**Advanced Complex Analysis-Part 2.**
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**Lecture-21.**
**Completion of Proof of Hurwitz's Theorem for normal Limits of Analytic functions in the Spherical Metric.**

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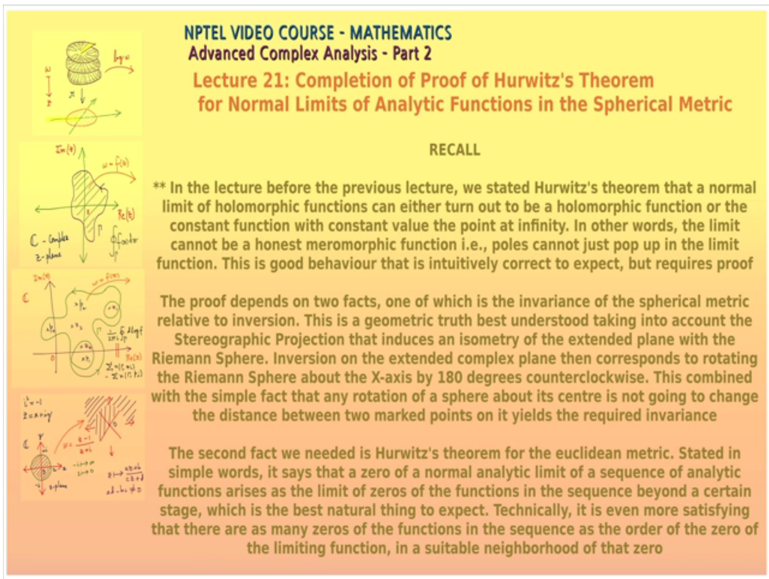


