**Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky**

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**Lecture No 18**

**Measuring Distances to Infinity, the Function Infinity and Normal Convergence of Holomorphic Functions in the Spherical Metric**

So see the point is that you know we are trying to look at families of Meromorphic functions, we are trying to look at trying to look at normally convergent families of Meromorphic functions, the reason is because you want to do topology on the space of Meromorphic functions okay and because that is the kind of you know set up that you need able to prove the big Picard theorem and the little Picard theorem okay. So you know let me briefly remind you if you take inspiration from topology alright, what is the topology that you will put on the space of functions, normally if you have a topological space and you have you are looking at real valued or complex valued functions then you will restrict yourself to continuous bounded real valued or complex valued chance that is that is a banach algebra and it is also a topological space it is complete as a metric space, the metric is induce by a norm and the norm is essentially convergence in that norm is actually equivalent to uniform convergence okay.

So the moral of the story is that if you are going to inspiration from topology okay then trying to do topology on the space of functions is the same as you studying functions under uniform convergence okay, now this is topological so that means that you are only worried about continuous functions okay but now suppose you come to complex analysis then you are worried about holomorphic functions or analytic functions okay and we are worried about something even worse, we are worried about Meromorphic functions which are actually which have the additional problem that they can have poles at finitely many at a set of isolated points okay and of course finitely many if your domain is compact is the whole Riemann sphere on the extended plane.

So if you are looking at say holomorphic functions on a domain, a domain in the complex plane or a domain in the extended complex plane that does not matter suppose if you looking at the analytic functions or holomorphic functions on the domain and you want to do topology on that set of functions okay. Mind you we have body seen that the set of functions it is a ring in fact and then if you look at Meromorphic functions it is a (())(4:09) okay. It has algebraic structure but we are now worried about the topology, so you want to do topology on the set of analytic functions on a domain or you want to do more generally topology on the set of Meromorphic functions on a domain which means analytic except for poles then you know if you try to draw inspiration from ology you would just say that this is the same as studying them under uniform convergence okay because topologically uniform convergence corresponds to convergence in the space of functions okay but if you come to the case of analytic functions okay this is not the right thing because you do not get uniform convergence.

So I was trying to explain to you last time that for example if you take the geometric series you take the functions that correspond to partial sums of the geometric series okay then they are of course polynomials and they converge absolutely in the unit disk to the sum of the geometric series which is one by one minus the variable okay, so the geometric series is 1 plus Z plus Z square and so on where Z is a variable and you are restricting Z to be in the unit disk that means you are making mod Z is less than 1 then one plus Z plus Z square and so on that converges to one by one minus Z that is the high school formula or geometry series. Now the point is that this convergence is absolute on the unit disk there is no problem about that but it is not uniform on the whole unit disk okay that was something that I told you I asked you to check it as an exercise I hope you have done it, it is very easy to do convergence is not uniform on the whole unit disk is only uniform on compact subset of the unit disk okay.

So you do not get uniform convergence but you get only normal convergence, so the moral of the story is that when you want to do topology on a space of holomorphic functions you should not look at them under uniform convergence you must look at them under normal convergence so that is the 1st moral 1st lesson to be learned that is what you should keep at the back of your mind, so that is one thing so you know more generally if you want to extend this to Meromorphic functions in serve more complicated because now you have poles okay and the other thing is that of course you know by going from uniform limits to normal limits things are going to be good okay because normal convergence is just uniform convergence on compact sets okay it is weaker than uniform convergence as it is okay but it is good enough for our purposes because as I told you in the last lecture if we have sequence of holomorphic functions which converge normally to limit function then that limit function is also holomorphic okay this is something that I explained last time okay.

Essentially it uses Clausius theorem and Morera's theorem and the fact that analyticity is or holomorphicity is a local property. So therefore there is no harm in relaxing the condition of uniform convergence with the condition of normal convergence. It means uniform convergence only on compact sets okay and I also told you philosophically why that is good enough for complex analysis because the moment you say uniform convergence on compact sets you get uniform convergence on closed disk because they are also compact and therefore you get uniform convergence on sufficiently small open disk okay that is good enough for the analysis for the differentiation theory and then for the integration theory also it is good because whenever you integrate on the contour, the contour is a compact set therefore you will get uniform convergence on the contour okay, so that helps in the integration theory, so for all practical purposes uniform convergence on compact sets that is normal convergence is good enough okay, so that is what we have to worry about.

