Advanced Complex Analysis-Part 2.
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Lecture-23.
What could the Derivative of a Meromorphic function Relative the Spherical Metric Possibly be?

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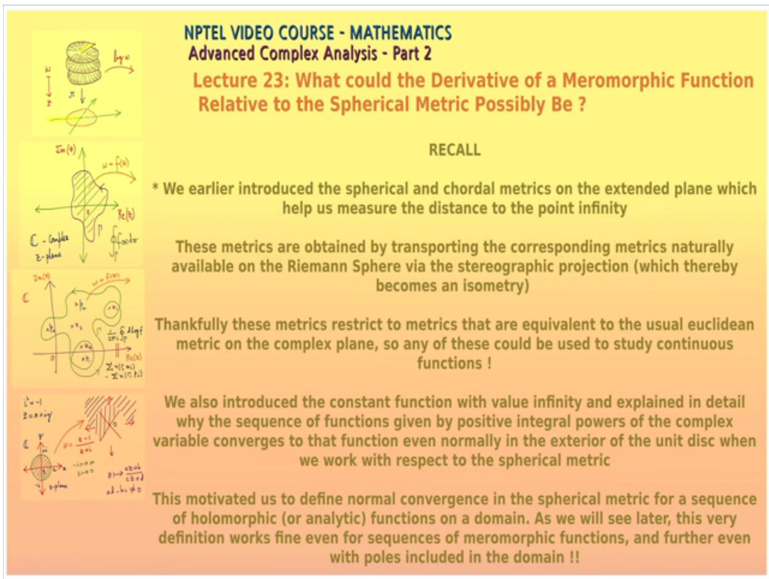

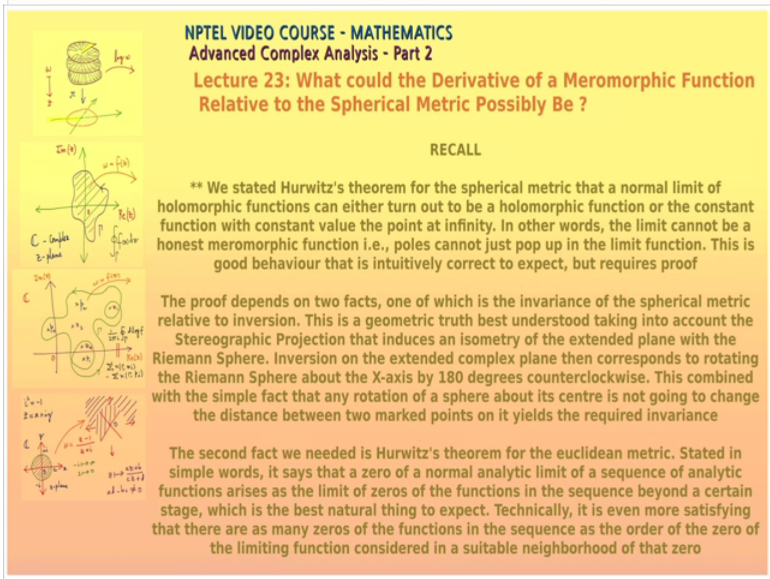

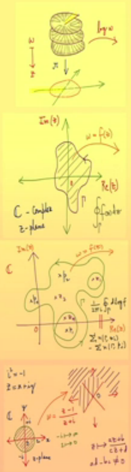

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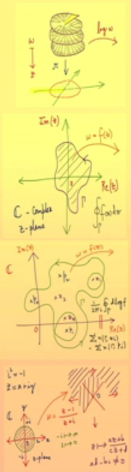
Advanced Complex Analysis - Part 2

Lecture 23: What could the Derivative of a Meromorphic Function Relative to the Spherical Metric Possibly Be ?

RECALL

*** We gave an introductory discussion on Hurwitz's theorem for the euclidean metric and gave a brief sketch of its proof. We recalled the Counting Principle or the Argument Principle which was needed in that proof. Then we began the proof of Hurwitz's theorem for the spherical metric that a normal limit of holomorphic functions with respect to the spherical metric is either holomorphic or the constant function with value infinity. We completed the proof in the lecture before the previous lecture

In the previous lecture, we extended the theorem to the case of a normal limit of meromorphic functions. These Hurwitz's theorems are important because they assert that singularities of normal limits cannot get worse and there is only one exceptional case -- when the limit is the constant function that is identically infinity



NPTEL VIDEO COURSE - MATHEMATICS

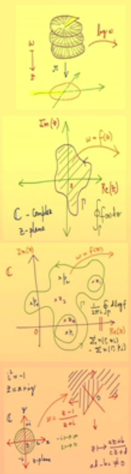
Advanced Complex Analysis - Part 2

Lecture 23: What could the Derivative of a Meromorphic Function Relative to the Spherical Metric Possibly Be ?

