Advanced Complex Analysis-Part 2.

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Lecture-22.

Hurwitz's Theorem for normal Limits of Meromorphic functions in the Spherical Metric.

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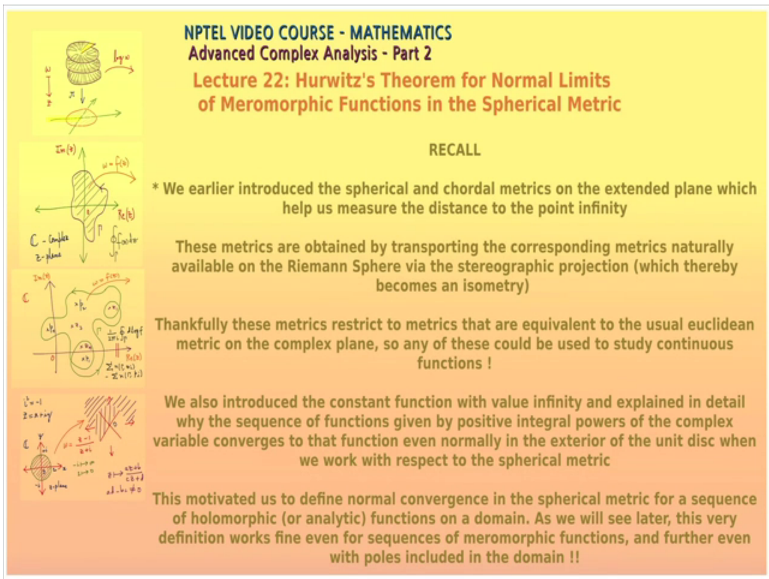
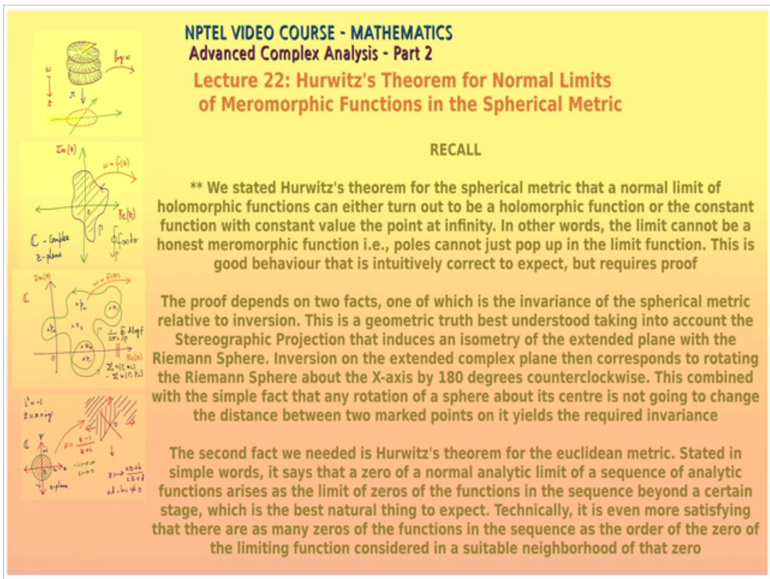
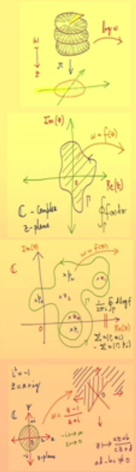






NPTEL VIDEO COURSE - MATHEMATICS

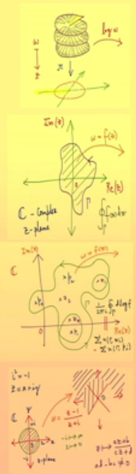
Advanced Complex Analysis - Part 2

Lecture 22: Hurwitz's Theorem for Normal Limits of Meromorphic Functions in the Spherical Metric

GOALS

*** In the lecture before the previous lecture, we gave an introductory discussion on Hurwitz's theorem for the euclidean metric and gave a brief sketch of its proof. We recalled the Counting Principle or the Argument Principle which was needed in that proof. Then we began the proof of Hurwitz's theorem for the spherical metric that a normal limit of holomorphic functions with respect to the spherical metric is either holomorphic or the constant function with value infinity. We completed the proof in the previous lecture

In the present lecture, we extend the theorem to the case of a normal limit of meromorphic functions. These Hurwitz's theorems are important because they assert that singularities of normal limits cannot get worse and there is only one exceptional case -- when the limit is the constant function that is identically infinity



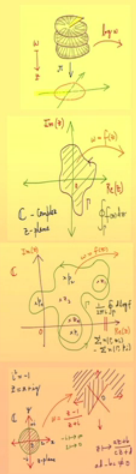
NPTEL VIDEO COURSE - MATHEMATICS

Advanced Complex Analysis - Part 2

Lecture 22: Hurwitz's Theorem for Normal Limits of Meromorphic Functions in the Spherical Metric

KEYWORDS & KEY PHRASES

one-point compactification, Riemann sphere, Stereographic Projection, complex plane as punctured sphere, meromorphic function, analytic except possibly for poles, set of poles is countable, euclidean spaces are second-countable, countable basis for topology, meromorphic on the extended plane same as rational, pointwise convergence, uniform convergence, locally-uniform convergence, convergence on compact subsets or normal convergence, Banach space or complete normed linear space, metric induced by a norm, a sequence of analytic functions could converge normally to the constant function infinity, a normal complex-valued limit of analytic functions is analytic, constant function with value infinity, metrics on the complex plane and the extended complex plane...