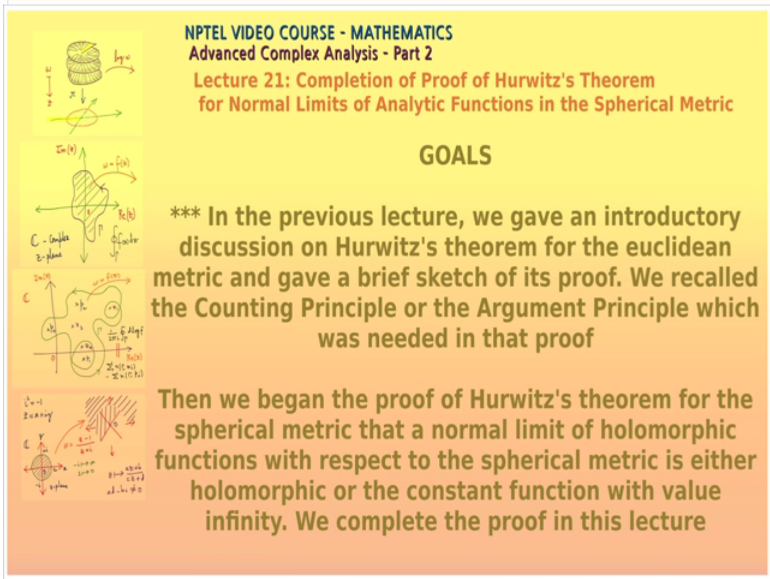

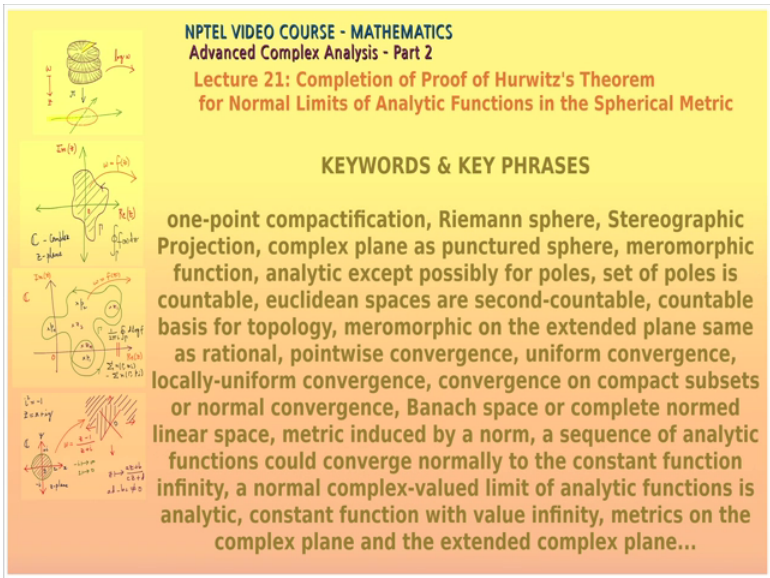


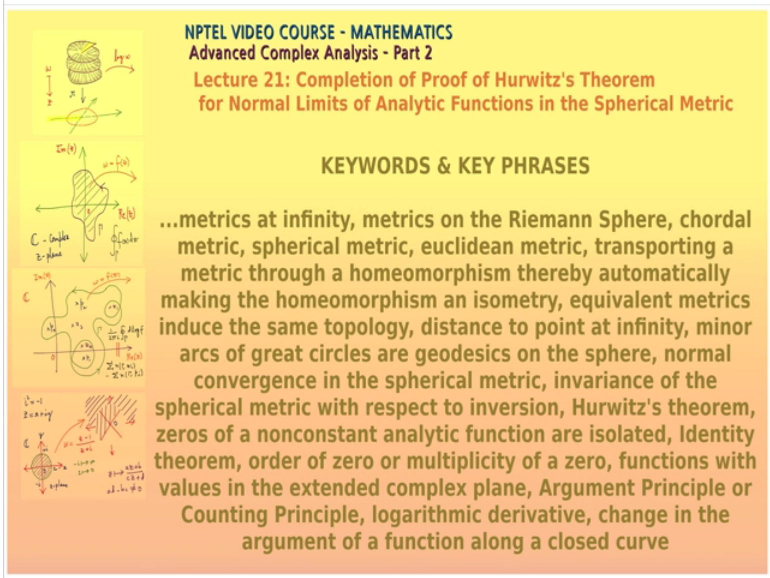


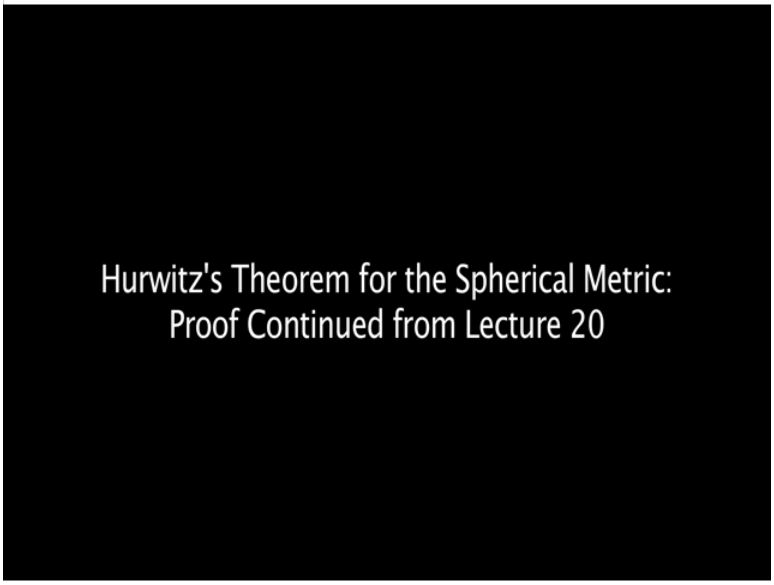


So this is the continuation of the last lecture more or less . So you see our situation is that we have taken a sequence fn of functions, analytic functions defined on a domain in the complex plane, okay. And we assume that fn converges to f, fn converges to f, where f is now a function again defined on the same domain, okay, on which each of the fns is defined. But now you are allowing the f to take the value is infinity. So f is considered as a function not into the complex plane, it is considered a function in c union infinity, the external complex plane, okay. And the convergence is normal and when we say the convergence is normal, we mean that it is converging, this uniform on compact subsets of the domain, okay.

You call the domain as d, okay, and since you have taken the value infinity, the convergence the point wise convergence is with respect to the spherical metric, okay. So fn of z converges to f of z in the spherical metric, this is, this is the same as saying that the spherical distance between fn of z and f of z, that goes to 0. And you wanted to go to 0 normally on d, okay. And you know the reason why we are using the spherical metric, because f of z can take the value infinity, then you will have to measure distance of the point on the complex plane to the point at infinity, okay.

And for that, you have to do it only on the riemann sphere, so essentially use a spherical metric on the riemann sphere, okay. Fine, so what we were trying to grow, we were trying to prove this very important theorem that you know if you take sequence of analytic functions, suppose it converges normally to limit function, then the limit function is either analytic, that is holomorphic or completely it is infinity, okay. And you do not get anything between, all right. So what we do, we try to prove this by using, by applying hurwitz's theorem and also by using the fact that the spherical matrix, the spherical metric is invariant with respect to inversion, okay, we have to use these 2 facts.