GOALS

*** * In the present lecture, we give an example of a sequence of meromorphic functions converging normally under the spherical metric to infinity, and another example where the sequence converges normally to a holomorphic function

We next recall the notion of infinitesimal distance or arc length in the euclidean metric and the integral formula for arc length. We explain how the modulus of the derivative acts as a scaling or magnification factor in the integral formula for the length of the image of a given curve. We then discuss these notions for the case of the spherical metric, namely we bring in the infinitesimal distance or arc length in the spherical metric and give an integral formula for the spherical distance. This discussion gives us the clue to guessing what the derivative of a meromorphic function with respect to the spherical metric could possibly be



NPTEL VIDEO COURSE - MATHEMATICS

Advanced Complex Analysis - Part 2

Lecture 23: What could the Derivative of a Meromorphic Function Relative to the Spherical Metric Possibly Be ?

KEYWORDS & KEY PHRASES

one-point compactification, Riemann sphere, Stereographic Projection, complex plane as punctured sphere, meromorphic function, analytic except possibly for poles, set of poles is countable, convergence on compact subsets or normal convergence, constant function with value infinity, metrics on the complex plane and the extended complex plane, metrics at infinity, metrics on the Riemann Sphere, chordal metric, spherical metric, euclidean metric, distance to point at infinity, minor arcs of great circles are geodesics on the sphere, normal convergence in the spherical metric, integral formula for the spherical distance, infinitesimal distance or arc length in the euclidean metric, rectifiable arc or curve, integral for euclidean arc length, modulus of the derivative as a scaling or magnification factor in the integral formula for the length of the image curve, spherical derivative

Alright, so we have been seeing about meromorphic functions and holomorphic functions converging normally, you know with respect to the spherical metric, so I want to continue with a few examples, so that you get a feel for what we want.

So let me write it down, examples of normal convergence under the spherical metric. So this is what I want to talk about. So the 1st question, 1st question that we ask is , we have already seen an example of a sequence of holomorphic functions which tends identically to infinity, okay. That is basically it is a sequence of functions it is that power n, okay and the domain is the exterior of the unit circle, okay. And z power n converges as n tends to infinity to the function which is identical infinity, okay, point wise and this convergence is in fact normal, okay, we have seen this.

now you can answer, you can ask the following question. Can you have a sequence of holomorphic functions tend to a meromorphic function, function which is honestly meromorphic, which means it has at least one pole? And the answer no, okay. A sequence of holomorphic function, if it converges normally under the spherical metric, either it converges to the constant function which is identical infinity or it converges to a good old holomorphic analytic function.

You can get meromorphic functions, proper meromorphic functions which are having some poles, you cannot, you cannot take normal limit of analytic functions and expect a pole to pop-up, okay, that will not happen, that is what the proof, okay. but we think of other possibilities, suppose we have sequence of meromorphic functions, so can you think of a sequence of meromorphic functions which which converge to say infinity? And then of course normally under the spherical metric and can you think of a sequence of meromorphic function which converts normally to a holomorphic function, okay. So they have already proved that if you take a sequence of meromorphic functions and suppose it converges normally under spherical metric, then the result is again a meromorphic function or it has to be the extreme case which is, a function which is identical infinity, okay.

And mind you, let me again stress the, this is so important because you see in the limit, the limiting function that you get could have developed bad singularities, it could have developed nonisolated singularity, okay. Or for example it could have had singularities on a curve, it could have happened no, then it would have been terrible. for it could have developed an isolated singularity which is essential and once you have an essential singularity, that function is out of our discussion because our discussion we are trying to work only with meromorphic functions.

And for meromorphic functions we allow the poles as singularities, okay. So such bad thing to do happen, you do not have an essential singularity popping up when you take the limit, normal limit with respect to spherical metric. You do not have for example a nonisolated singularity popping up when you take normal limit of meromorphic functions, okay. So this is, this is, I decide that this is what happens. And that is what we proved the last 2 classes, okay.

now I just want to give you a couple of examples where you can have a sequence of, I think you can have a sequence of meromorphic functions that goes to holomorphic function, okay. And you can also have a sequence of , so that, so that example is an example where, and of course you can also have a sequence of meromorphic functions, honestly meromorphic functions go to infinity, okay. So you know, so let me write down those 2 examples. So here is example 1.

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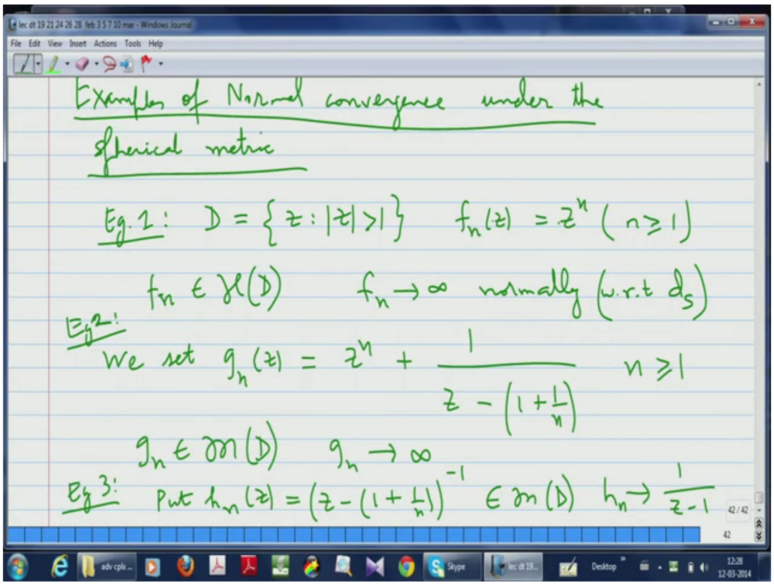