NPTEL VIDEO COURSE - MATHEMATICS

Advanced Complex Analysis - Part 2

Lecture 22: Hurwitz's Theorem for Normal Limits of Meromorphic Functions in the Spherical Metric

KEYWORDS & KEY PHRASES

...metrics at infinity, metrics on the Riemann Sphere, chordal metric, spherical metric, euclidean metric, transporting a metric through a homeomorphism thereby automatically making the homeomorphism an isometry, equivalent metrics induce the same topology, distance to point at infinity, minor arcs of great circles are geodesics on the sphere, normal convergence in the spherical metric, invariance of the spherical metric with respect to inversion, Hurwitz's theorem, zeros of a nonconstant analytic function are isolated, Identity theorem, order of zero or multiplicity of a zero, functions with values in the extended complex plane, Argument Principle or Counting Principle, logarithmic derivative, change in the argument of a function along a closed curve

Alright, so what we are going to do now is we are going to show that if you have a normal converging sequence of meromorphic functions, then the limit function is either a meromorphic function or it is identically the function of infinity, okay. And the importance of this theorem is that in the limit you can only get a meromorphic function and nothing worse, okay. Because you see meromorphic function is something that has special kind of singularities, there are is only poles as singularities.

But when you take a limit of meromorphic functions, anything could have happened, we could have got a function with, for example essentially singularity, such a thing could have happened, okay. Or you could have even got a function non-isolated similarities but all these things do not happen. The nice thing happens, namely if you take a normal limit of meromorphic functions, you can get a meromorphic function. And the worst thing that can happen is that it is a function which is identically infinity, therefore, okay.

So let us go ahead into the idea is similar to the idea that you know if you take a sequence of holomorphic functions, and then you take a normal limit of holomorphic functions. Then the limit function is either absolutely holomorphic, okay, it is holomorphic, perfectly holomorphic or it is identically infinity, okay. So let me start like this.

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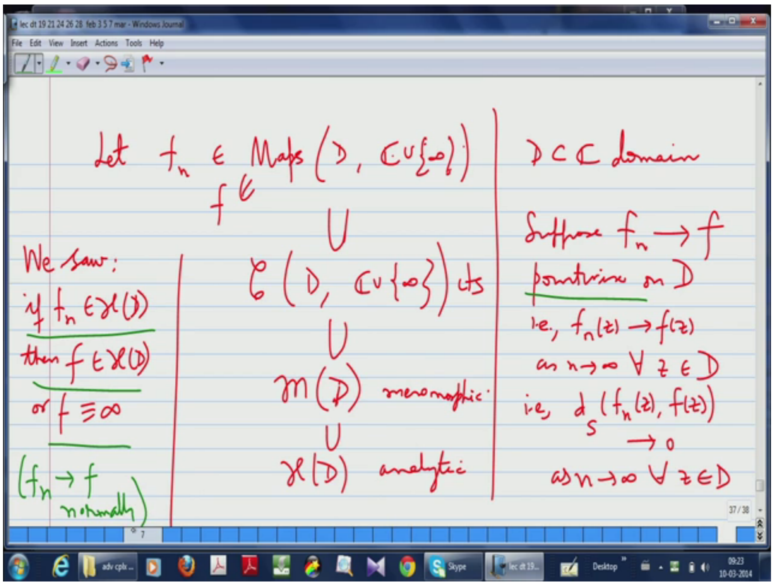

Let fn be a sequence of maps from D to C union infinity, D inside the complex plane is a domain, okay, of course nonempty set, nonempty open connected set. And I am taking maps with values in C union infinity which is an external complex plane and the reason for that is, of course you know we allow the value infinity now. And so suppose, so you know suppose, so let me write it here, suppose fn, suppose fn tends to f, suppose fn tends to f point wise, point wise on D, right. Suppose it is a point wise convergence, so that means that f is also here, so your f is also here, so let me write it somewhere here. f is also a map from D to C union infinity and fn tends to f point wise on D, you have to be, when you see point wise, it means for every point.

And when you say every point, it means that fn of z tends to f of z for all z in D, okay. But then what does fn of z tends to f of z for all z in D mean? It means that the distance between fn of z and f of z tends to 0, okay. As n tends to infinity for each z in D . And what distance are you going to use, you just cannot use the usual Euclidean distance, okay, because you are allowing the value infinity, you have to use spherical metric. The distance under the spherical distance has to be used. So let me write that down, i.e. fn z tends to fz as n tends to infinity for all z in D .