Now the other important thing that I want to tell you is that you see at least if you are working with Meromorphic functions you know the Meromorphic functions have poles okay, so at a pole a function is going to behave in a bad way in the sense that the modulus of the function is going to blow up to infinity okay, so for example that is one of the characterisations of a pole, the limit of the function as you approach a poll is going to infinity and by that time limit goes to the point at infinity okay and of course here you are using the topology on extended complex plane namely the complex plane the point at infinity given by the one-point compactification which makes it holomorphic to the Riemann sphere okay.

Now is another pathology and that is the pathology that I was trying to explain what is the and of the last lecture, so the pathology was the following, you take the exterior of the unit disk or Z greater than 1 that is a variable, we are on the Z plane the complex plane and you are taking the exterior of the unit disk mod Z is greater than 1 and what you are doing as you are looking at these functions powers of Z you are looking at one which is Z power zeros if you want then as Z square and Z cube and so on okay. Now that is a sequence of functions and a point is that this sequence of functions you can see point wise it will go to infinity because sense mod Z is greater than 1 mod Z to the n to the power of n is going to go to infinity because it is for a real number greater than 1 you know its higher powers will diverse to infinity, so mod Z…so Z power n so this sequence of function is going to converge point wise to the function with the constant function at infinity.

Namely it is a function which associate every point the value at infinity, so you have to worry about this crazy function okay so this is a pathology that happens that you have to take care of and the point is that therefore we are forced to introduce a function call infinity okay and this function infinity is what it is just the constant function infinity, namely it is the function which maps every value to infinity that is what it is and then if you think of that as a function I mean it is a function of course theoretically if you want it is a function from your domain to the extended complex plane because after all in the extended complex plane infinity is a valid point okay it is a member of that site, so you can really think of the function infinity as he constant function taking the value at infinity provided you extend your values to not just complex values but also the extended plane you include the value at infinity that is one thing.

So in that sense you can say that this sequence of functions f n of Z is equal to Z power n that converges to infinity you can say that and when I say that converges to infinity I mean that it converges point wise in the exterior of the unit circle to the function which is infinity okay. So you have this is very nice situation, it is a very nice pathology you have these Z power n which are all holomorphic functions in fact they are entire functions they are just polynomials and they converge to the function infinity in the exterior of the unit disk the convergence is again a normal convergence it is uniform on compact sets okay. It is still a normal convergence is not just a point wise convergence but it is in fact even a normal convergence in a way I will explain to you.

So what is the moral of the story? The moral of the story is you have (())(12:35) sequence of holomorphic functions you have (())(12:39) sequence of analytic functions which is converging to the function infinity normally that also happens. You see this is the extreme case that happens and this also has to be taken care of in our arguments okay and mind you if this is happening for holomorphic functions will happen also for Meromorphic functions because you know holomorphic functions are very good Meromorphic functions are worst because they have poles, so even for a family of holomorphic functions even for a family of analytic functions if you can get normal convergence the functions which is infinity okay you should expect the same thing happen also for Meromorphic functions.

So what I am trying to tell you is that if you sum up all this if you want to study topology on the space of Meromorphic functions 1st of all you must study with respect to normal convergence okay the 2nd thing is you have to introduce this function keeping mind this function which is the function infinity okay and then you have to justify this business of trying to make sense of normal convergence okay and so let me begin by trying to you know explain at least in this particular case where this normal convergence comes from okay, so let us do the following thing will worry about metrics on the plane and metrics on the extended plane which are transported from metrics on the Riemann sphere okay.