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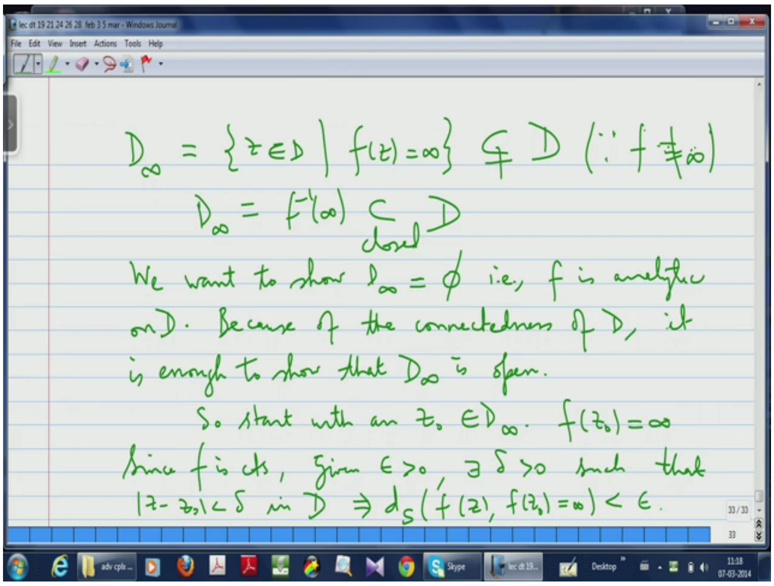


And how was the going to use it? That is what we are trying to do, so we have assumed that the limit function is not identically infinity, okay, so that means that this, so if you now look at what we have written in the last lecture in the last couple of lines. D infinity is the set of all z in d where f takes the value infinity, that is a proper subset of d and this is, this is because, f is not identically infinity . So d infinity is a proper set and mind you d infinity is a closed subset because you see i have already told you last time that f has to be continuous because uniform limit of continuous function is continuous and normal limit in the local uniform limit, so this also continuous.

And locally continuous is same as containers, okay, because continuity is a local property. So f is certainly a continuous function. And f inverse, inverse image of a closed set under a continuous map is a closed set, the point at infinity is a closed point, okay. The subset consisting of only a single point is closed in the external plane because topologically it is the same as the riemann sphere and the infinity, the point at infinity corresponds to the north pole. So f inverse of infinity is exactly d infinity and that is closed, okay. What we are trying to show is that we are trying to show d infinity is empty, okay.

We want to show, we want, we want to show d sub infinity is a null set, which means that d is actually, so that will mean that that is f is analytic on d. Because you know that the fact is that if you show t infinity is empty, then you are saying that every point s takes only a finite complex value, okay, it does not take the value infinity. So f is not actually mapped into c union infinity, it is actually mapped into c. And then you know if you have, if you have a complex valued function which is a normal limit of analytic function, then it is analytic.

This is something that you have already seen, this is part of for example the 1st course in complex analysis where you essentially have to use cauchy's theorem and you had to use morera's theorem, okay. So all you have to show is d infinity is empty. And now how do you, how do you show this, we exploit the fact that d is connected set. See d is a domain in the complex plane, in the complex plane, so d is then open connected set. Of course we are always worried about only nonempty sets, d is nonempty and mind you d, since d is connected, we use, try to use this very important property of the connected space.

For a connected space the only subspace that is both opened closed has to either be the null set or it has to be the full space. So what you try to do is that you try to show d infinity is open, okay. So d infinity is already closed, suppose we show d infinity is open, so d infinity has to be either, so it is an open and close circuit of d which is corrected, so d infinity either has to be a null set has to be all of d. But it is not all of d because i have assumed f is not identical infinity. So d infinity has to become the null set and you are done, and the theorem is proved.

So let me write that down, because of the connectedness, the connectedness of d it is enough to show that d infinity is open, okay. That is enough to, it is enough to show that, what does it mean? It means that if you give me a point of d infinity then there is a whole neighbourhood surrounding that point which is also in d infinity, that is what openness means. It means that every point is an interior point, so in other words what does it mean, it means that if you take a point, i did not in d infinity? Namely a point z0 where f takes the value infinity, then there is a small neighbourhood, that is a small disc surrounding z0 where also f takes the value infinity. That is what you have to show.

If f is infinity at a point, then f is infinity for all points in a small disc surrounding that point. This is, this is what we will mean to say that d infinity is an open set, okay. So that is what we are going to do, that is exactly what we are going to do, all right. And for that we are going to use the same, we are going to use the invariance of the spherical metric with respect to inverse. And how we will do this, you will see is a. So let us start with, and z0 is in d infinity, okay. And of course i should tell you that, of course i am assuming d infinity is nonempty to begin with.

If you want to be very logical, you can say that let us assume t infinity is nonempty because if t infinity is empty, we are anyway done, okay. And so you assume t infinity is nonempty and then prove and get a contradiction, okay. So if you want to be very radical you can say like that. So in any case am assuming, i start with the point, this is, this is often a feature of mathematics. See finally d infinity is empty, that is what you want to show. You want to show that the set is empty, you want to show that there is no point in that set. But then it is a roundabout way you do it, what you do is, you assume it is nonempty and then you try to, once it is nonempty, you try to get some properties of the set with will give you a contradiction.