And that is supposed to mean that the spherical distance of fn of z, spherical distance between fn z and f of z, that tends to 0 as n tends to infinity for every z in D . This is what point wise convergence D means. And of course you have to use spherical distance because the function values might be one of the points to which we are measuring from or to which you are measuring the distance could be the point at infinity, so then you will have to use the spherical metric, okay. That we have denoted by D sub S, all right. So you have a sequence of functions defined on a domain D and at this moment they are only maps, okay.

I am not even saying that they are continuous or something like that but then we have been looking at so many subsets of this collection of maps, so the 1st important subsets is those which are continuous maps from D to C union infinity, these are the continuous ones. So I will write CTS for continuous maps. And continuous maps from D to C union infinity makes sense because D is anyways a domain, it is a topological space, it is a subspace of the complex plane. And C union infinity is a very nice topological space, in fact even complete compact matrix space, okay.

Because it is, basically it is isomorphic to the Riemann sphere by the stereo graphic projection. So you have this and then there were smaller subsets, more important subsets, there was also the subset of meromorphic functions on D . So this is the subset of meromorphic functions on D and so this is meromorphic. So meromorphic means that you know these are functions which are analytic, that is holomorphic except for subset which is a subset of necessarily isolated points. And at each of those points the function on the phone, okay. So analytic except for pole, that is what meromorphic means.

And we have seen that this set of meromorphic functions is actually a field, it is a field extension of the complex numbers, algebraically it is a field. And then there is further smaller subsets of holomorphic functions on D or analytic functions on D . So let me write here analytic. That is a small subset and of course these are functions which are analytic everywhere on D, there is they do not have any singularities, right. And what we have, what we have seen is that all the fns are in H of D, then f is either in H of D or f is identically infinity, that is what we have seen, okay.

So let me, so let me write that here. We saw is all the fns are in H of D, then f is in H of D or f is identically infinity. This is what we are seen in the last lectures and it is a nice thing because it says that sequence of holomorphic functions, analytic functions can either go to infinity or it will go to analytic functions, that is all. You do not get, you do not get something weird in between. for example you do not even get a meromorphic function which is not holomorphic. Okay. You remember we used, basically we used 2 important things, we used the invariances the spherical metric that between version and the other thing we used was the Hurwitz theorem, okay, the theorem of Hurwitz.

Alright, now what I want to say is that similar theorem is true if if you take fns to be meromorphic. So if you take, what we will show today is that if all the fns are in M of D, then either f even M of D or f is identically infinity. So you are just extending this from the holomorphic case to the meromorphic case, that is what we want to do, okay. And that is also very nice thing. See the point is that, you know why all these theorems are important is that, you see on the one hand they are very believable, I mean you expect them to happen because under normal convergence good properties are preserved, okay.

You know from basic analysis, if you have normal convergence, which is locally uniform convergence, so essentially it is uniform convergence locally. And under uniform convergence all nice things happen. If you take uniform, if you take uniform limit of continuous functions, you get a continuous function, if you take uniform limit of analytic functions, you get analytic functions. If you take uniform limit of integrals, okay, then that is the same as integrating the uniform limit, you can interchange the limit an integral, you can interchange, you can do term wise differentiation if you are working with the series which is uniformly converging.

So uniform convergence is a very nice thing to suffer under uniform convergence you expect everything to go smoothly. So it is correct to expect that if you take a sequence of holomorphic functions, if it converges normally, that is locally uniformly, then the limit function is also holomorphic, that is correct to expect and that is what happens. Similarly if you have sequence of meromorphic functions, if it converges normally to limit function, the limit function is also meromorphic. The only thing that you have to worry about is the other extreme possibility which is that the function becomes infinity, that is a possibility you cannot ignore, okay.

But that is the worst thing that will happen and nothing in between these 2 happens. So it is important that, this is very very important because you know it should not happen that I start with a series of meromorphic functions of holomorphic functions. If there are holomorphic functions, they do not have any singularities, if they are meromorphic functions they have poles, okay. But in the limit suddenly I should not get a function which is crazy enough to have say essential singularities. Or I should not get a function which has a singularity which is nonisolated, I mean such horrible thing should not happen.

And you can expect such horrible things should not happen because of normal convergence, which is local uniform convergence and this is exactly what we are trying to show in these lectures. I mean this is, so you know this is always part of mathematics, you guess that something nice will happen but then to prove it you will have to do some work, he will have to use theorems, okay. So, well, so let us see. So 1^{st} of all I want to say that see if you take the 1^{st} thing I want you to understand is that if all these fns were not just maps but suppose they were continuous, okay, the f also becomes continuous.

