Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

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Lecture No 20

Introduction to Hurwitz's Theorem for Normal Convergence of Holomorphic Functions in the Spherical Metric

Okay so this is the continuation of the previous lecture and we were just discussing Hurwitz's theorem which (())(1:46), so let me again just recall what we are trying to show is the following if you have a sequence of analytic functions on a domain in the complex plane and suppose you assume that the sequence converges normally to a limit function and you allow this exception that the limit function can take values, the value infinity then what can happen is either the limit function is completely analytic function or it is completely the constant function infinity namely the function that takes every point infinity okay.

So in other words and of course here when I say I am including functions that take the value infinity it means that I am not just taking complex valued function, I am taking functions with values in the extended complex plane which is the complex plane along with the point at infinity okay denoted by this symbol infinity and of course the convergence whenever the point at infinity is concerned the convergence should be taken with respect to the spherical metric okay which is the spherical metric on the Riemann sphere transported via the stereographic projection to the extended plane okay and you must also remember that the convergence in the spherical metric is the same as the convergence in the Euclidean metric on the complex plane so far as domains of the complex plane are concerned alright so in the proof of that result we needed Hurwitz's theorem.

You see I wanted to understand the beauty of that result, the beauty of that result is following, see when you do a 1st course in complex analysis and you are not worried about the value infinity okay you are only worried about complex options then you know that if you say a uniform limit of analytic functions you will get an analytic function and more generally you need not take uniform limit but you can take our local uniform limit which is for example (()) (3:50) by taking a normal limit, a normal limit is a locally uniform limit. So even if you take a locally uniform limit or a normal limit of analytic functions you will get an analytic function this is what you study in the 1st course of complex analysis.

The proof involves just Clausius theorem and Morera's theorem okay at now what we have to do is you see we have our aim is to prove the great Picard theorem, the little Picard theorem, the Picard theorems and the problem is that to prove that you will have to worry about infinity as an isolated singularity okay and the problem is that you will have to worry about Meromorphic functions and families of Meromorphic functions you should do topology on a space of Meromorphic functions and the fact is that you allow the value infinity a Meromorphic function becomes a continuous map into the extended plane. See if you take an analytic function and suppose it has a pole at a point okay then as you approach the pole the modulus of the function goes to infinity, so the function goes to infinity.

So the function of course becomes discontinuous the usual sense but then if you allow the function to take the value infinity you think of infinity as actually a value and where it is the extra point that you have added get the one-point compactification of the complex plane. It is the point at infinity in the extended complex plane, mind you the extended complex plane is a nice topological space it is a compact matrix it is a complete metric space okay. It is because it is just equivalent topological isomorphic to the Riemann sphere which have all these properties and now the point at infinity can be seen on the Riemann sphere as the North pole.

So you can think of it as a valid point and now if you take a Meromorphic function and think of it as taking values not just in complex plane but also allow it to take the value infinity then the Meromorphic function becomes a continuous function because what you will do is at a pole you will define its value to be infinity and this definition is continuous okay, okay it is continuous for the topology on the extended complex plane or for example if you want to use a topology and that is of course the topology induced by the spherical metric on the extended plane okay. So now you see by allowing infinity we are value okay and taking continuous functions which can take the value infinity you are also allowing Meromorphic functions, now you see what you have done is? You have jumped from holomorphic functions on a domain to all the way to Meromorphic functions on the domain which means you are allowing even for functions with poles okay and then you go one step further and say that you also allow the constant function infinity.

Maybe the function that assigns every variable to the value infinity okay it is a constant function infinity and mind you constant functions are continuous in any sense of the term okay, so now since all this has happened now if you again take normal limit of analytic functions, a normal limit of holomorphic functions then you know of course if everything is happening in the complex plane you are only worried at complex values and if this is really a usual convergence okay then you will get the limit function to be analytic that is nothing more but the pointers now you are allowing the extended complex plane, you are allowing the value infinity and why could it not happened that sequence of analytic functions convergence in the limit to Meromorphic functions.

So suddenly a pole some poles can pop up in the limit that can happen right you do not expect it to happen by continuity okay but we have already seen that you can get a function which is identically infinity example I told you take the domain which is the exterior of the unit circle and you take the function Z, Z square, Z cube, et cetera so the nth function is Z power N. Now that sequence of functions, it is a sequence of analytic functions mind you in fact entire functions that sequence converges if you look at it in the usual sense it will not converge because you take any value Z with a mod Z greater than 1 Z power n will diverge because mod Z power N will go to infinity because mod Z is greater than 1 but if you now include infinity as a value and think of the extended complex plane, Riemann sphere in disguise okay.