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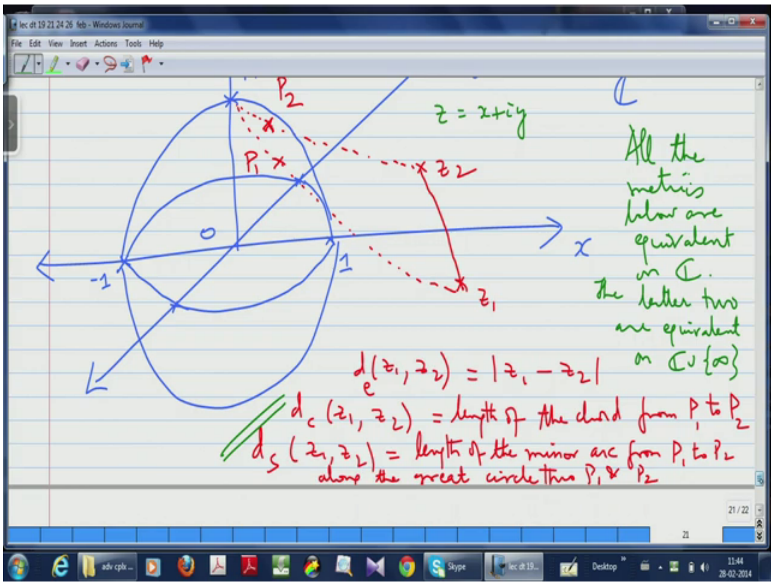


So here is what I am going to do so let me draw a diagram, I have this so this is my usual complex plane xy, so this is my usual complex plane which is the xy plane and then I have the of course you know we are going to compare everything with the Riemann sphere using the stereographic projection, so what I am going to do let me draw this thing here which is Riemann sphere, so this is supposed to be 1 this is minus 1 on the x axis and here is my sphere its cross-section on the plane is the unit circle, so it is going to be so this is my Riemann sphere as it is and then of course I have this 3rd axis which I will not call as Z I will call it as u because Z is supposed to stand for x plus i y okay Z is supposed to be x plus i y and well and here is the North pole okay suppose I start with 2 point Z 1 and Z 2 on the complex plane okay.

See what kind of the distances can I define on these points, what kind of metric and I define on the complex plane that is the usual metric which is D of Z 1, Z 2 so I will put d sub e for Euclidean metric and you know what that is? It is simply modulus of Z 1 minus Z 2 it is just the distance between these 2 points okay. This is the good old metric that we use always in Euclidean space right. It is actually the length of the line segment okay joining Z 1 and Z 2. Then the other thing you can do is you can take the images of these points on the Riemann sphere because of the stereographic projection and you can measure the distance between those 2 points and call that as the distance between these 2 points, so what you can do is so here is the stereographic projection, so p 2 is this point here on the sphere which is the unique point of intersection of the line joining n and Z 2 on the sphere okay and similarly p 1 is unique point on the Riemann sphere which is the intersection of the surface of the sphere with the line joining n and Z 1.

Now you see p 1 and p 2 lie on the sphere, now what I can do is that I can measure the distance between p 1 and p 2 okay. Now that distance I measuring in 3 space because now everything once you draw the Riemann sphere you are actually in R 3 and your R 2 which is xy plane corresponds to the complex plane alright, so what you can do is you could define the following new distance also d let me call this as d c is the chordal distance from Z 1 and Z 2 it is just length of the cord from p 1 to p 2 where p 1 the stereographic projection of p 1 is Z 1 and the estimated projection of p 2 is Z 2 okay so this is another distance that I can define it makes…see what this distance does is that actually it is the metric in R3 after all the cord joining p 1 p 2 is exactly the line segment p 1 to p 2 in 3 space and I am just taking the length of that line segment, so it is actually the metric in R3 it is metric in R3 and so it is a metric space you know whenever you have a metric on space and you restrict to a substance then the subspace also becomes automatically a metric space.

So this distance will make the Riemann sphere into a metric subspace of R3 okay and what we are doing is that to the stereographic projection you are transporting that metric to the complex plane because after all this geographic projection is the bisection between the extended complex plane and the Riemann sphere okay. The moment you have bisection of a set with a metric space it can transport the metric on the metric space to the set, so I will just transported the… basically what I have done is I have simply transported the metric on R3 restricted to the Riemann sphere. I have simply unsupported it to the plane that is what this d c is.

