Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

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Lecture No 2

Recalling Singularities of Analytic Functions_ Non-isolated and Isolated Removable, Pole and Essential Singularities

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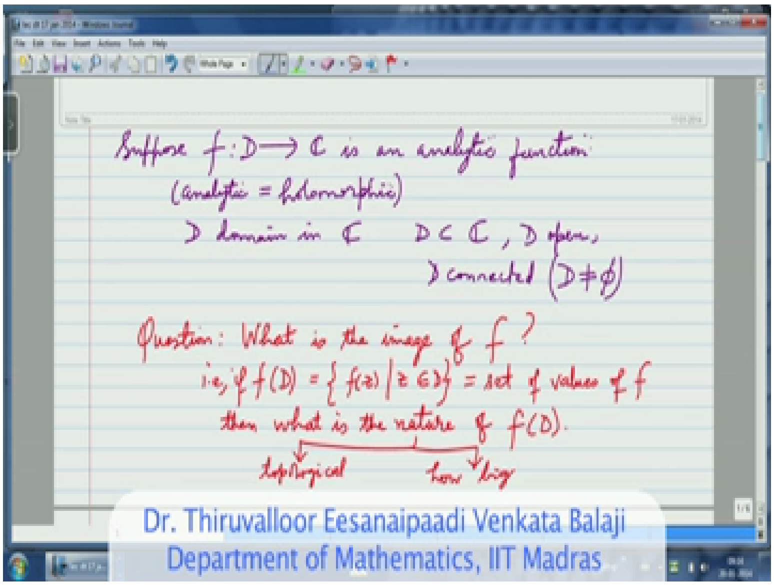

Okay so let us continue with whatever we were doing, select me very briefly recall what we have been saying, see you were been asking this question as to what will decide of the size of the image of an analytic map be, so basically you have analytic function, analytic is same as holomorphic function on the complex plane. It is defined on open subset of the complex plane and you want to know what is the image? We namely you want to know what is the set of values that the function takes and you want to know what this in the topological sense and also how big the set is and I told you that there is the to answer what the set is topologically, the set will the image domain will again be a domain, the reason being that any non-constant analytic function will be an open map okay so that is the so-called open mapping theorem right.

So the image of domain which is by definition an open connected set will again be an open connected set and of course you know the image of a connected set will be connected because that is the property of continuous function an analytic function is of course continuous, so at least you know that the images the image of the domain is a domain and the image is certainly an open set, now the question is the next question is that we asked was how big is the image okay so the answer to that at least in the case of let us say analytic functions which are analytic on the whole plane which are called as entire functions.

So in that case we have the so-called little Picard theorem or the small Picard theorem which says that functions which is analytic on the whole plane okay its image will be either the whole plane or it will be a punctured plane namely it will be… the whole plane minus 1 value so that is the only if at all it misses a value if at all it misses value, so it will miss only one value okay or it will miss no values okay and the case when it misses one value is in that case the image is a complex plane minus 1 value and that is called the punctured plane okay and the standard example is the exponential function Z going to E power Z which misses the value 0 and takes every other value which you can verify because any non-zero complex number has a logarithm okay.

So and of course you know if you take functions like polynomials okay then you will see that the image of the whole plane (())(3:30) again be the whole plane and this is one example is possible to reduce this using the fundamental theorem of algebra that any polynomial equation in one variable in one complex variable with complex coefficients always has all its roots as complex numbers okay, so well so see the idea is that you know in the 1st course in complex analysis the little Picard theorem straight stated okay but since we are since this is advanced complex analysis we would like to see a proof of the little Picard theorem and interestingly I was telling you last time that the key to that the key to the proof of the Little Picard theorem that you are going to see is actually having it deduce from the so-called big Picard theorem or the great Picard theorem and that is interestingly a theorem which involves singularity okay.