Then this Z power N will tent to infinity, it tends to a point a value in your set and a sequence of function Z power N at converges to the constant function infinity at every point outside the unit circle and low behold this convergence is even normal, this convergence is even uniform on compact subsets with respect to the spherical metric that is amazing thing. Once you include the value infinity even you get normal convergence okay but the point is that it is converging to the constant function infinity. Now what is the guarantee that something… instead of converging at all points where infinity why cannot it just converge at some points to infinity? Why cannot it converge on to infinity only at say an isolated set of points? That means you are going to get a Meromorphic function okay or why cannot it converge to infinity on some subsets which is for example not even isolated. Why cannot such strange things happen?

Okay so that is the theorem you trying to prove, the theorem we are trying to prove is that either what happens is normal what you expect normally that the limit function is actually nice complex valued analytic function or the other extreme happens. Namely the limit function is always infinity, it is a constant function infinity you do not get something in between, you do not get the Meromorphic functions with poles they will not come in between okay so you do not get that. So it does not go wrong in that sense and that is the theorem we are trying to prove okay.

See we have to be worried about all these things because you are allowing the value infinity okay once you allow the value infinity anything can happen and then you will have to be very careful and you have to prove things carefully. See these are all basically ingredients that you really need but to understand very well if you want to understand the Picard theorems, the proof of the Picard theorems that is the reason why I am doing this pretty slowly. So now in order to prove that so how will be prove that what we will do is we will take sequence of functions on some domain in the complex plane and assume that the sequence converges to a function uniformly on compact sets okay that is normally and of course this convergence will be…you are allowing the limit function to take the value infinity.

So the limit will be in functions with values not in the complex plane but functions with values in the extended complex plane as of course the convergence is going to be normal alright and what you want to show is that suppose the limit function is not the function which is identically infinity, you have to show that the limit function is actually analytic okay. The limit function is either analytic which means it is a (())(11:19) complex valued analytic function or it is infinity that is all there are only these 2 cases there is nothing in between okay that is what you are trying to prove.

So what we will do is essentially for that you know for proving that we need 2 facts one fact is the important property that the spherical metric is invariant with respect to inversion, so the map is Z going to 1 over Z that defines an automorphism, self-isomorphism of the extended complex plane okay it is a homeomorphism in fact isomorphism in the topological sense and that when you translate it to the Riemann sphere it simply becomes rotation by 180 degree around the x axis that is what I told you last time and of course distances on the sphere are not going to change if I rotate the sphere okay that is obvious. So moral of the story is that this tells you that the spherical metric is invariant under the inversion okay that is one fact that we need.

The other side that we need in the proof is Hurwitz's theorem okay which I thing some of you must have seen in the 1st course in complex analysis probably some of you have not seen but anyway I will tell you what it is. See roughly this is what I was trying to tell you at the end of the last lecture and I am now continuing roughly the philosophy of Hurwitz's theorem is the following. Suppose you have a uniform limit of analytic function and suppose that the limit function is also a (())(12:41) analytic function okay.

Then if you take a 0 of the limit function, mind you a 0 of the limit function as to be isolated because the limit function is analytic and you know zeros of an analytic function are isolated and you know this is equivalent to identity theorem if we have seen it in 1st course of complex analysis, so the limit you take a 0 of the limit function then Hurwitz's theorem says that you see since your sequence of functions converging to a limit function then the 0 of the limit function also comes as a cluster point or as a limit point of zeros of functions which converge to that limit function.

So that means there is a so you know geometrically what it says is suppose you have a 0 of the limit function at a point Z naught then there is a small neighbourhood around Z naught where that will be the only 0 this is because it is zeros of an analytic function are isolated and the limit function is analytic okay and then you can use this neighbourhood sufficiently small so that in that neighbourhood all the members of your original sequence which converge to this limit function beyond a certain stage that means for all sufficiently large indices subscripts okay. The functions in your sequence also have zeros and that disk okay and the number of zeros is also equal to the order of the 0 of the limit function at that point and these zeros as you make the disk smaller and smaller and smaller is 0 actually converge.

