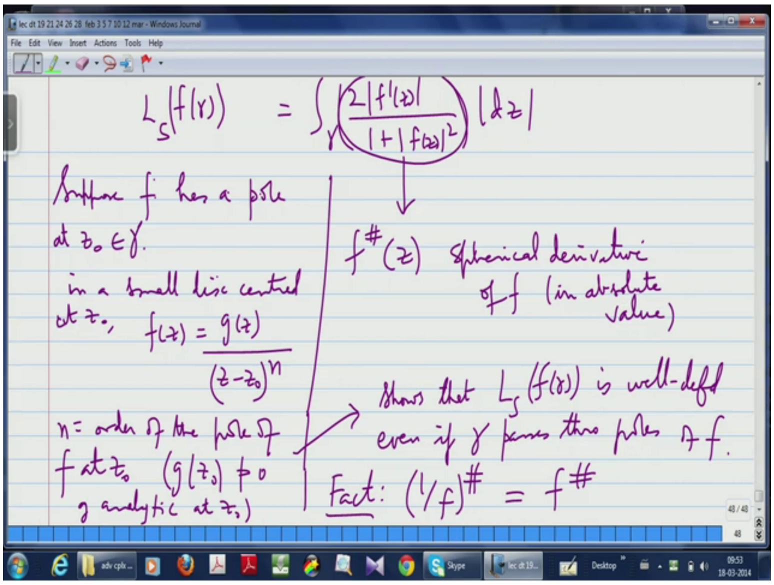

Suppose f of z, so, let me rub this and probably go down a little bit so that i have more space. Suppose, suppose f has a pole at z0 which is lying on gamma, this will create, this will not create any problems, this will not create any problems, why? Because you see in a small disc centred at gamma, centred at z0, you see f of z will look like g of z by z minus z0 to the power n where n is the order of the pole, okay. Now you calculate, you calculate this quantity, okay, notice that and of course g of z0 is not 0, okay. G of z0 is not 0 and of course this is how pole, locally a function looks locally at the pole of order n, all right.

And g is of course analytic, okay, g analytic at z0, okay. Now you see, just look at this expression that i have written about, if i take f dash, if i take the derivative, the derivative will continue to have a pole of one more order, if i differentiate this gz by z minus z0 power n, if you want using quotient rule, then i will get z minus z0 to the n +1 in the denominator. So what i will get, i will get, i will get a pole of higher-order, okay, of greater order. And if you go to the denominator, the denominator will have 1 by, it will have mod gz the whole square by z minus z0 to the power 2n.

And as z tends to z0, you will see that the numerator tends to , i mean this whole quantity will go to either 0 or 2 a finite value, okay. Because what, because what is actually happening is as z is tending to z0, z minus z0 is going to go to 0, so f is going to infinity, all right. But the fact is that the denominator will go to infinity faster than the numerator because the denominator contains f square. F square has a pole of order 2n, where as the numerator has a pole of order only n +1, okay. Therefore the denominator goes to infinity faster than the numerator as a result the integrand is bounded.

So the point is that this integral is valid even at a pole, that is a big deal here. So this integral is, you gamma can pass through poles of f, there is no problem. And geometrically also you should believe this because if gamma passes through a pole of f, its image will pass through the north pole on the riemann sphere. After all at a pole of f, f is taking the value infinity and that corresponds to the north pole on the riemann sphere, so after all what you are saying is that the image curve is passing through the north pole of the sphere, how does it matter. It is not going to affect, it is, north pole of the sphere is in no way different from any other point on the sphere, okay.

So the moral of the story is that, it shows that l s of f gamma is well-defined, even if gamma passes through poles, okay. So this formula always works, it works even for meromorphic function, it even works is gamma passes through poles, that is a very very important thing. And then there is, so this something you need to know. And there is another fact that it will expand into the next lecture, the factors that if you take the reciprocal of f, you see f is a meromorphic function, then 1f is also a meromorphic function. And the beautiful thing is that if you take it spherical derivative, we will get exactly the same as spherical derivative of f.

The reason is because of the fact that the spherical distance is invariant under inversion. I told you the inversion on the complex plane translates to a rotation of the riemann sphere about the x axis. And it leaves spherical distances invariant. Therefore the spherical derivative is also invariant if you invert the function, okay. So this is another important fact that we use. And the advantage that you can replace f by 1 by f is that whenever f has a pole, 1 by f has a zero. So you can reduce from the meromorphic case to the analytic case, so you can feel works with the analytic functions.

You see this is the advantage of having spherical derivative, okay. So spherical derivative does not distinguish between f and 1 by f, the advantage of moving from f to 1 by f is that you can, what is a pole for f becomes a zero for 1 by f and zeros are very friendly, more friendlier than poles. In the neighbourhood of zero you can usual, you apply usual analytic function theory because the function is after all analytics. So that is the advantage, so that is another motivation for having the spherical derivative. So i will stop here.