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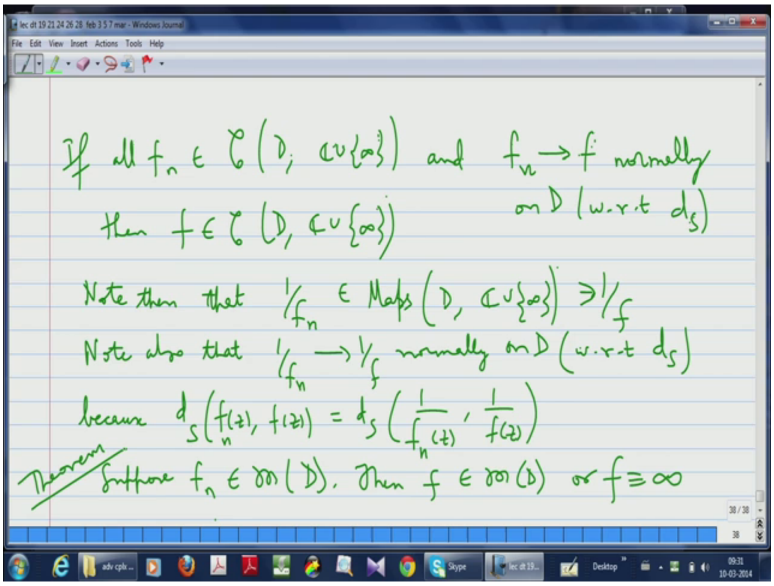

Because uniform limit of continuous function is continuous and a normal limit is a local uniform limit, all right. Therefore, so let me write that down, if, so let me use a different colour. See if all the, if all fn are continuous functions from D to C union infinity and the fns converge to f normally, normally on D and this always means with respect to the spherical metric. So you know here I must tell you that in the previous slide here when I say if fn, if fn are all holomorphic then f is holomorphic or f is identically infinity here. Of course I am assuming that the convergence is not just point wise but it is in fact normal.

fn converges to f normally, this is very important, this is very very important. Of course normal convergence I did not mention it then but it is important, okay. So point wise convergence is useless, point wise limits cannot be as good as you want them to be, okay. But normal convergence is very very important here, right. So suppose you take all the fns are continuous and assume that the convergence is normal on D, then of course f is also continuous and this is just the, the very well-known fact that the uniform limit of continuous function is continuous, all right, you know that.

And the point is that normal limit is something that is locally a uniform limit, okay, so it means that given any point, okay, because you are working on an open set, you can always find the closed disc surrounding centred with that point which is inside your open said, okay. And such a closed disc is of course compact and normal convergence promises you that the convergence will be uniform on that closed disc. In particular it will be uniform on the open disc which is a subset of the closed disc. So whenever you have uniform convergence on a set, you will always have, you will automatically have uniform converges on any subset, okay.

So you get locally uniform converges and therefore the locally the limit function f will become continuous. And if you and continuity is a local property, so if a function is locally continuous, discontinuous, if you have global functions. So that is why f is continuous, now what is very very important is the following thing. If you add this condition that, now notice that since you are working with values in the extended plane, this is very very important, not only fn makes sense, 1 by fn also makes sense. And not only f makes sense, 1 by f also makes ends, okay. So you see so let me say this, node then that 1 by fn also makes sense and 1 by f also makes sense.

So let me say, makes sense, so let me put something more general. 1 by fn certainly makes sense as map from D to C union infinity, okay. 1 by fn of z is very well-defined, if fn of z is infinity, then 1 by fn of z is 0, if fn of z is 0, then 1 by fn of z is infinity and if fn of z is a nonzero complex number than 1 by fn of z is the inverse of that complex numbers. So it is very well-defined. So this one by fn is also, these 1 by fns also makes sense, okay. And not only that similarly 1 by f also makes sense, okay. Any function on the domain which has values in the extended plane has an inverse, okay.

for example, if you take the worst-case which is the function which is identically 0, you do not have to worry, its inverse is the function which is identically infinity. But it is not an inverse in algebraic sense that you know the function multiplied by the inverse will not give you 1. You should not go ahead and write the function 0 multiplied by the function infinity is equal to1, that is not correct, okay. But this is a convention, to the convention, okay. So what we say is if you take a function which is identically 0, then its inverse function is defined to be the function which is identically infinity. But it is not an inverse in the algebraic sense, you have to be careful about that, okay.

fine but what is the advantage of this? Advantage of this is that you know if you are looking at meromorphic functions, then the inverses of meromorphic functions are also meromorphic functions, that is the advantage. So now what I want to tell you is that you see if, suppose I assume that fn converges to f normally, okay, then come on D, with respect to the spherical metric, then it also happens that one by fn converges to 1 by f normally on D with respect to the spherical metric. That will also happen, that is just because of the fact that the spherical metric is invariant with respect to inversion, okay.