So this this example 1 is as well, you know, let me say, I have just recall that example of holomorphic functions that goes to infinity and modify it carefully to make it into a sequence of meromorphic function that go to infinity. So this, 1st example is just recalling, you bought the domain D to be the, well the exterior of the unit disc, set of all z such that mod of z greater than 1, mod z greater than 1, this is the exterior unit disc. Mind you this is a nice domain in the external complex plane, okay. Well, well the fact is that you know if you consider this as a domain in the complex plane, did not simply connected, okay, because it has a hole in between. It has a whole unit disc closed, unit disc out as a hole inside.

but if you consider this as a domain in the extended complex plane, mind you it becomes simply connected. And how do you see that? It is actually, it is actually a neighbourhood of infinity, okay. And how do you see that? If you, if you just take its image under a stereographic projection, you will get a small disc surrounding the north pole on the Riemann sphere. And that is clearly is simply connected set, okay. So mind you including the point at infinity done this from, you know multiplying connected domain to simply connected domain, okay.

Anyway, that was just an observation, so you take this domain D and then you take fn of z to be z power n, n greater than equal to1, this is my sequence. Then you know that fn, then fn converges, the fns are all holomorphic functions on D, okay, they are in H of D, these are all. Of course as fns are, in fact entire function, that is polynomials, so they are holomorphic. And well, now but what happens, fn converges to infinity normally, so when I say normally this is with respect to this spherical metric, okay. And when I say fn converges to infinity, it means, by infinity I mean the function which is constantly equal to infinity at all points.

Mind you we have put that extra function there, okay. And this is the extreme case, all right. And now what you can do is you can modify this so that you can get a sequence of meromorphic functions which go to infinity for example and you can also modify this to get a sequence of meromorphic functions which goes to holomorphic functions, okay. You can do things like that. So for example I can make each of these functions to have a pole, that is, for example lying on the real line, and it is approaching 1, okay. And then if I take the limit, the poles are moving towards 1. So what will happen is that the in the limit I will get a function which will have, which will actually be holomorphic because I have included mod z, I have excluded mod z equal to1 from my domain.

Okay, so what you do, we said g of n of z to be well, z power n plus let us add a pole at if you want, well 1 + 1 by n which is a little to the right of the point 1, okay. So I put 1 by z dash 1 + 1 by n and let me make this n greater than equal to 1. So if you take these functions gns, they all have poles at the points, the gn has hole at, it has only one pole, that is at 1+1 by n and that is of course lying in mod z reference 1, all right. So for example z1 has a pole at 2, z, sorry g1 has a pole at 2, g2 has a pole at 1+ half , 1+ half which is 3 by 2 and then and so on.

You can see that as n tends to infinity, 1 by n tends to1, all right. And you can see that these gns are actually meromorphic functions on D because they only, they are not holomorphic because they have poles but and each has only one pole, each function has only one pole, so it is meromorphic function by definition. These are all meromorphic functions and you know if you if you let n to tend to infinity, then what will happen is that these zn, of course these z power n will go to identically to infinity. for any fixed z with mod z greater than 1, mod z power n is going to go to infinity as n tends to infinity. Therefore point wise these gns are going to go to infinity, except we will have to worry about the poles.

Mind you, in the neighbourhood, the poles are all moving away, so if you if you if you take any particular value of z, even if is one of these poles, okay, for example if you take 1+1 by n, that is a pole of zn but from gn +1 onwards it is not a pole. And of course it is not equal for gn minus1 and functions before that, okay. but mind you the value at the pole is by definition infinity. for a meromorphic function, we are allowing, mind you we are allowing the value infinity, okay. So somehow what you will see is that, if you now take the limit as n tends to infinity, okay, you will end up with you will end up with the function which is identically infinity, okay.

So this is just a modification of the original example, okay. So gn, gn tends to, so gn tends to infinity, right, so here is one example. So here is an example of Hurwitz meromorphic functions which go to infinity, okay. And well, and in fact it would, if I forget z power n and I took only the function 1 by z - 1+1 by n, okay. now that function if you see, if you look at it, that will be the function which is again meromorphic with only one pole at 1+1 by n. but now the point is that this function 1 by z dash 1 + 1 by n because z is in the denominator, as you let z tend to infinity, this will go to 0, so this function is bounded at infinity.

So as a result, you know if you, if you put Hn of z to be just 1 by z dash 1 + 1 by n, then Hn of z, they will all be meromorphic. That gives you, you get an example of sequence of Hurwitz meromorphic functions which tends finally function which is holomorphic and it is also holomorphic at infinity, it is analytic at infinity because at infinity it is bounded. So that will give me the next example, so this is, this is, this was example 2 which was the modification of example 1. And the part of, and then we have example 3, so example 3 is put Hn, Hn of z to be simply z dash 1 +1 by n and then you put it, put, take the reciprocal so that you get the pole at 1+1 by n for Hn.