So in this case you assume t infinity is nonempty, okay and then you get the fact that t infinity has to be everything and that is not true because f is not identical infinity, okay. So this often happens in mathematics. So you start with the point z0 in d infinity, so f of z0 is what it is. And, now mind you f is a continuous map, okay, f is a continuous map into c union infinity. And it is, therefore discontinuous at z0 also. And what i want to say is that since discontinuous at z0, okay, you can find the sufficiently small neighbourhood of z0 such that all the function values on that neighbourhood are close to infinity to within whatever epsilon distance you want.

And mind you you have to now use the spherical metric, okay. So i am using the continuity of f at z0. So since f is continuous, the given epsilon greater than 0, there exists it has to greater than 0 such that, well mod z minus z0 less than delta in d will imply that, i should write this with respect to spherical metric, d sub s, f of z, f of z0 by the way is infinity, this can be made less than epsilon, okay. So you see i have to use the spherical metric, okay. So i am just saying that since f takes the value infinity at z0, in a sufficiently small neighbourhood of z0, f has to take value close to infinity.

So the difference between the function f z and infinity which is f of z0, that can be made as small as i want if i choose sufficiently small neighbourhood, that the neighbourhood of z0. And of course i have to choose it in d of course, because i want f of z to make sense. F of z makes sense only for z in d, okay. Fine, now you see, mind you what this means, you see try to understand what it means. It means that if you take this wall disc centred at z0, radius delta, then the image of that disc lies in the neighbourhood of infinity, because the distance between f of z and f of z0 which is equal to infinity is less than epsilon means f of z lies in the neighbourhood of infinity.

That means that small this centred at z0 radius delta is mapped to the exterior of sufficiently large circle, okay. So you must think that as epsilon becomes smaller and smaller, you are getting the exterior of sufficiently larger and larger circles, okay. So if you are thinking in in terms of radii of circles, you should think of 1 by epsilon often by epsilon square of something like that, as epsilon goes to 0. Because then 1 by, are positive power of epsilon greater than 1 will go to infinity, okay. Fine, this is what it means. Now, you see, so you know, so let me tell you basically what the philosophy is all about.

See the idea is very very simple , let me give you the idea of the proof. You see f is, so there is this neighbourhood of z0, okay at z0 f is infinity, okay, and that is the small neighbourhood of z0 where f is close to infinity, okay. So you know if i if i draw a diagram, it is going to be something like this, so let me draw the diagram, it helps to draw a diagram.

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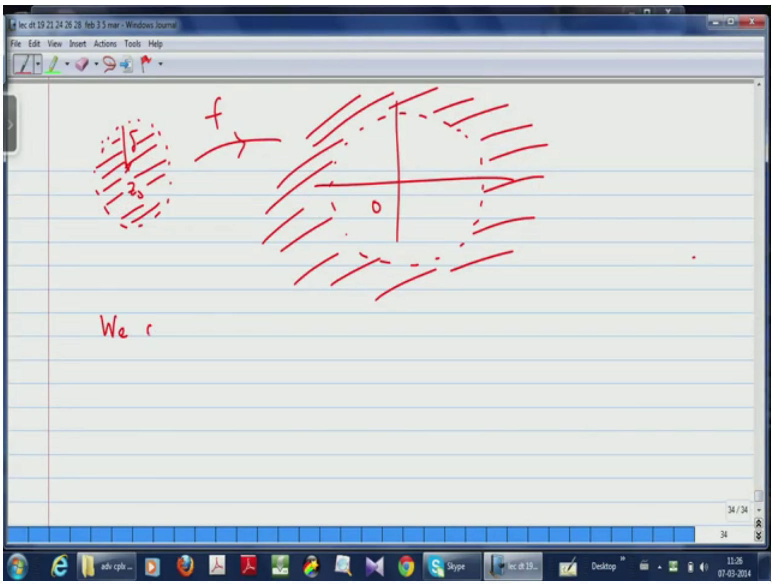

So here is z0 and this is a small disc surrounding z0, radius is delta and what f is doing is that it is mapping this onto neighbourhood of infinity which is you know the exterior of sufficiently large disc, this is what is happening. So the interior of this, okay is going to be exterior of the sufficiently large disc. And this is, this disc is centred at z0 is radius delta sufficiently small and the image of this disc, i am not saying the image is all of the you know exterior of that large circle but it is a subset of that. So yah, so this is a situation.

See what this tells you therefore is that f is certainly bounded away from 0 in the neighbourhood of z0, right. Because at see at z0 f is taking the value infinity, all right and in the neighbourhood of z0 it has to take value is close to infinity. And values close to infinity are certainly nonzero values, because values close to infinity are supposed to be values which lie on the exterior of the large, the exterior of a large circle. So f is going to be nonzero in particular, okay. And now imagine your fn converges to f, uniformly on compact subsets of t, that is given to you, that is given to us from because we have assumed fn converges to f normally on d, all right.