So let me write that, note also that 1 by fn converges to 1 by f normally on D with respect to spherical metric, because the distance, spherical distance between fn of z and f of z is the same as the spherical distance between 1 by fn of z and 1 by f of z. This is just the invariance of the spherical metric under inversion, okay. I told you that the inversion on the complex plane, if you transport it via the stereo graphic projection to the Riemann sphere, what what it will give you is it will give you rotation of the Riemann sphere. It will give you a rotation of the Riemann sphere by 180 degrees, okay.

So and under under rotation, under rotations of a sphere, the distance, the spherical distance between 2 points on the sphere will not change obviously. So this is something that we have seen before. So the point is that it is beautiful, if fn converges to f normally, then one batsman converges to also 1 by f normally and this is, these are equivalent mind you. fn converges to f, if and only if one by fn converges to 1 by f, okay, it is a simultaneous statement. these 2 simultaneous statements and both of them are equivalent, okay. So the point is that, therefore what I am trying to tell you is the philosophy is as follows.

Whenever you are looking at functions with values in the extended plane, always think automatically of the reciprocal functions also, the inverse of those functions. They also make sense, okay, that is the advantage. now what I want to tell you is that you see, suppose, now let me do the following thing. Suppose all the , suppose all the functions fn were all meromorphic, okay, what I want to actually say is that f will be meromorphic and that is important result that we want to see. So now let me, let me assume that. Suppose fn are all meromorphic functions only, okay, then the theorem is that f is also meromorphic only.

So here is a theorem, then f is meromorphic or the other extreme case is that f is identically infinity, okay. So this is the theorem, okay. If fn is meromorphic, each fn is meromorphic, than the normal limit of fn namely f, that is meromorphic or it is identically infinity. What it means is that the limit function will have, if it all it has singularity is, they will only be isolated, they will be poles and you will not get anything worse than that, okay.

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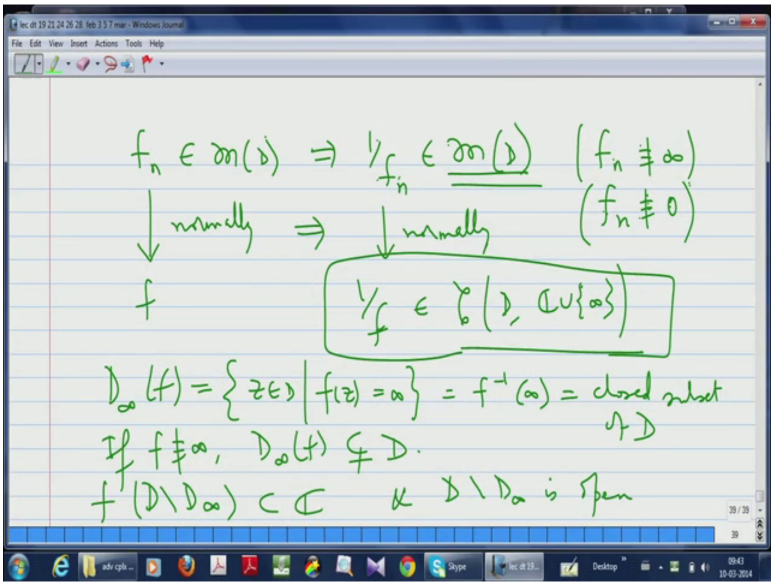

now what I wanted to understand is that, you see 1st and foremost fn are meromorphic, okay, and that will imply that 1 by fn are also meromorphic, okay. See because, see reciprocal of a meromorphic function is also a meromorphic function, right. And the, you see, so this, so this makes sense for. All this is okay if, with a small modification, all right. This is provided fn is not identically infinity, okay, because a function which is identically infinity, we do not, I mean we do not, you know it is an extra functions at we add, all right. We do not consider it to be meromorphic, we consider to be an extreme case, the function which is identically infinity, okay.

But its inverse will be the function which is identically 0 and that is a nice constant function, it is the holomorphic function also, okay. So when I write this, I am assuming that the fn that I am looking at is not identically infinity and when you may ask what if fn is identically infinity, in that case 1 by fn is defined to be the function that is identically 0 and that is by definition it is a constant function, the holomorphic okay. So that case is always, you know you keep it separate when a function is identically infinity, okay. And also this case that if fn is identically 0, then 1 by fn is defined to be identically infinity, all right.

And so you know you should also , I also throughout the case where fn is not identically 0, when fn is identically 0, okay, I throw out that case, because then 1 by fn becomes the function which is identically infinity and the function which is that typically infinity is not considered as the meromorphic function, okay. Because it has, meromorphic function should have only poles only at, you know it has only poles and they should be an isolated set of points. If suppose it goes to infinity but it cannot go to infinity everywhere, right. now what I want you to understand is that, see I want you to see that therefore, since 1 by fns are all meromorphic, 1 by fns are, including the case of 1 by fn being infinity, okay, all these 1 by fns are of course continuous.