Then these are all of course in, they are all meromorphic functions in D, so Hn tends to1 by 1 by z - 1, okay, as n tends to infinity. In 1 by z minus1 has a pole at 1 and that is not a problem because one is not in our domain, all right, it is in the boundary of our domain. fine, good. So, yah so now what I want to tell you is that, what are we going towards next. See we need to understand several technical concepts in order to get the proof of the Picard theorem which is our aim, okay. And one of these concepts, one of the technical things that we have to worry about is the so-called spherical derivative, okay, the absolute spherical derivative of a function, all right.

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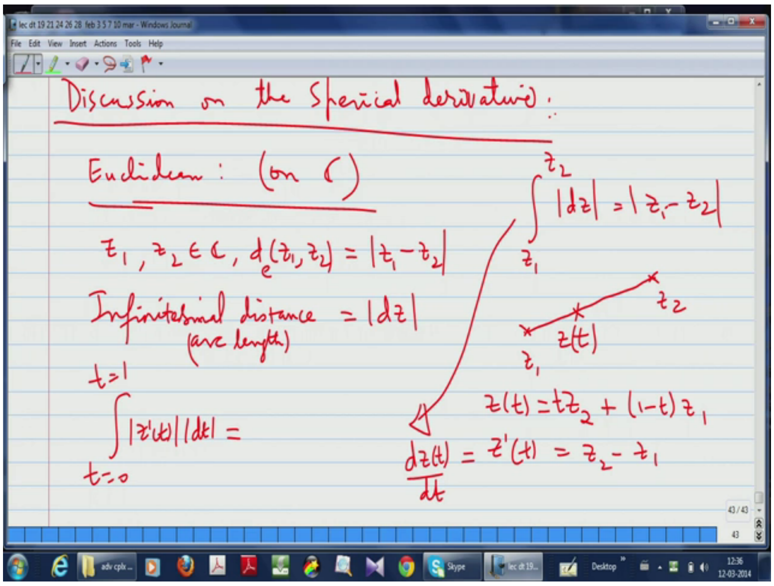

now let me try to explain that and with reference to what we already know. So the 1st thing is that, so I, I will tell, I will put this heading as, so let me change colour and put this heading as discussion of the spherical metric on the spherical metric and spherical derivative because this something that we need, okay, we need to worry about this. So the 1st thing is, let go 1st Euclidean, so that is on the complex plane, okay. So you know if you give me 2 points z1 and z2 on the complex plane, then the distance, Euclidean distance between these 2 points the z1 and z2, if well, mod z1 minus z 2, okay, this is the usual distance formula.

And the infinitesimal distance, how the given, you get the infinitesimal distance by putting z1 as say z and z2 as z plus Delta z, okay and then mod z1 minus z2 becomes mod Delta z, okay. And then you change the Delta to D, okay, to get the infinitesimal. So the infinitesimal distance , distance, so this is arc length, this is mod ds, okay. Let me use, let me use the right rotation dz, these are claimed small dz and roughly the way you think of it if you put z1 equal to z and you put z2 equal to z plus Delta z. So the difference mod z1 minus z2 becomes mod of Delta z and then change the Delta to D.

The idea is you ignore, if you do such an operation, you have to ignore higher powers of Delta z than the 1st. If we get Delta is at whole square, Delta is at whole cube, okay, all these terms have to be ignored, okay. And then you replace Delta by the, so this is a heuristic way of getting the infinitesimal distance. And, well and how do you , and how do you for example check that this is, this infinitesimal distance is correct. So for example you know if I take, if I take integral from z1 to z2 of mod dz, if I do this, I will get mod z1 minus z2, this is what I will get. And that tells you that the infinitesimal distance you have computed is correct.

So how this calculation is done, so you have for example, of course when I say this integral from z1 to z2, I am going along this, this integral is along the straight line segment from z1 to z2. So whenever I write it, mind you whenever you write an integral, there is always a path involved, without the path if you write it, then it is very ambiguous. So here when I say integral from z1 to z2, I mean the linear distance, the Euclidean distance between z1 and z2. That means I am taking straight line segment from z1 to z2, okay.

So what you have to do is, here is z1 and earlier z2 and then you are taking this path, the straight line segment okay and then you can parameterise this path. You can parameterise this path and then you evaluate that integral, okay. And if you evaluate it, you will get mod z1 minus z2. So for example you know if you take point t here, how will you, how will you write that, we write the parameter? If you take the parameter as t, so I should write z of t, so what will happen to z of t, z of t will be, so you know z of t should give me, for example t is the parameter varying from 0 to 1, then you know z of t has to be a complex combination of z1 and z2.

So when t is 0 I am supposed to get z1 and when t is 1 I am supposed to get z2, okay. So I will get z1 plus I will get 1 minus t times z2. So in this if I put t equal to1, I will get z1, sorry I will, it should be the other way round, it should be the other way round, let me change it. It is z2 +1 minus t, I think I should put t here also, so you put t equal to 0, I get z1 and you put t equal to 1 I get z2, this is a parameter equation of this line segment, okay. And it is a function of t, it is a differentiable function of t, in fact it is a linear function of t. And what is z dash of t, z dash of t gives just z2 minus z1, okay, did the constant.