They converge to the 0 of the limit function, so in other words what it says is that you know if the limit function has 0 at a point then all the functions in your sequence beyond a certain stage should also have zeros in a neighbourhood of that 0 of the limit function okay. So a limit function cannot get is 0 just like that okay it cannot happen that you know all your limit functions never have any zeros then suddenly you know in the limit is certainly the limit functions as 0 pops ups out of the blue out of nowhere that does not happen okay. So you see intuitively this is very nice to believe but the problem with mathematics complex analysis of mathematics is that you have all these intuitive things you believe that such thing should not happen by continuity you believe at such nice things should always happen but then prove them is the (())(15:06) that is where the meat lies you have to really sit down and work it out and that is why all these analysis is being done okay.

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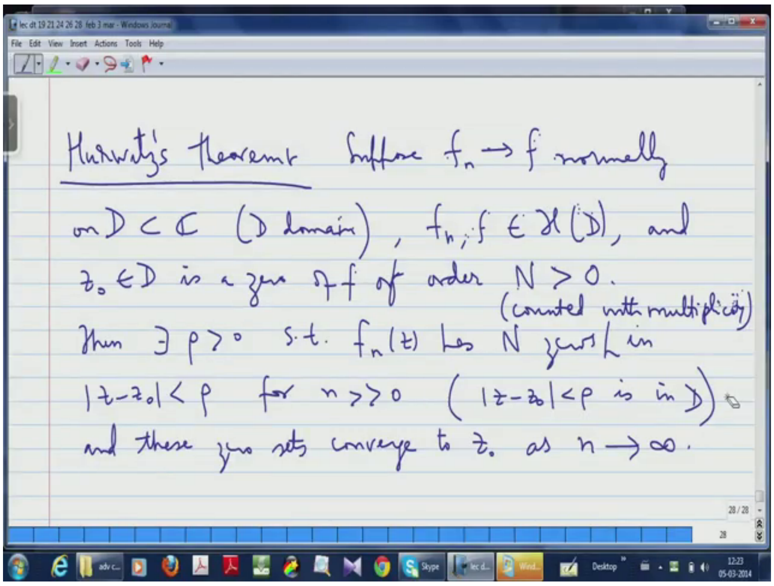

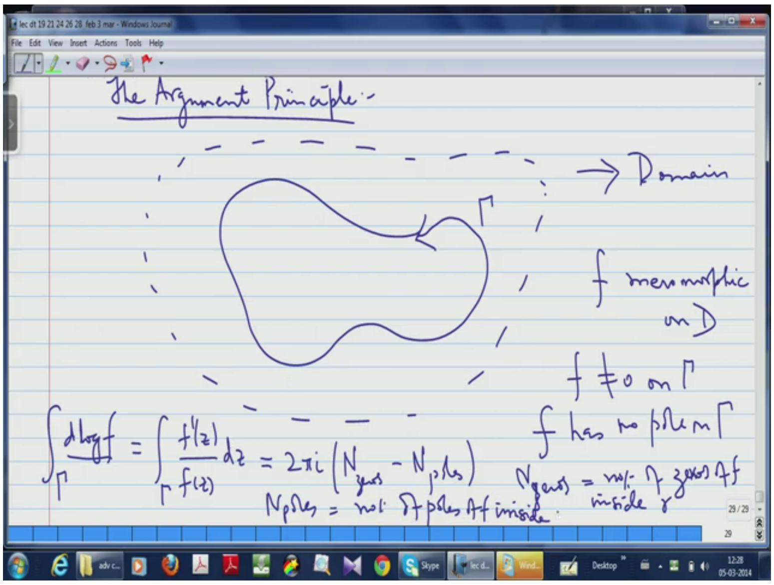

So the key to Hurwitz's theorem is the so-called argument principle which I will try to recall, so what is this argument principle, so the argument principle is as follows, what is this argument principle? So the situation is like this you are in some domain, so there is a domain here and well and inside the domain there is some there is a simple closed contour gamma okay and of course the interior of that on 2 are also lies in the domain okay.

So both gamma and the interior of gamma okay they lie inside the domain and you have a function f which is Meromorphic on the domain, it is Meromorphic on the domain means that it is analytic with the exception of an isolated subset where it has poles that is what it means okay you have a Meromorphic function and assume that, so you know the function f has only singularities it has poles okay and of course it will have zeros also but you know any way zeros of an analytic function are isolated you know that, so there is some subset of the domain which is disjoined from the subset of poles which were the function has zeros okay, now what you do is you make sure that this contour gamma does not pass through a 0 or a pole, so you assume that f is not equal to 0 on gamma f has no pole on gamma okay. Now what is argument principle if you recall it?