So because of, so in particular if i choose this delta sufficiently small so that even the boundary of that circle, mod z - z0 equal to delta is also inside d, i can do that if i make this is little bit smaller you want, okay. Then mod z minus z0 less than equal to delta becomes a compact set because you know it is now closed and bounded, it is a compact subset of d. And therefore fn will converge to f normally on that, right. Because if, in fact uniformly on that because it is a compact set. So if i choose delta sufficiently small, i can make sure that on this close small closed disc centred at delta, centred at z0 radius delta, the convergence of f n to f is uniform, okay.

But then f is never 0 there, f is nonzero on that common that closed disc common that small disc. Because of this, f values are in the neighbourhood of infinity, so f is not 0. So that means because of uniform convergence, fns is also nonzero beyond a certain stage in that, in the closed disc. And if fns are nonzero, they are nonzero analytic functions, so 1 by fns will become holomorphic, they will become analytic, okay. And you know fn converges to f in the spherical metrics because of the property of the invariance of the spherical metric with respect to inversion, fn converges to f in the spherical metrics will tell you that 1 by fn converges to 1 by fn, 1 by f in the spherical metric, okay.

So 1 by fn will converge to 1 by f and the 1 by fns are all analytic in the disc and 1 by f therefore will become analytic in the disc. But what is 1 by f of z0, it is 1 by infinity, it is 0. So z0 becomes 0 for 1 by f and 1 by fn is the sequence of analytic functions that is converging normally to1 by f in the disc. Apply hurwitz's theorem, what it will tell you that all the 1 by fns beyond a certain stage, they will have zeros, as many zeros with multiplicities as the zeros z0 of 1 by f, okay. But then you see if one by fn has zeros, that means fn will have more poles, all right.

But fn are all analytic, how can they have poles, that is the contradiction. Therefore the moral of the story is that you get a contradiction and therefore, so you can, see you can either see there is a contradiction or you can go one step further and say that see you can look at where hurwitz theorem will go wrong. See hurwitz theorem can go wrong in the following sense. If you take a sequence of analytic functions, if they are converging normally to limit function, okay, then hurwitz theorem says that 0 of the limit function come from the zeros of the functions which converges to that limit, okay. It comes as a limit point of zeros are functions that converge to that limit function.

But then there is one extreme possibility, the limit function itself could have been identically 0, okay. If the limit function is identically 0, okay, then that, then that is a case which we do not deal with in the actual hurwitz theorem. In the hurwitz theorem we always assume that you have an isolated 0 of the limit function, so the only way this hurwitz theorem can fail to apply here is that 1 by f becomes identically 0. But then if you think, if you see that 1 by f becomes identically 0, you are just saying that f is identically infinity in the neighbourhood of z0.

And that means that wherever f is infinity, if you take a point where s is infinity, that there is a neighbourhood surrounding the point where again f is infinity, so d infinity is open and we are done, okay. So there are both, there are these ways of looking at it which will give you the proof of the theorem, okay.

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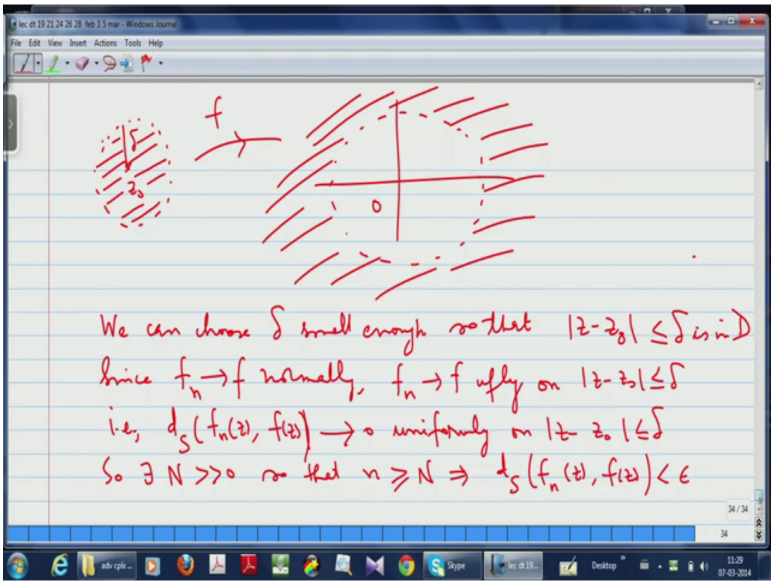


So now let me write down things inverse, okay. So let me write this, we can, we can choose delta small enough so that mod z minus z0 less than equal to delta is in d, okay. And now you see you also have this uniform convergence, since fn of, fn converges to f uniformly, i should say normally, fn converges to f uniformly and i am abbreviating uniformly to ufly on mod z - z 0 less than equal to delta, okay. And well note that, so what does this mean? This means that if for the spherical metric if you take fn of z and f of z, this, see this distance can be made lesser, i mean this distance goes to 0 uniformly on mod z minus z0 less than equal to delta, because of course mod z minus z0 less than equal to delta is compact, okay.