So this d c also will also d subsea that is also a metric you can check that that also makes the complex plane into a metric space okay and but the big deal is that all these metrics are all equivalent okay namely the topologies that they induce on the complex plane there are all the same that is the whole point okay and that is very important because what it tells you is, it tells you the following thing if I want to study convergence of functions you know as long as you are worried about continuous functions I can use any of these metrics and the point is so let me tell that in advance, why I am worried about these extra metrics is because I can also define the distance of a point on the complex plane to the point at infinity okay because that the point at infinity will correspond to a finite point namely the North pole on the Riemann sphere and distance to that is something that I can measure okay, so that is the advantage.

The advantage is you see I want to be able to measure the distance the point at infinity R I cannot do it with the Euclidean metric because 1st of all the point at infinity is not in my set okay it is an extra point I have added for compactification and once I had these extra points I have to add this extra topology, the topology of the one-point compactification but then that is not enough I have to even make it a metric space and where will I get the metric structure? The only way is I will have to get this metric structure from the Riemann sphere which is what is holomorphic to the extended plane okay and therefore I am lead to look at the metrics on the on the Riemann sphere, so this is the chordal metric okay, so d subsea is the chordal metric and then here is the 3rd metric which is the spherical metric so d sub s of Z 1, Z 2 this is the length of the arc of the minor R from p 1 to p 2 along is a length of the minor arc from p 1 to p 2 along the great circle through p 1 and p 2.

So you see this is a spherical distance, what is a spherical distance? This is spherical distance actually am trying to measure distance on the Riemann sphere on the surface of the sphere, so it is the curve distance okay and I am trying to measure the shortest curve distance and you know you can imagine this…basically what one is doing is that one is doing a kind of some kind of Riemann in geometry, what is happening is that you have a surface okay you imagine some nice smooth surface you have 2 points okay then you can try to connect those 2 points by many arcs by many arcs on the surface passing on the surface okay and then you can measure the lens of each of the arcs and you can take the smallest length okay and the arc of smallest length is called a geodesic okay.

So now what is happening is that if you take the sphere it is quite easy to see that the if you give me 2 points on this sphere, on the surface of the sphere then the geodesic is exactly the following for those 2 points you get a big circle, a great circle a circle of largest radius on the surface of the sphere passing through those 2 points and you take the minor arc okay any 2 points of a circle will split the circle into 2 arcs and you take the minor arc the one of smaller length and take the length of that, that is exactly the spherical distance okay and that is what I am denoting as d of s it is a geodesic for the sphere for any 2 points on the sphere, so the great circle are the geodesic minor arcs of the great circle are the geodesic for points on the sphere.

So what is happening is that now I have all these 3 metrics, the beautiful thing is that these 3 metrics give you metrics not only on the complex plane the point is they give you metrics on the extended plane okay see what I have drawn here is for Z 1 and Z 2 imagining Z 1 and Z 2 as points on the complex plane but I can very well make Z 1 or Z 2 to be the point at infinity. Now when I say I make the point Z 1 or Z 2 to be the point at infinity I cannot see it on the complex plane okay but I can see its image on the Riemann sphere it is the North pole, so basically what I am doing is I simply taking 2 points on the Riemann sphere and I am measuring their distance, the distance between them either the chordal distance or I am measuring the spherical distance, so the moral of the story is that these distances help you to give a metric on the extended plane and the topology induced by all these metrics is one and the same, all these metrics are equivalent okay.

So this is the fact that you need to check from you know this is very easy fact to check are logically let me tell you how to do it how will check the 2 metrics are equivalent for a topological space it is very simple, what you do is that you show that you take an for each point of the topological space you take small open balls with respect to one metric and show that it contains a small open ball with respect to the other metric and do this for both metrics symmetrically and then you are done okay, so you can see pictorially you can see that it is true okay if for all the 3 metrics, so therefore all these 3 metrics will give you one and the same topological space structure on the extended complex plane which is the same as the one-point compactification and that will be exactly holomorphic to the Riemann sphere by the stereographic projection okay and the advantage of doing all this, why do all this? You can ask me why do all this?