So the argument principle is if you integrate over gamma the logarithmic integral of f okay which is by definition and integral over gamma f dash of Z by f of Z D Z okay because you know D log is a suggestive notation, the derivative of log f is one by f times the derivative of F, so it is f dash by F. So integrating D log means you are integrating f dash by f alright, so you calculate this integral of f dash over f you calculate this integral over gamma okay and mind you this integral is very well-defined because you see notice that the integrant is f dash by f the only problem is that for the integrant is where f has zeros because when f has zeros then f is in the denominator, so f dash by f will have poles okay but I have not allowed any zeros of f to lie on gamma.

Mind you when you integrate thing the variable of integration is varying only on the region of integration, here the region of integration is gamma which is the contour and on the contour f is not going to vanish. So that function I have written down f dash by f there is going to be no problem with the denominator and the numerator there is going to be no problem because you see if a function has a pole at a point then its derivative will have a pole of order 1 more at the point okay because if a function has a pole at a point Z naught it locally looks like some g of Z by Z minus Z naught power N where N is the order of the pole at Z naught okay and if you take this g of Z by Z minus Z naught our end and differentiate it once you will get Z minus Z naught power n plus 1 in the denominator, so that means you know if it has a pole at a point then its derivative will have poles of 1 higher-order at that point.

So the only way this integrant and get into trouble because of the numerator is because of the poles of f but then I have also told you that none of the poles of f should lie on gamma, so the integrant has got nothing has no problems on gamma, so this integral is well-defined okay and the argument principle is that this integral is going to give you 2 pi i times the number of zeros minus the number of poles inside gamma okay that is the argument principle, so this is equal to 2 pi i times number of zeros minus number of poles and here of course n zeros is number of zeros of f inside gamma and n poles is number of poles of f inside gamma okay.

So what I am saying here is that when you say number of zeros or number of poles I want you to realize that you have to count them with multiplicity that is very important, so For example if you have a pole you may have only 3 poles inside gamma but each pole may have different orders okay suppose you had a 3 double poles inside gamma then the number of poles will become 6 because we have to count the double pole twice even though it is one and the same point okay, similarly 0 should be counted with multiplicities okay for example if you take Z power N at 0 the multiplicity is N as a 0, if you take one by Z power N at the origin, the multiplicity of the pole is N, you should think of it as n poles the point is geometrically it looks like only one point at actually algebraically there are nth of them because there are n factors alright.

So when you write number of zeros minus number of (())(21:40) you have to count multiplicities. It is not just the number of points which are zeros minus the number of points which are poles that is not correct okay. Now well this is called the argument principle and you know the point is that if your function is actually analytic there are going to be no poles and what you are going to just get is 2 pi i times number of zeros okay and why is this called the argument principle, the reason why it is called the argument principle is that if you take this number on the right side and divided by i will get 2 pi times an integer okay and this 2 pi times an integer as you know it is actually an angle 2 pi n is an angle, so 2 pi itself is one full rotation, so actually this 2 pi times is integer is actually the it is an angle and what is that angle? That is the change in the argument of f as you go around right that is the change in the argument that you get as you go around and in fact there is a change in argument of f as you go around and the way to see it is like this you know.

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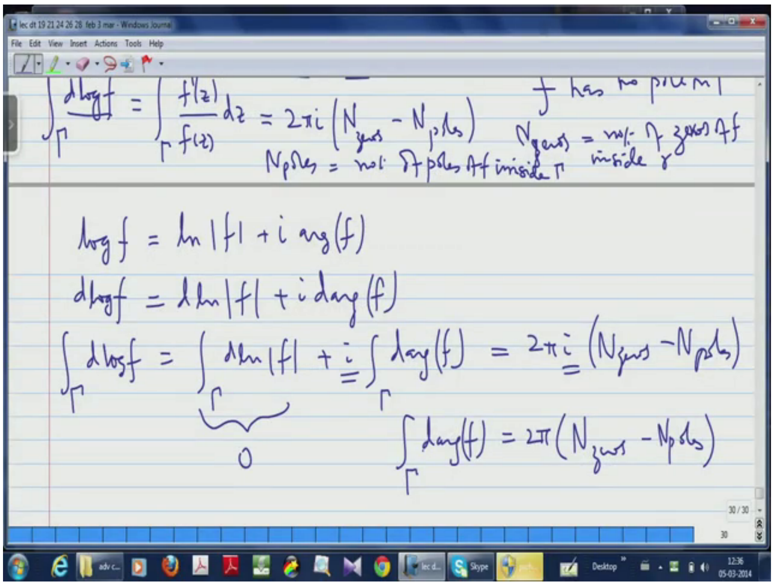