And what is uniform convergence means? Uniform convergence means that you can an index, capital n large enough such that for all small n greater than capital n, you know you can make this spherical distance is less than epsilon differential you want. So mind you feel already started with some epsilon, let us keep that epsilon, so there exists an n sufficiently large so that n greater than equal to, n greater than or equal to, small n greater than equal to capital n, greater than equal to capital n implies that the distance, spherical distance between fn of z and f of z can be made less than epsilon but here is the uniform, here is the uniformity of the convergence.

For all z in mod z minus z0 less than equal to delta independent of n. So the point is that you can choose this capital n in a way that it has got nothing to do with this z, that is the uniformity of the convergence, okay. In general if it is point wise convergence for the capital n will depend on epsilon of course but it will also depend on the z, particular z that you are looking at, the point z. If the uniform convergence is that naye you have this capital n depending only on epsilon and not depending on z, z could have seen anything.

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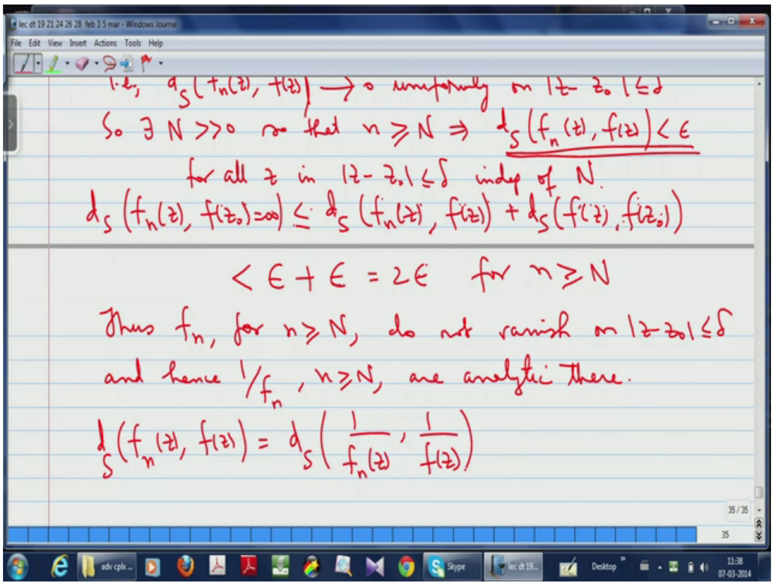


So let us look at this, d sub s of, i am trying to compare fn of z and f of z 0 which is infinity and this is by triangle inequality it is d sub s fn of z and now you could f of z, okay. Okay, and then put d sub s, f of z and f of z0, this i can do, put fz, okay, you introduce this fz as the 3rd point, the final apply the triangle inequality, okay. If you do this, see this is certainly less than, ds, fn z, fz, that is less than epsilon as i underlined above, so will get an epsilon. Plus the ds, fz fz0 is also less than epsilon, that is because of continuity of f at z0.

So i will get epsilon plus epsilon is to epsilon, okay. So what this will tell you is that, this will tell you what we want. And you know this is for, this is for n greater than equal to capital n. So, what this will tell you is that beyond a certain stage, all the fns, they are in neighbourhood of infinity, so they do not vanish and that is what i want. I want all the fns not to vanish your dissidence taste. Why, because then i can invert them and say that they will, the inverses will also be analytic. So thus, so this is what i want, the fn for n greater than equal to small n greater than equal to capital n do not vanish on mod z minus z0 less than equal to delta and hence 1 by fn, n greater than equal to capital n are analytic there, analytic or holomorphic there, okay.

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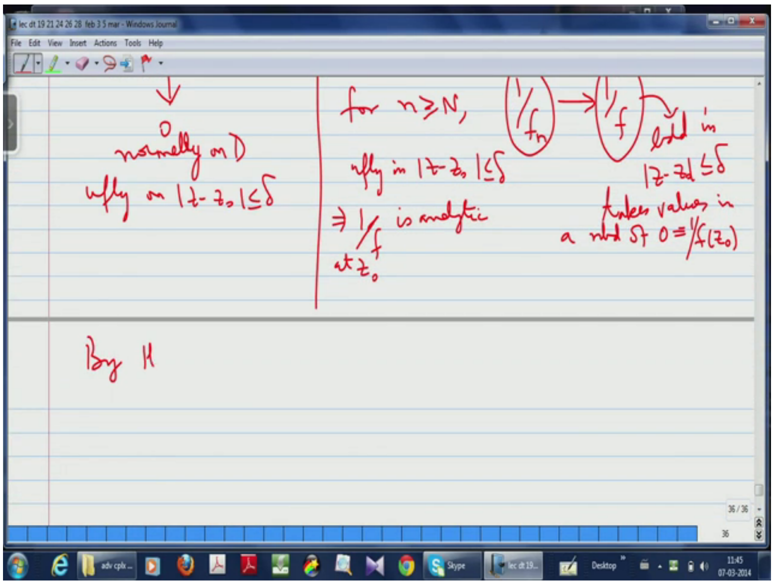