And what is the, what does this integral come out to, this integral comes out to, if you ballot it,, you have to, you know you have to transform the integral in terms of the parameters. So you will put t equal to 0 to t equal to1 and you will plug-in this mod dz, so it will become, so dz is, z dash of t is dz of t by dt, okay. So you will think of mod dz at, mod z dash of t into mod dt. Okay. So I am going to get mod z dash of t into mod dt and you know, well instead of mod z dash of the I am going to put mod z minus z1, that is a constant, that will come out and I will simply get integrals 0 to 1 mod dt, and that is going to be 1, so I will get mod z2 minus z1, okay.

So that is how this formula is verified and it is a very very simple calculations but I am just trying to justify that mod dz is the correct infinitesimal version of the of the arc length, okay. And you know, you would have seen this also in complex analysis, how do you measure the length of an arc on the plane? It is done by the same thing.

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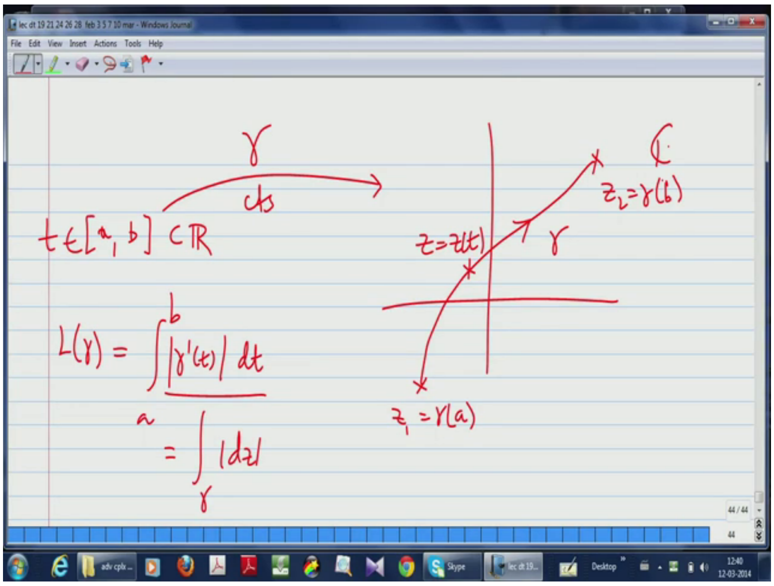

So you know suppose you have an interval, suppose you have this, say this interval, let me when would a, b, suppose this is closed bounded interval in the real line and suppose I have I have a continuous map gamma from that into the complex plane so you know it traces an arc. Sometimes the map is also called gamma, the image is also called gamma by abuse of notations. So this is, this point is that one which is gamma of a and this point is z2 which is gamma of b and, well outdoor get the length of the arc gamma. Well you get it by simply integrating from a to b gamma dash of t, mod gamma dash of t into dt, okay.

And this is the same as integrating over gamma mod dz. And basically the integral over gamma mod dz is transformed to the previous integral, integral from a to b gamma dash of t dt, mod, mod gamma dash of t dt because you just plugging in z equal to z of t. because if you take point on gamma, the point z there is given by that of t, it is a function t where c is the variable here on the real line. When t is a, you get gamma of a which is z1, starting point. When t is b, you get gamma of b, which is z2, the ending point. So basically if you want to, mod dz is the infinitesimal distance and you integrate an arc integrate that over an arc, you will get the arc length.

And of course it is very important that you are should be reasonably good, so that you get a finite number, okay. So arcs for which the arc length turns out to be finite, they are called rectifiable arcs, okay, rectifiable arcs. And one has to worry about these things because there are, if you just, of course whenever you say ask, it is always, this gamma is always continuous, all right. but then you know, even to write that integral, you see that integral actually means this, mod gamma dashed of t dt, so you know gamma dash should exist.

So gamma is not only continuous but actually it is differentiable and you can relax it to be piecewise differentiable, piecewise continuously differentiable, I, mind you whenever I integrate, it has to be continuous. So 1st for gamma dash has to exist and gamma dash has to be continuous, only then I write that integral, okay. If I am being very naive, of course if I use measure theory, I can do much more, I allowed is continuity on a measure, measured 0 set. but we are not going to that level of you know complication. We are, we assume the naivest point of view and even for the naivest point of view developed assume that gamma dash exist and it is continuous.

And of course all Riemann integrals can be evaluated even if there are a few finite, finitely many discontinuities which are with finite jumps okay. So that is the reason we use the word contour, we said that gamma should be a contour, which means gamma should be a piecewise continuously differentiable function, okay. fine. So this is the length of gamma and of course there is this important restriction that gamma should be a rectifiable arc. You have to worry about such thing because there are strange things like which are called space filling curves and these are curve which can fill out the whole region of space.

And obviously the length of such a curve will be infinite, okay. So you do not want to end up with such horrible things. So that is the reason you have to worry about computing this integral only for rectifiable arcs, okay. now, now what is it, how is this connected with analytic functions?