So we have the so-called great Picard theorem, the great Picard theorem says that you know you take you take an analytic function when you take a point which is an essential singularity of the function okay and then you take a small disk about the essential singularity, small open disk about the essential singularity such that in that disk the function is analytic of course leaving out the singular point and then the image of that disk no matter how small is going to be again the whole complex plane or the whole complex plane minus a single value namely the punctured plane and that is what the great Picard theorem says and that is an amazing fact, so what I want to tell you is that…so this leads us to understand what singularities are?

And eventually let me tell you that you know when we try to prove the big Picard theorem we have 2 study families of analytic functions with singularities and in particular singularities which are poles and such functions are called Meromorphic functions so we have to study families of Meromorphic functions we have to do study of topology of the space consisting of elements which are actually Meromorphic functions that is the generality in which we will have to go to understand the great Picard theorem okay but to begin with we need to worry about singularity so what I am going to do now is I am going to tell you I mean I am going to recall events about singularities which you have probably seen the 1st course in complex analysis okay but anyway it is good to recall them.

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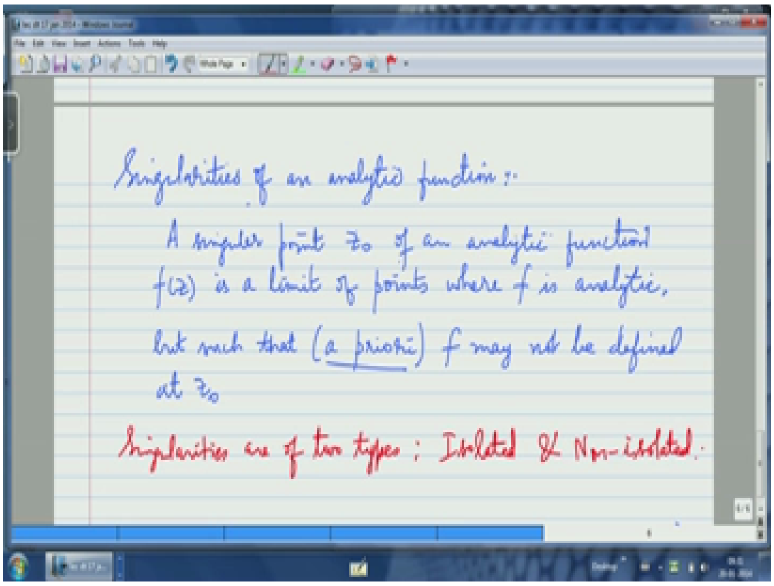

So let me, so let me write here, so singularities of an analytic function, so singularities of an analytic function is what we are going to worry about, so 1st of all let me recall so what is a singular point of an analytic function? So you see by definition a singular point is defined only for a function which is analytic okay if the function is not analytic then there is no question of talking about singularities, so the idea behind defining the singular point is that the singular point should be approachable by points where the function is analytic. A singular point should always be a limit of good points, good points for the function means good point I mean for points where the function is actually analytic okay.

So you know in a way when I say singularity of an analytic function it seems to be you know a misnomer or you know on the one hand an analytic function is supposed to be analytic at all the points where it is defined and then I say singularity of an analytic function it sometime looks odd but that is not the point, the point is that you see a function which is analytic is usually analyticity is defined on open set okay that is an open set of points at each of which the function is analytic and of course you know for the definition of analyticity you need an open set you need every point to be an interior point of the domain of analyticity okay. So but the question is that by move to the boundary of this open set, if I go to a point in the boundary of this open set then how is it that the function is going to be here?

That point may be a point where the function might continue to be analytic or it may fail to be analytic okay and it is only about these boundary points, in the boundary points of the domain or the open set where the analytic function is defined these are the points that we have to study for singularities okay. So the very 1st thing is that you know a singular point is defined only for an analytic function okay and by definition it is a point such that it is approachable or this limit of points where the function is analytic okay, so let me write that down a singular point Z naught of an analytic function f of Z is a limit of points where f is analytic but such that a priori f may not be defined at Z naught.