If you (())(23:02) wright log f as Ln mod f plus i times argument of F, suppose you write it like this okay then you know you can see that if I put a D to this then I will get D log f is D Ln mod f plus i d arg f, this is what I will get and now if I integrate this over gamma okay if I integrate this over gamma, what you will get is? So I will get integral over gamma d log f is equal to integral over gamma d Ln mod f plus i times integral over gamma d arg f, now you see what is this? This will be 0 see d Ln mod f it will be an exact differential, it will be 0 and basically you see it is a derivative of Ln of mod f and Ln of mod f is a nice real valued function okay.

So it is the exact so the 1st integral is exact and for an exact integral by fundamental theorem of calculus the integral will be final value minus the initial value of the anti-derivative. The anti-derivative the 1st integral is Ln mod F, so it is Ln mod f final value minus Ln mod f initial value but this is a close look, so final value will be the same as initial value okay therefore it will be 0, so 1st integral is just 0 and what is the 2nd integral? The 2nd integral (())(24:46) is the change in the argument of F, it is a change in the argument of the function f of Z as Z goes one around gamma okay and that is what this 2 pi i times number of zeros minus number of poles okay and now you see that you know there is this i here there is this i here alright, if you get rid of this i you now see the reason why it is called the argument principle, what it actually says is that integral over gamma d arg f is actually 2 pi times number of zeros minus number of poles.

So what it actually says is that if you integrate the logarithmic derivative what you are going to get is just the change in the argument of f and what is that change in the argument of F? It is 2 pi times an integer and so the change in the argument of f is mind you it is multiples of 2 pi, it is a multiple of 2 pi and what is that multiple? That multiple is actually number of zeros minus number of poles, and that is what the argument principle says and one needs this for Hurwitz's theorem and how does one get the proof of Hurwitz's theorem on this? It is very simple you just you in use uniform continuity. Let me go to the proof of Hurwitz's theorem I am just giving you a short sketch details can be filled in later.

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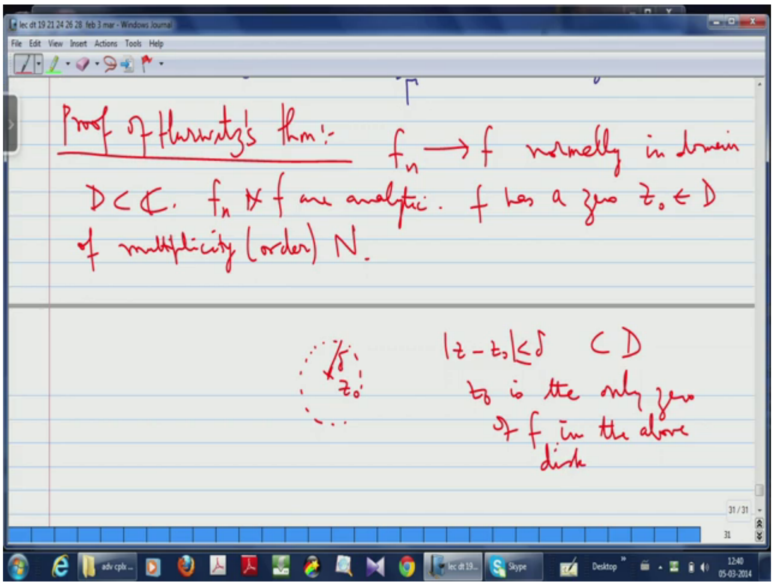

So here is the proof of Hurwitz's theorem so what is the situation in Hurwitz's theorem? You are given that f n converges to f normally, f in the domain D, this is in a domain D in the complex plane and f n and FR analytic okay f has zero Z naught in d of multiplicity otherwise order N okay this is given and what is Hurwitz's theorem, Hurwitz's theorem is that all the f n also will have n zeros okay, they will have n zeros counted with multiplicities in a small neighbourhood of Z naught and these as n tends to infinity the zeros will come close and closer and closer and in the limit they will all coalesce to this 0 Z naught that is Hurwitz's theorem, so Hurwitz's theorem says that the 0 of a limit just does not pop-up like that, it pops up as the limit of zeros of the original function which gave that limit okay that is what Hurwitz's theorem says.