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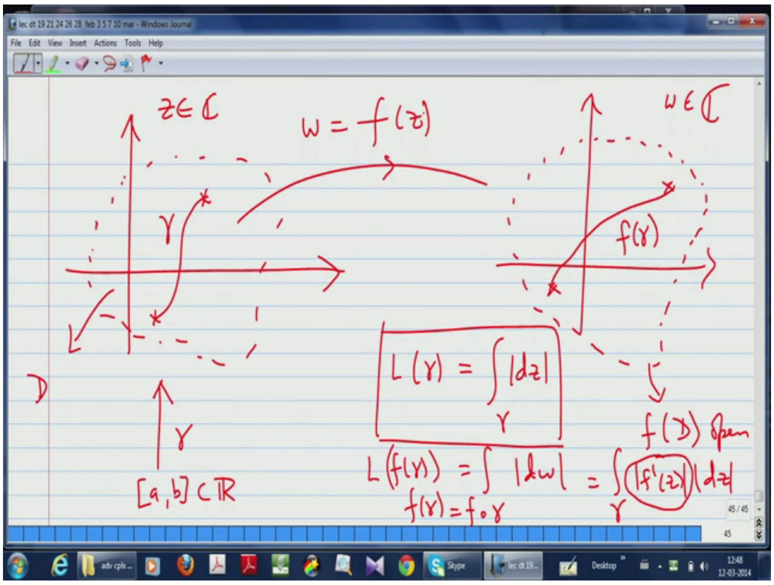

So you see, see the point is, suppose you have, suppose you have complex plane and suppose you have domain here, there is some domain, okay, so here is some domain D in the complex plane. And suppose there is a, there is an analysis function f, analytic holomorphic function f which is defined this domain. If suppose I have, suppose I have a path, I have an arc in the domain, okay, which means that, you know I have this continuous, piecewise continuously differentiable function gamma from an open, I am in a closed bounded interval in the real line.

And it is giving me this arc, with this as the starting point and this as the ending point and this is also, the labelled arc also is gamma, where, actually it is an image of gamma but I will, we will use, in abused notation we also call that gamma. So we use gamma for the image of the arc, we also use gamma for the function that parameterises the arc, okay. So well, you know what is, what is length of gamma, length of gamma is as a told you just now, it is integral over gamma mod dz, this is the length of gamma, okay. now what will you get if you integrate over f circle gamma mod f dash of, mod f dash of z , so let me write this , so let me write something here, mod dw, okay.

So you see, so you know I am writing this mapping as W equal to f of z, which is similar to Y equal to f of X, if you are looking at functions for a sphere variable. X is the independent, independent variable, Y is the dependent variable. but now z is the independent variable and

W is the dependent variable, W depends on z by means of the function f. So this z varies over this copy which is the source complex plane and in fact z is varying in D, that is where you are function is defined. And then you have, well here is the target complex plane, okay.

And W is in the target complex plane, and mind you that the image of f, I am of course assuming that f is not a constant function, I am assuming f is a nonconstant analytic function. We have this theorem, that is nonconstant analytic map is an open map, okay. Therefore the image of f is an open set, okay, that is a very deep fact. So this is the, this is f of D is E1, that is open actually. And of course we are not looking at the case when f is a constant function, okay, f is not constant. I do not want to worry about that case, okay because the image is only one point, okay.

So f of d open and, well, what happens to, what is f of, what is the image of gamma, well I will get another curve here. So I get this, I get this as f of gamma, okay, this is just the image of the curve gamma under f. And you know if I now take the variable in the target, if I take the variable in the target plane which is W and if I integrate f circle gamma, okay, f circle gamma is just mind you f of gamma, it is just, what is f circle gamma, you 1st apply gamma and then apply f. Which is the same as taking the image of gamma under f, okay. So f circle gamma is in principle f gamma, f of gamma, okay.

And if you integrate f of gamma over f of gamma dw, then you should get the length of f of gamma. So this is going to give you length of f of gamma, okay, by what we have seen, right. 1st what is, but what is, now you plug-in what dw is. W is fz, so mod dw will become mod f dash of z into mod dz. Okay, so what will happen, so let me write that down, I will get, I need a little space, so let me rub this off and write it on the open side. So this will be integral over f circle gamma of mod f dash of z into mod dz, this is what I get.

Okay, so if you look at it very carefully , you see that if you normally integrate over dz, okay and of course you know since I change the variable from W to z, I should change the curves from f circle gamma to gamma itself because now my z is in the source and what is in the source is gamma, not f circle gamma, f circle gamma is in the target, okay. So I should change this to gamma, so now, now look at this. now compare the 1st equation which says that L of gamma is gotten by integrating over gamma just mod dz and look at the 2nd equation, what you have done is instead of simply integrating mod dz over gamma, you have multiplied it by this mod f dash of z, okay.

And what does it give you? It gives you the, it gives you the length of the image of gamma under f, okay. If you simply integrate over mod dz, you get the length but if you integrate over mod dz, the derivative of your function, okay, then you get the length of the image of the curve under the function, okay. So this, so this is what I want to tell you, whatever you put, whenever you integrate over gamma, mod dz and you put something in the integrand, that something in the scaling factor, okay. It is a, it is a factor of scaling, all right.