So you see so look at this definition very carefully what it says that you know point is it not in the complex plane is a singular point for analytic function, if you can approach that point it is a limit of points where the function is analytic okay but at that point itself the function may not be defined okay and force the phrase a priori means is that to begin with or in advance you do not know whether f is defined at Z naught or where the f can be defined at Z naught these are things you do not know okay so it is a point which is outside, so singular point is a point who is outside the domain of definition, it is outside the open set that the function is defined and your question is whether the function, how does a function behave close to that point?

You see what you must understand is reason why we define a singularity like this is because you see I want to study a function at singularity okay and I told you what is the motivation, why should be at all worry about functions with singularities, the answers because singularities occur, not all functions are going to be entire okay not all functions that you are going to study are going to be defined on the whole complex plane. A lot of functions they come out naturally with singularity so for example you take the identity function f of Z equal to Z that is the identity function f of Z equal to Z at is the identity function is of course entire okay but the moment you invert it if I take f of Z equal to 1 by Z okay then you see immediately at 0 it is not defined okay.

So the problem is that there are it is very easy you know to get hold of functions which you cannot define a point at that point was surrounded by points at the functions analytic okay therefore such a point is a singular point okay, so singular point will come when naturally. They are the most natural things that you have to come across you have to study and of course I told you the 1st motivation was that you know we are trying to prove the big Picard theorem which is actually a theorem about the mapping properties of a function around an essential singularity, so that is also a motivation as to why worry about singularities okay, so let me come back to this definition of singularity see the point is that I have a point that there is a point Z naught where the functions not defined okay but I would like to study the function close to that point okay and why should I study function close to that point because that is the only way I can study out the function behave as I go closer and closer to that point okay.

So it means that no matter how close I get to Z naught I should be able to study the function okay that means the function should be defined no matter how close I get to Z naught that is the only if the function is defined can I study it okay so that is the reason why our singular point is always defined as a limit of good points okay, so what I want to tell you is that there are functions for which you know singularities per se do not exists, so for example take the example of f of Z equal to let say mod Z the whole square okay and if you take f of Z equal to mod Z the whole square this is of course defined on the whole complex plane and if you check mod Z the whole square is Z into Z bar where Z bar is a conjugative Z okay and you will see that the Cauchy Riemann equation I satisfied only at the origin okay.

So if at all this function is differentiable will be only at the origin okay, so certainly the function cannot be you can find a single point where this function is analytic okay because analyticity means the function should not only be that point it should also be in the whole neighbourhood around that point but there is no such point, the only point where this function f of Z equal to Z square mod Z the whole square differentiable is the origin and at that point is not analytic because at no other point it is differentiable so if you take this function what is a set of singularity which the empty set, I mean singularities not even defined because it is not even analytic okay so what you must understand is the singularity are defined for only analytic functions okay we are not worried about functions which are not analytic in the 1st place okay.

So that is one thing that is 1 point then the 2nd thing is that you know singularities comes in 2 categories if you want or 2 types okay and one is friendly one is less friendly and other is more friendly okay, see the more friendly ones are called the so-called isolated singularities okay, what is an isolated singularity? It is a point where the function has a singularity but there is small open disk surrounding that point but the function has no other singularities, so it means that there is a deleted neighbourhood of the point that the function is analytic okay such singularities are called isolated singularities okay and then the less friendlier singularities are the so-called non-isolated singularities okay and these are more difficult to study okay.

The standard example of non-isolated singularity is that of the log function okay if you take f of Z equal to log Z to be the principal brands of the logarithm okay you know that to make it analytic you have to make it happen throughout the negative real axis along with the origin of course origin will now come into the picture because you cannot define log 0 okay and then you will have to cut out the negative real axis okay and then you get the so-called slit plane, it is the plane minus the real axis from the origin to the going to minus infinity that whole line segment that whole ray is cut off okay.