I wanted to invert the fns and why i wanted to invert the fns because you see i want to use this you know the invariance of spherical metric with respect with version. So now ds of f of fn of z, f of z, this is the same as ds of 1 by fn of z, 1 by f of z, you have this. Okay, here is where i am using the invariance of spherical metric with respect to this inversion. And mind you what is it that we have, what is basic assumption, our basic assumption is that this fellow on the left goes to 0 normally on d, okay. This is our assumption, original assumption.

But that quantity is equal to the quantity on the right side, okay. And you see but there is a small thing, this is, this, the quantity on the left side goes to 0 normally on d, it goes to 0 uniformly on mod z minus z0 less than equal to delta because it is a compact subset of t. So uniformly on mod z minus z0 less than equal to delta, that, let me write that. But see this equality is evident, that is valid everywhere but the only problem is, you know i want, see i want 1 by fns to be analytic, okay. And that does not happen everywhere, that happens anyway on mod z - z 0 less than equal to delta.

So see i am worried about equality only in mod z minus z0 less than equal to delta because 1 by fns are analytic there. For small n greater than equal to capital n, okay, 1 by fn converges to 1 by f, okay and this is uniformly in mod z minus z0 less than or equal to delta, all right, this is uniform convergence. And the 1 by fns, these 1 by fns are all analytic. They are all analytic in mod z minus z0 less than equal to delta. Okay. So you have a sequence of analytic functions that is converging to limit function, okay and this is, this converges is actually uniform convergence.

And mind you the limit function is also complex valued, that is the big deal. So you notice, f is, f takes values in the neighbourhood of infinity, right. F at z0 is infinity and in that small disc surrounding z0 f takes value in the neighbourhood of infinity. That is the reason why the distance, spherical distance between f of z and f of z0 which is infinity is less than epsilon, that is what it means. So f takes values also in the neighbourhood of infinity there and if f takes values in the neighbourhood of infinity, f is not 0 of course. Okay. And 1 by f will become very small because f is very large.

And 1 by f is bounded, so the moral of the story is that this is a 1 by f is a not in that neighbourhood mod z minus z0 less than equal to delta, 1 by f is a complex valued function, that is it is a bounded complex valued function. That is a big deal there. It does not take the value infinity, 1 by f never takes the value infinity, okay, because f takes values in the neighbourhood of infinity. So 1 by s, this is bounded in mod z minus z0 less than equal to delta, okay.

Now we are again using this standard theorem from our 1st course in complex analysis that your normal limit of analytic functions, if the limit function is also complex valued functions that the limit function is continuous and in fact it is analytic. So the moral of the story is, see all this pays to say that 1 by f makes sense and 1 by f is complex valued. Of course 1 by f, 1 by f z0 is going to be 0 because 1 by f z 0 is 1 by infinity which is 0 according to our convention. So this 1 by f is in the neighbourhood of 0 actually, it takes values in neighbourhood of 0 because f takes values in the neighbourhood of infinity. So okay.

So let me write that here, it takes values in a neighbourhood of 0 which is actually 1 by f z0. Okay. Now what does all this tell you? All these things will now tell you that 1 by f is analytic at z0 and of course, at z0 what happens? At z0 1 by f is 0 because it is infinity, f of z is infinity, f of z0 is infinity. Therefore z0 becomes a 0 of analogy function and what is the property of analytic function? It is isolated. So the moral of the story is that 1 by f is analytic at z0, z0 is the zero of 1 by f, now appeal to hurwitz theorem.

Now appeal to hurwitz theorem and what will hurwitz theorem then say? Hurwitz theorem will say that if 1 by f is not identically 0 then z0 will be an isolated zero for 1 by f. And suppose it has a certain order l, then all the fns, all the 1 by fns for n sufficiently large will also have l zeros in the neighbourhood of z0. But zeros of 1 by fn are the same as poles of fn, right. And that is not allowed because fns r: analytic, can they have poles. So you cannot apply hurwitz theorem, you should not be able to apply hurwitz theorem. The only way out is that 1 by f should be identically 0 in that neighbourhood.

And 1 by f being identically 0 in that neighbourhood is the same as f being identically infinity in that neighbourhood and therefore we want some okay. So you see here is where hurwitz theorem comes in, all right. So let me write that down.