So if you for example you know if I put 2, if I instead of integrating over gamma, integrating over gamma mod dz, suppose I integrate 2 mod dz, I will get by the length of the curve. Okay. So and similarly if I integrate it with some constant times mod dz, I will get, of course the constant should be positive constant because you are worried about length, then you will get positive constant times that length, okay. If I integrate, if I integrate let us say K mod dz, I will get K times the length of the curve. That is because mod integrating just over mod dz is going to give me the length of the curve as it is and it is multiplied by K, the K will come out.

but then this is all right if I am putting a constant but instead of a constant I can actually put a function which varies as the point z varies on the curve, okay. And then what will happen is that you are scaling the, you are just scaling the, you are getting a scalar version of the length of the curve. And the fact here is that multiplying by mod f dash of z gives you the image, the length of the image curve. And the image curve is part of the original curve scaled by the map f. The map f maps gamma, the original curve onto the image f gamma, it scales it.

And that scaling factor is mod f dash of z and that is what appears in this integral, okay. So why I am saying all this is, I am trying to say that, you know, you look at the coefficient of mod dz in the integral, that should give you the modulus of the derivative, okay. The modulus of the derivative mod f dash of z is the coefficient in the, it is the coefficient of more dz in the expression for the length of the curve, okay. now what you have to do is, if you try to mimic this with the following modifications, namely you take again a function f, defined on a domain in the complex plane. but now I assume that the function is taking values on the Riemann sphere, I mean which is essentially taking values of the external complex plane, okay.

And on the external complex plane you use the spherical metric, then you know you would like to have an idea of what is the derivative of the function with respect to the spherical metric, okay. What we are looking at is trying to get an idea, we are trying to guess at what is the derivative of a function with respect to the spherical metric, okay. And what does it mean?

You are taking a function which is defined on the domain of the complex plane, it is taking values in the external complex plane because that is where the spherical metric makes sense and then you want to take the derivative of the function with respect to the spherical metric, okay.

It will be ideal if we get the, if we can guess formal for that. but then more importantly what this argument tells you that, if you take a curve in the domain, you take its image under the function in the extended plane, that means you are actually looking at its image on the Riemann sphere, okay. And if you find the length of that curve in the image, in the Riemann sphere, that is with respect to the spherical metric and in that if you take out the coefficient of, the coefficient of mod dz, okay or if you want mod d of whatever variable you are using then that coefficient should give you the modulus of the spherical derivatives, okay.

So I am just trying to tell you how this argument tells you how to guess what the modulus of spherical derivative is, okay. The only difference is you are not looking at an analytic functions, you are looking at a function which is, which is having values in the, not in the complex plane but in the extended complex plane. And mind you the advantage is that, because you are allowing external complex plane, your function can take the value infinity, okay. And mind you once you do that, you can also deal with meromorphic functions, okay.

The whole purpose of allowing the value at infinity is to deal with meromorphic functions. So you can now generalise all this to taking the image under a meromorphic function of a curve in a domain and taking its image in the external complex plane thought of as a Riemann sphere and trying to measure the spherical distance. So whatever we did with the Euclidean distance, you have to do with the spherical distance. So let me, let me try to quickly tell you how that goes.

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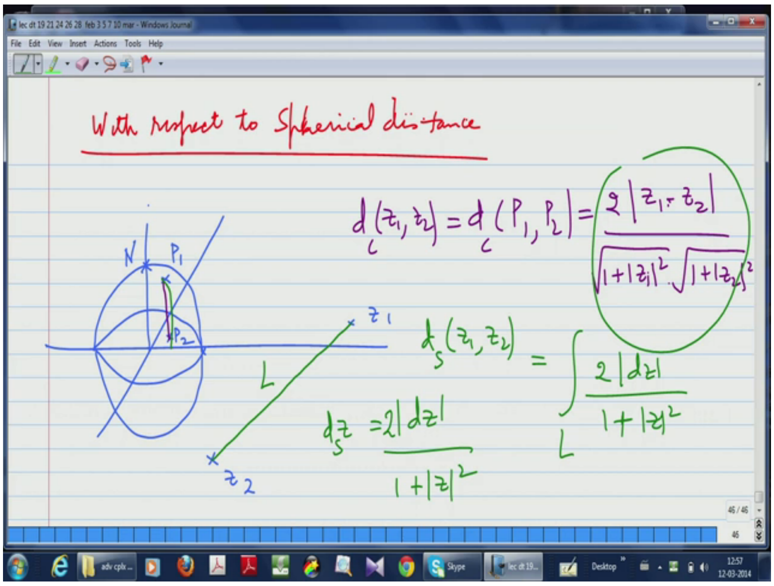

So you see if you take the, so now, with respect to spherical distance, so you see I want to do 1st check that. Well so if you take the, if you take the Riemann sphere, which we draw like this, it is just a unit sphere in 3 space, real 3 space and where they XY plane is thought of as the complex plane, the 3rd axis is called the U axis and this is the north pole which corresponds to the point at infinity, okay. now you know you give me 2 points on the, on the Riemann sphere. Suppose these are 2 points that corresponds to 2 points on the plane, so, well, so there is a point z1 and there is a point, well , well if I go by the diagram, then I will have to be careful.