So this is the function which this is the domain the slit plane is the domain where the principal brands of the logarithm be defined and is analytic there and every point on the negative real axis is a singularity by definition because the function is not defined there and it is not analytic at those points okay, so well in fact the truth is that function can be defined at each of those points but you cannot define it in such a way as it becomes analytic okay on the whole punctured plane okay, so the negative real axis in the case of along with the origin is all the points on this ray they are all examples of non-isolated singularities okay.

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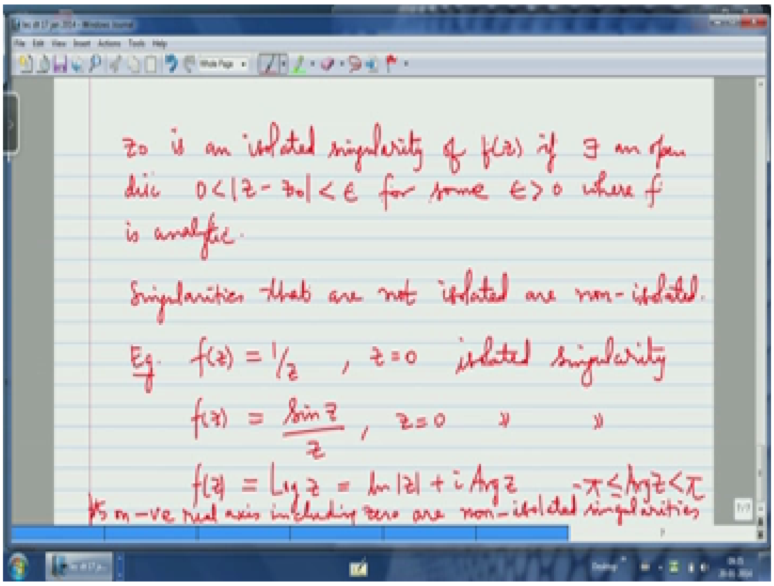

So let me write that down, so singularities are of 2 types isolated and non-isolated well so let me write that down Z naught is an isolated singularity of f of Z if there exists an open disk 0 less than mod Z minus Z naught less than Epsilon for some Epsilon greater than 0 where f is analytic. This is just another way of saying that there is a small neighbourhood around Z naught where the function f is analytic okay and so and what about non-isolated singularity well singularities which are not isolated are non-isolated singularities okay as the name says so singularities that are not isolated are non-isolated okay and of course the examples well you take f of Z equal to 1 by Z then Z equal to 0 yes of course an isolated singularity.

Then I can take f of Z equal to if you want sin Z over Z again Z equal to 0 is an isolated singularity okay but you will recognize immediately that the lemmatize Z tends to zero sign Z by Z is one, so is a singularity that can be really removed okay we will see about that very soon. Then let me give you the principal branch of the logarithm f of Z equal to principal branch of log Z which is (())(19:56) mod Z plus i times principal argument of Z with principal argument of Z varying from minus pi to pi, minus pi included pi not included, plus pi not included and this is so let me write here the negative real axis including 0 or let me say points on the negative real axis including 0 are non-isolated singularities, okay.

So the logarithm of course you know you have 2 so keep in mind the domain of definition so in the case f of Z the 1st example f of Z equal to 1 by Z the domain of definition is the punctured plane, the complex plane minus the origin and origin is the isolated singularity. In the 2nd case also it is positive plane is a complex plane minus the origin and the 3rd case of force the principal branch of the logarithm it is the slit plane, so it is the plane minus the negative real axis, with the origin removed okay fine, so that is that.

Now what are we going to do, so let me tell you that we are worried only about isolated singularities, we will not be worried about these non-isolated singularities but then let me also tell you that what is the way to study non-isolated singularities, one of the theories that helps in the study of non-isolated singularities is the theory of Riemann surfaces okay, so the point is that when you have non-isolated singularities then you basically have usually you have a curve where which is full of points where there are singularities okay.