The advantage of doing all this is that now I can say, now I can make sense of the following statement. A sequence of f function f n converges normally to infinity with a function infinity it makes sense now because I can say I can say convergence with respect to this metric one of these metrics namely the 2$^{nd}$ and 3$^{rd}$ one which are also defined for the point at infinity okay and that is the reason why we need to use that okay, so let me write this down, all the so let me write somewhere here may be use a different color. All the metrics below are equivalent on C alright and the last 2 metrics are equivalent on the extended plane. The latter to so let me make some space let me get rid of this, the latter two namely these 2 are equivalent on the extended plane C union infinity okay and of course I rubbed of Z equal to x plus i y so let me write it here okay and now here comes the here comes our the advantage of this.

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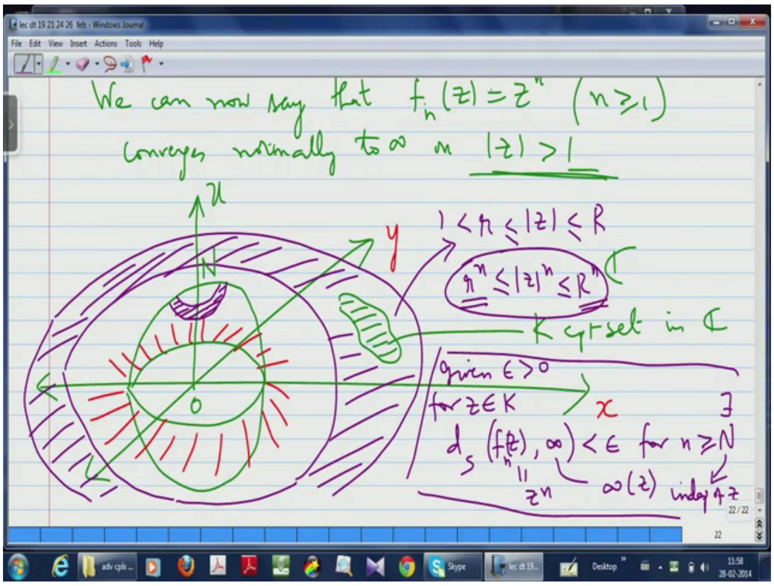


We can now say that f n of Z equal to Z power N, n greater than or equal to 1 converges normally to infinity on mod Z greater than 1 you can say this you can make this statement it makes sense okay and why is that correct? You have to do a little bit of you know why I am spending so much time on this is because you see this normal convergence the function which is infinity is something that is hard to…as it is if you do not analyse it is very hard to digester, so it is very important that you understand what is happening here and you must understand that therefore even if you are looking at the normal convergence of (())(28:54) holomorphic analytic functions can still end up with a function which is infinity okay. So you see what is happening?

So again let me draw another diagram, so you see here is the here is your complex plane as it is in R 2 and this is the origin and of course I have, so let me draw the Riemann sphere here okay so this is the situation and you see of course mod Z greater than 1 is this is the region of the plane exterior of the unit circle okay, so basically so this is mod Z greater than 1 yes it is very hard to draw that in three-dimensional diagram, so let me do the following thing let me use another color you know I just looking at so it is all these, so it is this exterior of this unit circle on the complex plane which is thought of as xy plane okay and of course you know the 3rd axis is I am calling it as u.

Now you see, so let me change color again so I have this let me draw this also so that this is u and I have the North pole here okay, now you see these red lines that I have drawn they are supposed to extend outside unit circle the whole exterior of the unit circle and what they are going to give me? They are going to give me this domain mod Z greater than 1 if you consider it as a domain in the extended plane it is a neighbourhood of infinity okay and what is its image under the stereographic projection it is exactly the upper hemisphere okay it is the whole upper hemisphere which you can see clearly is neighbourhood of the North pole okay that is the reason you are looking at alright.

Now you see take a compact subset of mod Z greater than 1 take a compact subset is a close and bounded subset of mod Z greater than 1 okay and so you know if you are looking at a compact subset on the plane okay then any compact subset in mod Z greater than 1 on the plane to be has to lie within a sufficiently well-chosen annulus okay it should lie within an annular region consisting of an inner circle and an ouster circle centred at the origin sufficiently small inner radius greater than 1 and sufficiently large radius greater than of course the inner radius. So you see if I take some compact set here, so here is some compact set, k compact set in the complex plane then you see this is k of course lies inside suitable annulus, so it is going to look like this you know I am going to get something like this, I am going to get this annulus here so I am going to get this annular region mind you this is annular region on the complex plane consisting of the region between these 2 circles and am also including the boundaries go make it compact okay.