So let me, let me redraw this little bit. So let me have 2 points, so this is z1, let say this is z2 and you know by the stereographic projection z1 will correspond to a point here which is P1 and z2 will correspond to say another point here which is P2, okay. And, well and you know now on the Riemann sphere, you know if I join this, if I join this cord, so let me use a different colour, if I this cord from P1 to P2, so the cordal distance between P1 and P2 which you can define to be either cordal distance between z1 and z2, okay, you can define it like that because you just transported distances on the Riemann sphere to the external complex plane because of the stereo graphic projection.

And what is the cordal distance between between P1 and P2, I want to do check this, it is a very simple exercise. It is actually, this is actually equal to 2 times modulus of z1 minus z2 by root of 1+ mod z1 square into root of 1+ mod z2 square, okay, this is what it is. I want, this is just analytic geometry, okay, I want you to calculate this, okay. And then what happens is that, you see, as, this is the cordal distance between 2 points and what it is, is actually, this is actually the distance of those 2 points P1 and P2 in R3, okay. Mind you this whole diagram is in R3, they XY plane in the complex plane, okay.

You can check that this is the distance in R3, it is a very simple analytic geometry calculation, right. now what is the, what is the spherical distance between P1 and P2? It is the length of the arc, minor arc from P1 to P2 on the major circle that passes through P1 and P2 on the Riemann sphere. So let me use a different colour for that and that is going to be, let me use something else, let me use green, okay. So here is the, so here is this, so I also have this d sub, so there is this d sub s, this is the spherical distance between z1 and z2, okay and how do, how do you get the spherical distance?

Mind you it is the length of an arc, it is the length of this arc, the circular arc along the great circle passing through P1 and P2. And how do you get the length of the arc, how do you get the length of an arc, you get the length of the arc by integrating an element of the infinitesimal distance, okay. now the cordal distance, as P1 and P2 comes closer, the cordal distance infinitesimally becomes a spherical distance, okay. Therefore and you know if you make z1 and z2 closer, if you put z1 equal to that and put that vehicle to z plus Delta z, you know what you will get is that you will get the element of spherical distance, you will get this, you will get this, you will get this expression.

You will simply get mod dz by 1+ mod z the whole square, okay. because you see in this in this expression here, in this expression you put the z2 equal to z1 plus Delta z, okay and you look at only Delta z terms and you change the Delta to d, you will get this, all right. And of course, of course there is 2, I forgot 2, that is 2 here. And therefore the spherical distance between z1 and z2 is simply given by integrating, okay along the arc of the great circle from b1 to P2, that is the same as integrating along this line segment L, because we image of this line segment L on the Riemann sphere is exactly that great, that minor arc of the great circle passing through P1 and P2.

So this integral over L, 2 times mod dz by 1+ mod z the whole square, okay. So this is the spherical distance between 2 points on the, on the complex plane. And this works even if you assume one of the points to be the point at infinity. Mind you because your points can vary on the external complex plane, so both points could be the point the north pole itself, okay. fine, so now what we can do is that we can use this to guess what the spherical derivative should be. We can use this and the previous argument to guess what the spherical distance should be. And I will just state it and stop.

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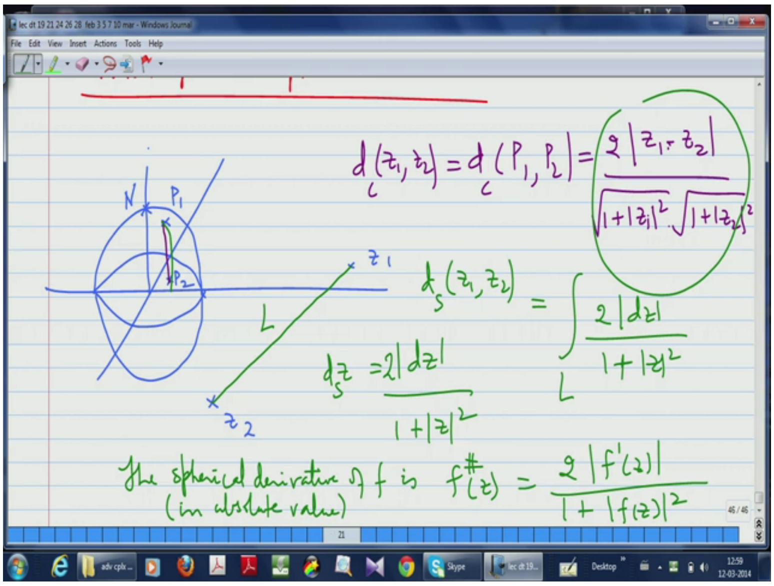

The spherical derivative is, of f is f hash of z, it is 2 into mod f dash of z 2 by 1+ f mod fz the whole square, this is what it will be. This is the, this is the, this is absolute value with respect to derivative. The spherical derivative, so I should say absolute value. This is what you will get if you think about it and I will explain this in the next lecture, okay.