So what do we do? We prove this just by using the argument principle is pretty easy, so see you have Z naught here alright and then what you do is you choose a sufficiently small disk centred at Z naught radius delta okay, so this say radius delta mod Z minus Z naught less than delta is contained in D you take a sufficiently small disk and in fact you also make sure that I will also put mod Z minus Z naught less than or equal to delta is D which means that I am also including the boundary, the reason is I want compactness by including the boundary I am making it a closed disk, closed disk is compact because it is closed end bounded and why do I need that because I can now use normal convergence because normal convergence means that whenever you have a compact subset it is uniform convergence.

So inside this closed disk and in particular on the boundary of the closed disk which is a nice circle centred at Z naught of radius delta mind you that is a compact set that is also closed end bounded. Even on the circle I have uniform convergence because it is a compact subset of D okay that is the reason why I am including a circle and now you see you can of course choose this disk in such a way that Z naught is the only 0 of f that is because you know f is analytic function, zeros of an analytic function are isolated which means that if you give me 0 of an analytic function I can find a sufficiently small disk surrounding that 0 where there is no other 0 okay, so you make that also. So let me write that Z naught is the only 0 of f in the above disk and of course am not removing that there is any other 0 even on the boundary of that disk okay mind you alright then what does your…

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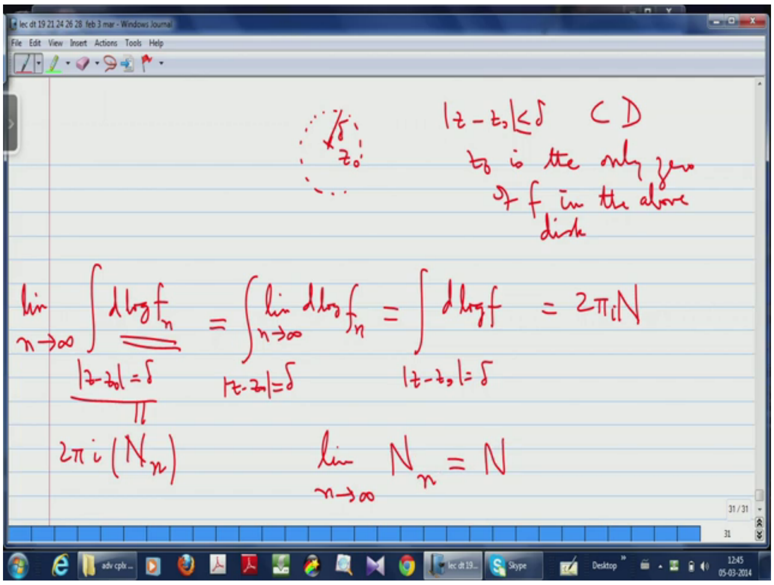

Now what does your now let us write out the argument principle integral over mod Z minus Z naught is equal to delta, D log f is going to be 2 pi times in this is what I will get I think I will also get an this okay 2 pi i n this is what I will get okay. If I write the change in argument and I have to remove i if I am just writing the logarithmic integral then I will get 2 pi i okay. Actually it is 2 pi i times number of zeros minus number of poles mind you the limit function f is analytic, so it has no poles okay so it has only zeros and the only 0 it has is at the center which is Z naught that is because I have made sure that there are no other zeros nearby, the zeros of an analytic function are isolated and what is the order of that 0 at Z naught, it is an okay that is what I have used here that is one part of the story.

Now look at the following thing, f n if on the other hand on this side suppose I write integral mod Z minus Z naught is equal to delta and if I write d log f n suppose I write this, what will I get? I will get number of zeros minus number of poles of f n times 2 pi i where I count zeros and poles of f n inside that open disk with multiplicity okay that is what I am going to get and essentially what will happen is you see, so this is going to be some you know let me use better notation okay so let me put this as capital N sub n small n okay times 2 pi i and again you know I will not have any poles because the f n are all anyway analytic functions there are no poles. Normally I should write number of zeros minus number of poles times 2 pi i but there are no poles here okay.

So I get 2 pi i times N n and now notice this is the big deal if I take the limit as n tends to infinity of this integral okay, now you see this is the point f n converges to f normally on D and the set on which I am doing that I am worried about is the set where I am doing this integral it is this circle because you see I am worried about this integrant, the integrant the variable of the integration varies over the region of integration, the region of integration is the circle, so I am worried about this circle but this circle is compact it is a compact subset of D and on this circle therefore there is uniform convergence.