Mind you because the set of meromorphic functions is actually contained in the set of all continuous maps. It is continuous because mind you at the poles you are allowing the value infinity for a function. So what you must understand is that all these 1 by fns are in fact continuous functions and therefore if you take, if fn tends normally to f, then because of the invariance of the spherical metric under inversion, 1 by fn will tend normally to 1 by f and 1 by f will certainly be continuous, that because each one by fn is certainly continuous. Even is 1 by fn is a function which is identically infinity, discontinuous, mind you, even the extreme case.

So what I wanted to , fn tends to f normally implies 1 by fn tends to 1 by f normally, this happens. And this implies that this 1 by f is certainly a continuous function from D to C union infinity and this includes the case when 1 by fn is, I mean when, when 1 by f is even identically the function which is infinity. Mind you the function which is identically infinity is here but it is not here, did not in the meromorphic functions by definition, right. fine, so what we want to show is that, we want to show this 1 by f is meromorphic, that is what we want to show.

now I want you to, now let me say another thing that I wanted to say but I did not. And that is, that how do you see 1 by fn is meromorphic, it is very very simple, you look at the places where the function fn has poles, this is an isolated set of points and 1 by fn will have 0 that those points, okay. Is a function has a pole at the point, then it is reciprocal will have 0 at that point and order of the 0 will be exactly equal to the order of the pole, okay. And if a function has a 0 at a point, then its reciprocal will have a pole. So you see if you take fn to be a meromorphic function, then there is, it is analytic outside isolated set of points, which are poles, okay.

And outside and isolated set of points you will get a open subset of D and inside that again the set of zeros will again be an isolated set of points. Because you see the set of zeros of an analytic function is always isolated. And this isolated set of points which are the zeros of fn, they will be the isolated set of points which are the poles of 1 by fn. So 1 by fn has only poles as singularities. So 1 by fn is meromorphic, you have to understand that, okay. So now you see what I want to say is that, so we had defined this set D infinity of f. This D infinity of f is the set of points in D where f takes the value infinity. So D infinity of S is, so let me write that, it is a set of all z belonging to D such that f of z equal to infinity and this is just f inverse of infinity and it is a closed set, it is a closed subset of D, okay.

And well and it is closed because the point infinity is the closed point in the extended plane, because it corresponds to the north pole on the Riemann sphere, okay, under stereo graphic projection. And the inverse image of a closed set under continuous function is closed, f is continuous, and so D infinity of f is a closed set. And the point is that if, if you assume that f is not identically infinity, then you are just saying that D, this is the same as saying that D infinity of f is a proper subset of T. D infinity of f is a proper subset, okay, because if D infinity of f is all of D, then it is the same as saying f is identically infinity.

So D infinity of f is a proper subset of D . What we want to show is that we want to show f is analytic except for poles. So you want to show that this D infinity of f which is a closed set, D s are certainly points where f takes the value infinity, okay. So you want to show that they are all poles and in particular you want to show that this D infinity of f, mind you, it is isolated. That is the, see that is the important point, it could be connected, it could be a curve, after all it is a subset of a domain in the complex plane. A subset of, a domain is an open set and an open set can contain curves.

It can even contain a sequence of points which has a limit, it can contain so many things. So it is not even clear that D infinity is isolated, is an isolated set of points, okay. So that we have to prove, he that is the important part, okay, that you to observe. So of course you know the, the way you handle it is by looking at 1 by f, okay. Because 1 by f is now there for you to, to, to play with because 1 by f is already continuous, you see that is the point. So you have it here, 1 by f is here, it is already continuous. All right, new and used it, all right. And 1 by fn converges normally to 1 by f, so you can use that. So let me say the following thing.

Let us take, let us take this D minus D infinity. If you take this D minus D infinity, this is would take f of D minus D infinity, this will go into the complex plane, okay. Because you have thrown out D infinity which is a set of points where f takes the value infinity, so at the other point which is in the point of D minus D infinity, f will take value other than infinity. So the only values other than infinity in the external complex plane complex values. So that means, and mind your D minus D infinity is open, D minus D infinity is open because D infinity is closed, all right.

And now you have the function f, you have the function f which is defined on this open set D minus D infinity and it is taking complex values and what is this f, this f is again a normal limit of fn. f is a normal limit of fn or all of D, so it is also a normal limit of fn on a subset of D, okay. If fn tends to f normally on D itself, so fn will tend to f normally or any subset of D. So since D minus D infinity is a subset of D, fn will tend to f normally on D minus D infinity. And what is this f on D minus D infinity, it is a complex valued function, all right. Just let me think for a moment. I am trying to look at, I am trying to locate the point where f is a common has values, takes the value infinity.

So I am trying to look at a point z0 in D infinity, so I want to say it is a pole. So I want to say is z0 belongs to D infinity, I want to say f has a pole z0. But the way to verify that f has a pole at z0 is to, is equivalent to verifying that 1 by f has a 0 at z0. So I will have to show that 1 by f has a 0, it will get 0 at z0, okay. But then what I will have to show is that 1 by f is analytic in the neighbourhood of z0, in the Delta neighbourhood of z0, I will have to show that 1st. And I must make sure, and then I will get that z0 is an isolated 0 of 1 by f.