So in the case of the principal branch of logarithm this curve is actually the negative real axis okay and such curve is called a branch in curve or branch locus okay of your function and the way to study that is to do what is called to go to what is called the Riemann surface of the corresponding function okay so there is a key to study studying simple non-isolated singularities which lie on a curve is a study of…will lead you to a study of Riemann surface okay but anyway we are not going to do that but this is just tell you at non-isolated singularities can also be studied alright and then you can also have a very strange situation like there may be function which has only one singularity and that one singularity alone may be non-isolated and all the other singularities maybe isolated you can have all kinds of examples, so here is so let me give you one example.

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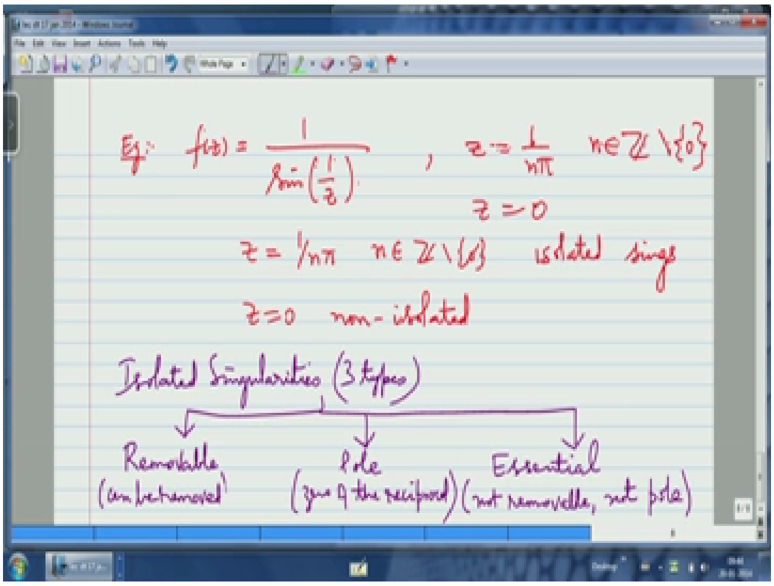

So here is another example you take f of Z to be 1 by sin of 1 by Z okay, look at this function 1 by sin 1 by Z you see the point is that whenever you take the reciprocal of a function your reciprocal is always in trouble whenever the function vanishes okay so when I write 1 by sin 1 by Z this is cosecant of 1 by sin okay and the problem with this function is whenever the denominator which is sin 1 by Z vanishes and you know sin 1 by Z vanishes when 1 by Z is n pi, so the problem is that the problem is at point Z equal to 1 by n pi where n is an integer okay so this is the these are the points among these you know you can see that if you take function sin 1 by Z that is already involves 1 by Z and 1 by Z is not defined it is 0.

So 0 is already a problem or the function sin for the function 1 by Z so it is also a problem for the function sin 1 by Z okay therefore you see of course when I write 1 by n pi I must make sure that n cannot be 0 because it does not make sense, so n cannot be 0 but then I should also include Z equal to 0 because this is a point where the function even the function that denominator is not defined namely sin 1 by Z is not defined okay. Now if you look at it carefully see this as n becomes larger size okay 1 by n pi come closer and closer to the origin okay and therefore you see but all these 1 by n pi for various n not 0 they are isolated singularities in fact they will be simple poles as we will see later okay but the origin will be an non-isolated singularity so here is an example of function which helps one singularity which is non-isolated and all other singularities are isolated okay.