Now you know because of uniform convergence I can interchange limit and integral okay, now this is one of the important properties of uniform convergence. Uniform convergence allows you to do things like changing limit and substitution which is continuity and then limit and differentiation which is differentiability term by term and it also allows you to change limit and integration which is the same as saying that you can integrate term by term okay. So because of uniform convergence I can push the limit inside okay and when I push the limit inside what will I get? I will get the thing on the right I will get simply integral mod Z minus Z naught is equal to delta limit n tends to infinity d log f n and that is equal to actually this limit n tends to infinity d log f n is just d log f okay, so what I get is, look at the moral of the story, the moral of the story is that limit as small n tends to infinity capital N sub small n is capital N this is what I will get finally this is just by applying argument principle but you see what does it say?

Capital N sub small n is a sequence of integers okay and a sequence of integers is tending to a constant means that the sequence becomes constant beyond a certain stage okay. See what does this mean, this means that for n sufficiently large N n is the same as capital N sub small n is the same as capital N, see sequence means it should come closer and closer but when these are integers closer means they are the same literary okay. If you say one integer is within an Epsilon of another integer they have to be the same okay if Epsilon is less than 1 alright, so N n is equal to N for n sufficiently large, what does that mean? It means this f n they have what is this capital N sub small n? It is the number of zeros of f n inside that this and what have you got?

You have gotten that beyond a certain stage all the f n have exactly n zeros and these n zeros and that n is the same as multiplicity of 0 of the limit function f at Z naught that is what it says and now mind you whatever I have done here will work if I make delta is smaller. After all if I make delta smaller the right side is not going to change because I am always going to get only the order of 0 of f at Z naught, so therefore this whole argument will work if I make delta smaller and smaller and smaller and smaller, so that means what? All the zeros of f n beyond a certain stage you see they are all you know they are going to go and cluster they are going to go closer and closer and closer and there are going to cluster and they are going to finally coalesce into 0 at Z naught and that is what Hurwitz's theorem.

It says that the 0 of a normal limit of analytic functions that 0 does not just pop-up just like that, it comes zeros of the original sequence of functions beyond a certain stage okay that is what Hurwitz's theorem. So now we will have to use this to give the proof of a fact that if a sequence of analytic functions does not converge to an analytic function then it has 2 completely converge to infinity, the constant function infinity. So now let us go ahead with and to prove whatever we were trying to prove.

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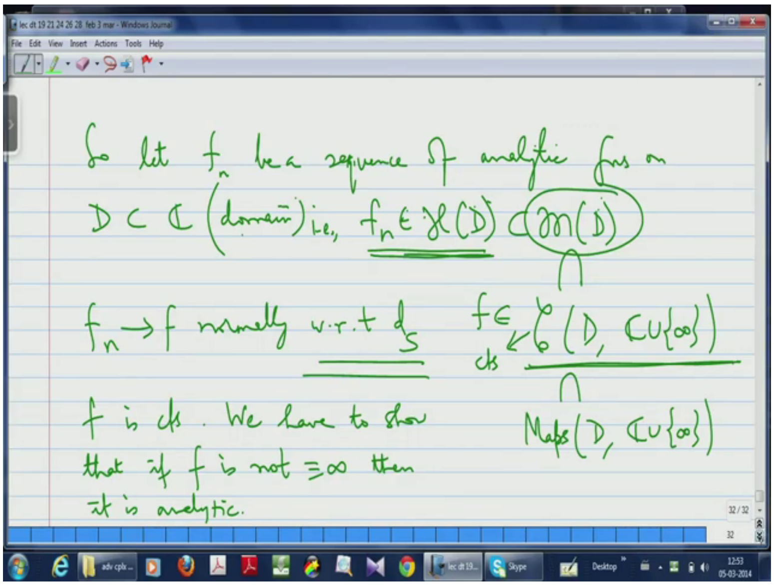

So we go back to the old problem that you are worried about, the statement that we 1st gave, so let f n be a sequence of analytic functions on D which is in the complex plane in domain. So f n is in our notation get is in H of D, H of D stands for the analytic function or holomorphic function on D and mind you this H of D is contained in m of D, m of D is a set of Meromorphic function on D that is further contained in the set of all continuous functions from D to C union infinity okay and mind you here are when I am writing it like this I am treating the Meromorphic functions also as functions which can take the value infinity and the Meromorphic function the value at a pole is defined to be infinity mind you okay and the sequence f n is in here and you assume that f n converges to an f point wise okay.