1 by f being analytic in the neighbourhood of z0 will tell me that f is analytic in the Delta neighbourhood of z0 and z0 is a pole. And if, and now, this will tell me that f is meromorphic, this is what I will have to do. So how do I show that, see the ideas of the proof are similar to the ideas that we used in the earlier lectures, okay. So let me do the following thing.

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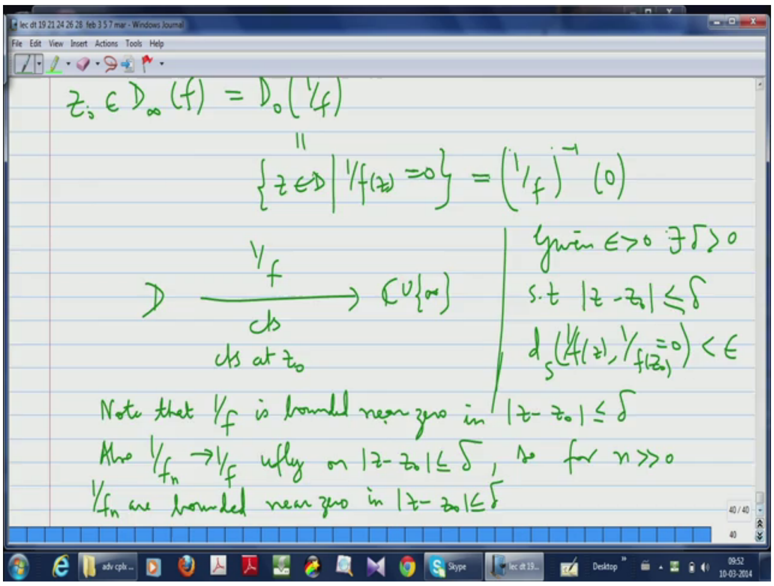

So D infinity of f is actually D0 of 1 by f, okay. So D0 of 1 by f is the set of all z belonging to D, such that 1 by f of z is 0, okay. Mind you 1 by f makes sense, it is a continuous map, okay.

And therefore 1 by f of z equal to 0 will be the locus of points where 1 by f takes the value, it is inverse image of 0 under 1 by f, which is continuous. So it is a, mind you it is a closed set, okay. So the singlepoint 0 is a closed set in C union infinity, any singlepoint is a closed set. So this is, so if you want let me write this, this is just 1 by f inverse of 0, okay, so take a point z0 in D infinity of f.

Take a point z0 in D infinity of f, then what happens, this is the point where f takes the value infinity, so 1 by f takes the value 0, okay. Since 1 by f take the value 0 and 1 by f discontinuous, okay, there is a small disc surrounding z0, in fact even a closed disc surrounding z0, where 1 by f is bounded, okay, because it is just continuity. C1 by f is a map from, it is a continuous map from D to C union infinity, okay. And if you use continuous, so in particular it means a discontinuous at z0 also, okay. That means that and 1 by f of z0 is 0, okay. So what it means, what is continuous, what will continuity tell you, it will tell you that given an Epsilon greater than 0, there exists a Delta greater than 0 such that if you make the distance between , so I can use the usual distance because now I am going to, because I am on D, if you make the distance between z and z0 less than Delta, okay.

Then the distance, the spherical metric between f of 1 by f of z and 1 by f of z0, by the way which is 0, this quantity is 0, that can be made less than Epsilon. This is just continuous, definition of continuity of 1 by f at z0. Given an Epsilon, okay, I can make 1 by fz to be as close as I want to 0 to within a maximum distance of 0. If I choose z sufficiently closed z0 and how sufficiently close, that is what is decided by the Delta, that is done, okay. And you know I can, I can, I can even put less than or equal to, that is because you know I can choose a smaller disc, such that, smaller disc centred at z0, radius is Delta, such that the closed disc including the circle, the boundary circle, that also like inside D, because D is after all an open set.

I can always choose this disc sufficiently small in D, okay, because D is an open set and then I make sure that even the boundary of the disc is inside D, I do that, right. And you see, what does, what does this tell you, this tells you that you see, the distance between 1 by f and 0, 1 by f is very close to 0 and you also have this fact that 1 by fns, they converge to 1 by f, mind you, normally. So the 1^{st} thing I want you to understand is that, since 1 by fn converges to 1 by f normally, 1 by fn will converge to 1 by f uniformly on this closed disc, that is because this closed disc is compact.

And because of that all the 1 by fns beyond a certain stage, they will also take values in the neighbourhood of 0, okay. And that means that in a neighbourhood of 0 all the 1 by, in the neighbourhood of z0, okay, all the 1 by fns beyond a certain stage, they are all going to be bounded. And that means that they are all going to be analytic, because mind you about the more in the lab and fns were all meromorphic. So 1 by fns are also meromorphic but you know a meromorphic function, if it is bounded at a point, then that has to be a good point.