So let me write this down Z equal to 1 by n pi, n an integer which is different from 0 these are all isolated singularities and Z equal to 0 is non-isolated, so you see you can have so this is another (())(26:00) example okay, fine. So what we will do is that we will start worrying about only isolated singularities okay and so we will leave out the case of non-isolated singularities and go to the case of isolated singularities so how do you classify isolated singularities, so this is again something that you should have done in the 1st course in complex analysis, the isolated singularities are classified as removable singularities poles and essential singularities, so let me let me recall what these things are, so let me first say in words what is a removable singularity, removable singularities essentially a singularity that can be removed namely that is an isolated singularity okay but the function can be extended to the singularity in a way that it becomes analytic okay.

It is like it is the analog of removable discontinuity that you study in 1st grade analysis okay so well then you have these so-called polls, what kind of singularities are poles these are going to be… Poles are supposed to be thought as the 0 of the denominator is okay so the point is that you cannot divide by 0, so whenever the denominator becomes 0 the function is not defined so all the places where the denominator becomes 0 these are the polls okay and they should be…and when I say 0 it should be 0 of a certain order okay and in general you think of poles as zeros of the denominator, the other way of saying it is that 0, the poles are actually zeros of the reciprocal of the function okay.

So 0 of the reciprocal of the function okay is exactly what a pole of the function is okay and then what are essential singularities by definition these are the singularities which are neither poles nor removable okay that is the clever way of defining them because then you do not have 2 you have to define them separately, so let me write down these definitions. So isolated singularities are of three types so the 1st one is they are called removable singularities the 2nd ones are called poles and 3rd ones are the essentials okay and by definition so if you go to definitions, essential is defined as not removable not pole okay that is how you define essential singularity and you may be wondering why the name essential singularity well let me tell you they are really essential because they kind of completely distinguish the function, the behaviour of the function.

The neighbourhood of the essential singularity can distinguish the function from other functions, so it is an though it is the singularity of the function it is like it is very essential for the function it can distinguish the function, it holds all the information more or less about the function okay that is why it is call essential. The behaviour of a function in a neighbourhood of the essential singularity completely holds the holds the full information about the function okay that is why this call essential, we will see more about this later and of course pole is let me say this is 0 of the reciprocal.

This is what the pole is and removable is well to say it in simplest words it can be removed so this is this is as simple as it goes but then you know so there are many ways of categorising so-called removable singularities, poles and essential singularities and one of the key is to that doing these things for that matter one of the key is to studying a function around an isolated singularities so-called Laurent expansion of the function okay so this is how you would have you would have all gone through 1st course in complex analysis where you have used Laurent you have come across Laurent series and then you view the residue theorem often trying to find out the residue at a pole and so on and so forth, so did Laurent series is one concrete way of trying to get formula for the function as a series of some of powers both positive and negative around the isolated singularities, so let me state the following thing.

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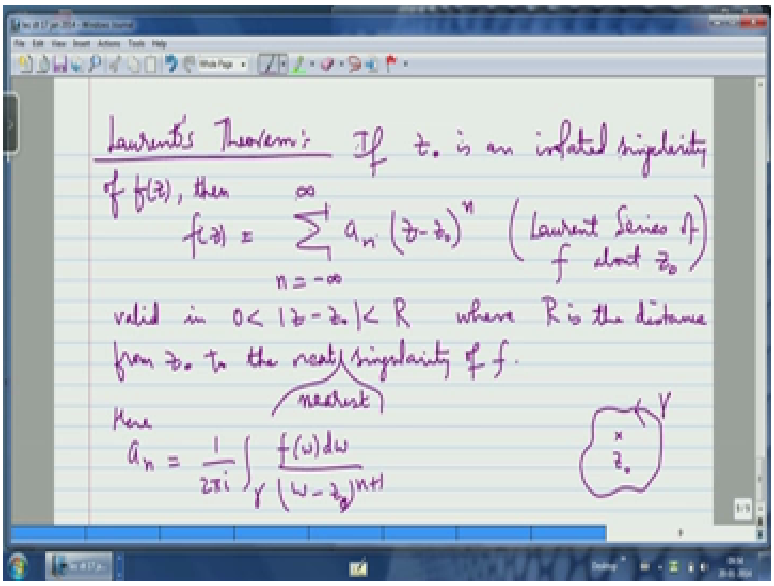