So f n converges to f normally with respect to the spherical metric okay and mind you I have to allow the spherical metric because I have the point at infinity also as a value alright and mind you this is sitting inside the set of all maps, not necessarily continuous from D to C union infinity okay and this script C is continuous maps, this is continuous maps alright and mind you let me again insists this is convergence with respect to this spherical metric okay point wise convergence with respect to the spherical metric. Why I need the spherical metric is because the limit function at some point can be infinity then I will have to measure distance with respect to infinity and I can do that only with the spherical metric okay.

Now I have already proved a lemma last time it is something that you or you know that whenever you have a normal limit of continuous functions it is continuous, so there is quite clear because a normal limit is a locally uniform limit and therefore and you know uniform limit of continuous functions is continuous and the function which is locally continuous is also continuous because continuity is a local property, so this f this limit function f is certainly continuous okay f is continuous, so f is actually here. f is continuous map from D to C union infinity and what is the claim? This is the big theorem that you trying to prove, the claim is if f is not analytic then f is identically infinity that is our theorem, that is what we are trying to prove okay.

So we have to show that if f is not analytic or let me write more logical way if f is not identically infinity then it is analytic this is what we have to show okay so let me find out you have situation where…so the limit function f either lies it is either the function here which is the constant function infinity or it is here itself it is in H of D itself. It cannot go into this it cannot become Meromorphic that is it cannot become honestly Meromorphic okay so you know it is like saying that what is the meaning of saying it cannot be honestly Meromorphic it means that suddenly a pole cannot pop-up okay in the limit you have sequence of analytic functions it is converging to limit function.

A pole simply cannot pop-up out of the blue of course the original sequence does not have poles because they are analytic functions okay by continuity you should expect this also to happen but again you know this is the point is has to be proved and you can see it is like there is already the flavour of Hurwitz's theorem which says that when you have limit of analytic functions 0 of the limit just does not just pop up out of the blue it comes from a limit of zeros of the original functions. So that is what we trying to prove, so assume that f is not identically infinity then show that f is analytic okay and what does it mean? It means that if f does not is not going to be the identically the function infinity, it cannot assume infinity even at a single point mind you. It has to assume only complex values (())(41:59) point there is a strong thing there. If it assumes value infinity at a point and if that point is isolated, point where it is assumed the value infinity it means it is a pole okay but that does not happen that is what it says, alright. That is what we have to prove.

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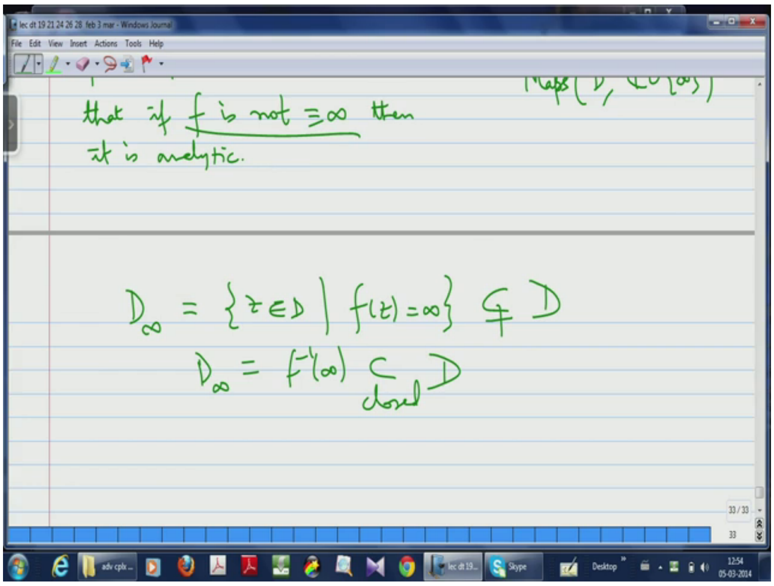

So how does one see that well you see you take the subset D sub D infinity this is a set of all Z in the domain where f of Z is infinity okay and this is a proper subset of your domain it is the proper subset of the domain because you have assume that f is not identically infinity. If f is identically infinity then it is the same as saying D infinity is D okay but when you say f is not identically infinity it means that the limit function f has some point where it has some finite complex value, a value which is different from infinity in the extended complex plane okay. Now mind you this is the closed set actually okay because it is the inverse image of infinity under f which is a continuous map I already told you that f is continuous and infinity single point is always closed okay in one-point compactification, so if you take the inverse image of infinity that will be D infinity, so D infinity is just f inverse infinity and that is closed inside the it is a closed subset just by continuity of f alright. So I will stop here.