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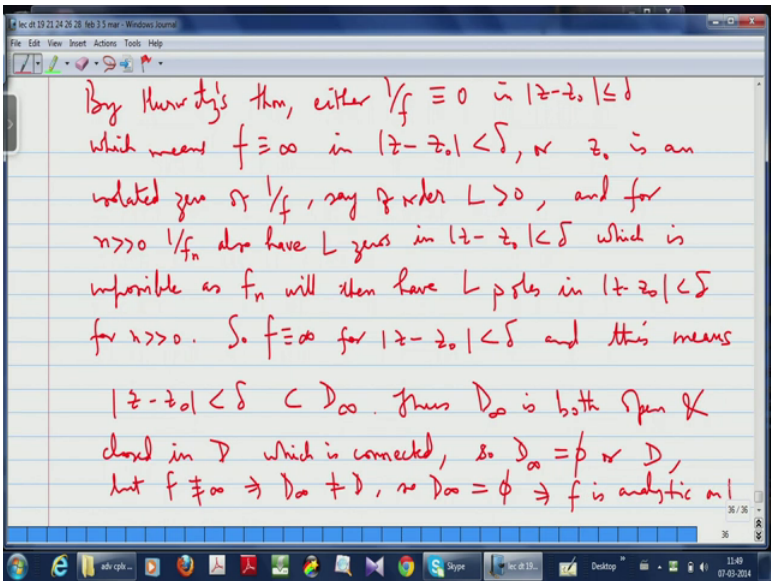


So let me write this here, by hurwitz theorem either 1 by f is identically 0 and mod z minus z0 less than equal to delta which means, so i should say, yes which means f is identically infinity in mod z minus z0 less than delta, right. Or z0 is an isolated of 1 by f free of order l and for n is recently large, 1 by fn also has l zeros in mod z minus z0 less than equal to delta which is impossible, which is impossible as fn will then have l pole in mod z minus z0 less than delta for n sufficiently large, okay. So what this tells you is that the only way out, so f is identically infinity, that is the only way out for mod z minus z0 m delta, okay.

And this mean that mod z minus z0 less than delta is contained in d infinity, okay. So you start with, start with z0 in d infinity, i am able to find whole disc surrounding z0 which is also in d infinity, so d infinity is open. So t infinity is open but already we know d infinity is closed, so d infinity is both open and closed subset of, it is both open and closed subset of d which is connected. So there is no other choice, it has to either be d or empty but it is not d because f is not identically infinity, so it has to be empty, just the empty set and we are done.

Thus d infinity is both open and closed in d which is connected, i should remove the comma here, so d infinity at the null set or d but f not identically infinity implies that d infinity is not d. So d infinity is a null set and this means f is analytic on d. And that finishes the proof, okay. So this is a very very nice theorem, okay. So if a sequence of holomorphic functions on a domain converges the spherical metric normally, then the limit function is either holomorphic, that is analytic or you go to the other extreme, the limit function is identically infinity, you do not get something in between.

And the idea is what could you get in between, well if you want something mild you can have a meromorphic function which means that you get some points which are isolated where the limit function develops poles, okay. But what this theorem says is that it simply cannot develop a port somewhere, okay. And you know now the reason, philosophically why f cannot develop a pole at a point because you see if fn converges to f in the spherical metric and f develops a pole at a point, then because of the invariance of the spherical metric with respect to inversion, 1 by fn will converge to 1 by f. But then 1 by fn will converge to 1 by f and if f has a pole at the point, 1 by f will have a 0 there.

So 1 by fns will start having zeros by hurwitz theorem. And they will give rise to poles of fn, means it is not possible, okay. Of course there is a much worser thing that you can expect, that this fn converges to f, of course you know this f, if you look at the locus where f is not infinity, that is of course is an open set, because that is the complement of d infinity. And on that open set f is going to be analytics, there is no problem, it is a honest complex valued function which has a uniform limit of analytic function, so it is analytic, there is no problem. But what kind of d infinity is very very, you know, it is very mysterious.

What would have happened if this d, what is, what, what prevents d infinity from being a set with nonempty interior, okay? Why should it be an isolated set of points? If d infinity is isolated set of points, it means that f is a meromorphic function but why should it be an isolated set of points? Why cannot d infinity be a curve? Okay, if it is a constant d infinity, then it means that your f has nonisolated singularities, okay. Why should, why should d infinity, you know if you want f to be meromorphic, the condition is that d infinity must be an isolated set of points.

So the next theorem that you are going to prove in the next lecture is that if you draw drop the assumption that the fns are holomorphic, assume that they are meromorphic, then also the limit function f will be meromorphic, it will not be any worse. That is d infinity will only be a set of isolated points. And it cannot be, it cannot contain nonisolated points, so you cannot have a sequence of, sequence of meromorphic functions. If it goes to a limit function, then that limit function, you know it cannot have horrible non-isolated singularities. It can only isolated singularities and they have to be only be poles, okay. This is again a very good thing and we will prove this in the next lecture, okay.