So this is let me just recall this is Laurent's theorem which is kind of a very helpful to study functions around there around an isolated singularity okay, so here is Laurent's theorem if Z naught is an isolated singularity of f of Z then f of Z is equal to the Sigma a n Z minus Z naught to the power of n, n equal to minus infinity to plus infinity this is the Laurent series of f about Z naught okay valid in 0 less then mod Z minus Z naught less than R okay where R is the distance is the distance from Z naught to the next singularity of f, so what do I mean by next singularity of f what I mean by that is the next nearest singularity of f, so maybe so let me write next nearest okay.

So this is the Laurent's theorem okay I have stated Laurent theorems for an isolated singularity okay but Laurent theorem is also valid in an annulus actually okay, so and you know a deleted a punctured disk is a special case of an annulus with the inner radius 0 you know an annulus about a point is the open region between 2 circles centred at that point of different radii okay and if you make the inner radius 0 okay then you get a punctured disk which is also a special case of the annulus, so and in that if you make the outer radius infinity then you get the punctured plane.

So a punctured plane is also a special case of an annulus okay so for example if you take the function e power 1 by Z okay and you are right out the use simply take the what is the Laurent expansion, the Lauren expansion is you know e powers Z has a Taylor expansion which is valid for all Z and that Taylor expansion simply replace Z by 1 by Z and that continues to be valid the whole plane except the origin and the whole plane except the origin is again an annulus with inner radius 0 out radius infinity okay punctured plane is also a special case of an annulus okay.

So this is Laurent's theorem and when I write f of Z is equal to Sigma n equal to minus infinity to a n Z minus Z naught power n, notice 1st of all that what this is supposed to mean is that the series on the right converges and it converges to f that is what it means okay and technically what are these a n, the Laurent coefficient they are given by integral so here a n is 1 by 2 pi i integral over gamma fw dw by w minus Z naught to the power of n plus 1. This is the these are the values of these a n and what is this gamma, see gamma is a simple so here is a Z naught and gamma is some simple closed curve, gamma is some simple closed curve going one around Z naught okay and of course in the region enclosed by gamma Z naught is the only singularity and there is no other singularities on gamma for the function okay.

So this is the Laurent theorem okay and of course you know you must keep in mind that whenever you write an integral like this when you write integral over gamma you know it is very important that or such an integral to be defined of the curve should be contour so it should be piecewise smooth which means piecewise continuously differentiable curve okay and the integral should be piecewise continuous at least on the contour for the integral to be defined okay so of course I can deform gamma little bit and the integral will not change that is because of cautious theorem, so in particular if you want to make calculations you can take this gamma to be a small circle centred at Z naught okay with sufficiently small radius okay.

Really the shape of gamma does not matter okay there is only the fact that gamma should be simple closed curve, simple means that it does not cross itself okay and it goes around 1 exactly one around Z naught and this is Laurent's theorem and the point, the important thing about Laurent theorem is that as you would have learnt in the 1st course in complex analysis is the most important thing about Laurent's theorem is the coefficient A minus 1 okay when I put n equal to minus 1 what I get a is a minus 1is 1 by 2 pi i integral over gamma fw dw okay and that is important that is the residue of f at Z naught it is the A minus 1 and it is important because it gives you it tells you what the integral of the function this around a singularity okay.

If I put n equal to minus 1 I get a minus 1 is equal to 1 by 2 pi i integral over gamma fw dw, so integral of fw dw over gamma where gamma is going around singularity that is a very important thing okay. Cautious theorem tells you that if you go around a point where the function is analytic okay if you try to integrate a function around a closed curve so the function is analytic inside and on the curve then you are going to get 0 that is what cautious theorem says it says you will not get anything so but then even now a question, what will happen if you integrate around that singularity if you take a function and you integrate it along the singularity around the singularity what will you get?