If a meromorphic function is bounded in the neighbourhood of a point, okay, then you know, it cannot, it can assume, that point has to be a good point by the Riemann's removable singularity's theorem. What can happen in the neighbourhood of that point, I have some poles because it is a meromorphic function but it cannot have a pole because at the pole it will take the value infinity. I am putting the restriction that it is taking value close to 0, so it means that in the neighbourhood of z0, all these fns, 1 by fns, they are all going to be analytic. And since all the 1 by fns are analytic and they converge uniformly to 1 by f in this disc centred at z0, 1 by f becomes analytic at z 0.

And once 1 by f becomes analytic at z0, z0 becomes a 0 of 1 by f which is now in analytic function, therefore it is isolated. So z0 becomes an isolated 0 of 1 by f and that is, that will tell you that z0 will be an isolated pole for f. So what this argument tells you is that every z0 where f takes the value infinity is actually a pole of f and therefore f is meromorphic. And that, that ends the proof, okay. So let me write that down. So let me write these things in words.

note that 1 by f is bounded near 0 in more z minus z0 n equal to Delta, also 1 by fn converges to 1 by f uniformly, so I am abbreviating it as ufly on mod of z minus z0 less than equal to Delta because of normal conversation because mod of z minus z0 less than equal to Delta is compact. So for n sufficiently large, the 1 by fns are bounded near 0 in mod of z minus z0 lead than or equal to Delta, okay.

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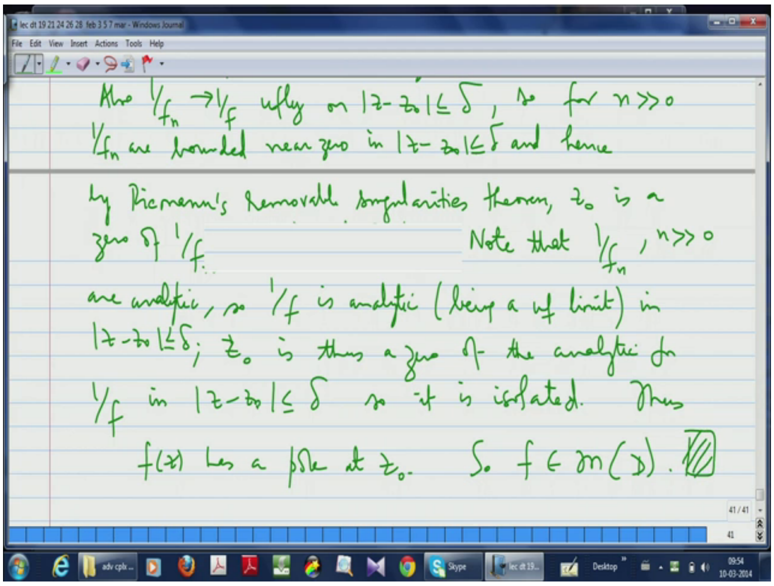

And, and hence, if you want Riemann's removable similarities theorem, z0 is 0 of 1 by f and of course there I am saying that 1 by f is, 1 by f is analytic because 1 by fns are all analytic, beyond a certain stage all the 1 by fns are analytic, 1 by fns tend to 1 by f and it is uniform convergence, therefore 1 by f is analytic. So 1 by f is analytic and it has a 0 at z0. So it is an isolated zero, okay. note that 1 by fn, n sufficiently large, are analytic, so 1 by f is analytic being a normal limit, being a uniform limit, so uf is abbreviation for uniform. So all this happens in mod z minus z0 less than equal to Delta, you may need to make Delta smaller if you want but that is not a problem.

The point is that, z0 is thus a zero of the analytic function 1 by f in model of z minus z0 in mod of z minus z0 less than equal to Delta, so it is isolated. Thus f of z has a pole at z0, so f is actually meromorphic function and that is the end of the proof. So it is a very nice fact that if you take a normal limit of meromorphic functions, barring the extreme case that the normal limit is identically infinity, the limit is again meromorphic function, okay. So that ends the proof of this fact, okay. And what we saw last class was that if you put the additional condition that fns are all holomorphic and you assume that the limit function is not identically infinity, then the limit function is also holomorphic.

And why is that too, now you can see, that if you take the fns additionally to be holomorphic, you know and if it is not, and the limit is not identically infinity, you know, by whatever you proof, now you know that the limit is meromorphic, okay. But if it is really meromorphic at a certain point, namely if it has a full letter certain point, then what will happen is that it is reciprocal will have a 0. And Hurwitz theorem will say that if 1 by f has a 0, then 1 by fns will start having 0 as n becomes large. But the moment 1 by fns start having zeros, fns will start having poles and that is not possible if you are assuming fns to be holomorphic. Therefore you see that all the, if all the fns are holomorphic, then f also has to be holomorphic, or it has to be identically infinity, okay. So I will stop here.