The answer is the residue so that is why residue is important, they help you to calculate the integral of a function around singularity okay so and that is so essentially that the residue is 2 pi i, so a minus 1 is the residue and 2 pi i times a minus 1 is equal to integral over gamma fw dw that is exactly the residue theorem okay. Residue theorem actually says that the integral around a…if you go ones around if there is a curve which goes once around a singular point then the integral of the function along that curve is going to give you 2 pi i times the residue that is if you are going around 1 singularity and then the residue theorem in general says that if you have several singularities and you have to take sum of all those residue it will be 2 pi i times sum of all the residue.

So this is the residue this is the residue theorem okay which helps us to compute lot of integrals even real integrals which you would have seen the 1st course in complex analysis okay so very well. This is Laurent's theorem, now what I am going to do is I am going to you know go back to our study which is the study of singularities and I am going to tell you know we saw 1st that they were that there are 3 types of singularities there are the removable singularities, there are the poles and then there are the essential singularities.

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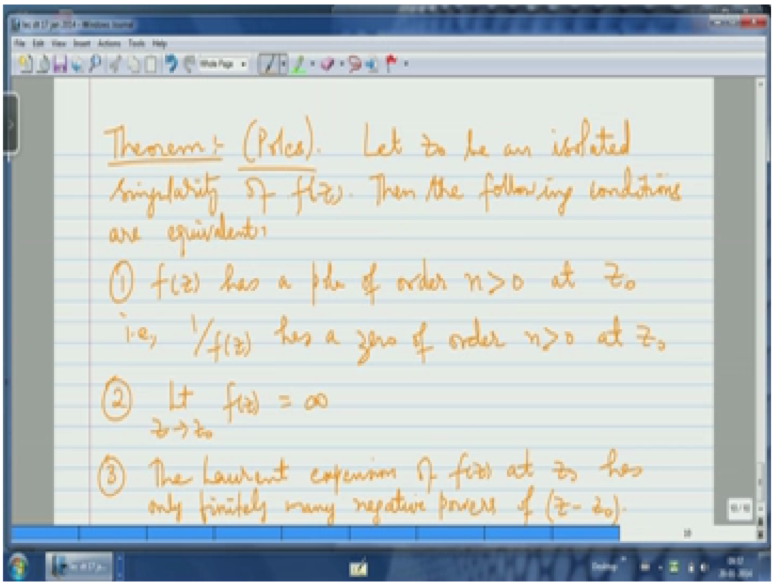

Now let me say something about poles which is something that you would have which you would have come across but you should try to now do this is an exercise will help you to revise your basic knowledge of complex analysis, so here is the theorem, so here is the theorem. So this is theorem about poles, let Z naught be an isolated singularity of f of Z then a following conditions are equivalent number 1 f of Z has a pole of order n greater than 0 at Z naught, so and that is 1 by f of Z has 0 of order n greater than 0 at Z naught, so the definition of a pole this is one of the definitions of a pole. In fact a pole can be defined in many ways and what this theorem says is that it gives you various equivalent conditions.

So the 1st thing is the definition of a pole which is the which is as 0 of the reciprocal, so f has a pole of order n if one by f which is reciprocal of f has 0 of order n okay and so what is the 2nd one, the 2nd one is limit Z tends to Z naught f of Z is infinity okay, so this is this is another condition for a pole, the function becomes arbitrarily large in modellers as you approach a pole okay and here is the 3rd one the Laurent expansion of f at Z naught has only finitely many negative powers of Z minus (())(43:37). So this is the way you define a this is the way you define a pole using Laurent expansion, see a Laurent expansion helps you also classify singularities, so what I want you to do now is that you should you need to I want you to go back and as an exercise of that all the 3 statements are equal okay at least you should have seen this in the 1st course but I want you to recall the proof that is an exercise and so let me stop here and we will continue in the next lecture.