So it is a closed and bounded set it is compact and this is a compact set and the point I want to make is that instead of just considering any compact set k in the complex plane which is lying in this domain mod Z greater than 1 it is enough to just consider such annuli which lie in the exterior of that circle okay the exterior of the unit circle and you see if you watch carefully how is this annulus going to be given by well this annulus is going to be given by mod Z less than small r I mean less than capital R less than or equal to capital R less than or equal to small r which is greater than 1 this is how it is going to be where small r is the radius of the inner circle capital R is the radius of the outer circle okay.

This is what this annulus is going to be given by and well what is its image going to be on the Riemann sphere (())(33:39) stereographic projection I am going to get this, see I am going to get something like this here this is what I will get, I will get a curved annular region centred at the North pole alright and now you see that now you can see why the f n converges to the function infinity normally okay because our definition of convergence is in the following sense okay. So the definition of convergence will be of course point wise convergence okay but then point wise convergence if you try to write it in the metric it will create a problem when you put the point at infinity okay, so I cannot say that or each point Z naught f n of Z naught converges to infinity as n tends to infinity I cannot say that. I can say that in a topological sense but I cannot say that in a metric sense but if I use Euclidean metric but then if I use a spherical metric I can say that okay.

So the moral of the story is that if you look at the distance the spherical distance between a point Z I take the point Z in K my compact set okay. If I look at the spherical distance between f n of Z which is in this case Z power n okay and the point at infinity okay and here you see what I mean here is infinity of Z. See infinity of Z I am thinking of the function which is constant function which gives the value infinity to every point so what I have written there is actually infinity of Z and I am saying that it is spherical distance between f n of Z and infinity of Z that can be made uniformly less than Epsilon irrespective of Z if I choose n sufficiently large, so I can make this less than Epsilon okay for an a grade than or equal to capital n irrespective of…

So given if I start with an Epsilon greater than 0 okay for Z in K I can make the spherical distance between f n of Z which is Z power n and infinity I can make it less than this Epsilon for n whenever small n is greater than or equal to a large enough n such a large enough n exist the point is that this large enough n does not have anything to do with the Z. It will work for all Z in the compact set that is the uniformness that is uniformness of the convergence on the compact set, so n this n is so there exit this n and this is independent of Z and you see this fact is true this fact is very. See suppose I give you an Epsilon okay what is the spherical distance between f n of Z and infinity it is actually the spherical distance between Z power n and infinity alright and Z power n is going to lie where, it is going to lie in mod Z power n is going to lie in this annulus okay and you know if you see if I…

The inner radius of this annulus is r power n small r power n the outer radius is capital R power n and you know if I and since R is greater than 1 if I increase n R power n is small R power n itself is going to shoot up okay, so moral of the story is that this region if I have its image in the on the Riemann sphere, what I am going to get is sufficiently small annular region surrounding the North pole and clearly the spherical distance can be made less than Epsilon for any point in that region okay, so that is you know that is pictorial justification or the statement.

So the moral of the story is now you know you are able to justify that this sequence Z power n converges normally the function infinity okay on the exterior of the unit disk in the extended plane okay and the point is that you are using the spherical metrics okay that is the advantage you are using the spherical metrics because it allows you to give you to measure the distance even to the point at infinity from a finite point in the complex plane which you cannot do with the Euclidean method and the way all this is done it will also extend the usual definition of normal convergence. Suppose you have a sequence of analytic function which converges to a (())(39:22) analytic function itself okay a finite valued function.

A function that does not take values infinity then what is the usual convergence that we talked about, the usual convergence that we talk about when you for example when you do a 1st course in complex analysis the usual convergence that you talking about is with respect to Euclidean metric okay you have to only worry about the Euclidean metric, nobody is worried about the point at infinity to begin with okay. Now if you take a usual (())(39:56) sequence of holomorphic functions on a domain, analytic functions on a domain suppose it is converging normally again to an analytic function on the domain okay. Then suppose this convergence is in the usual sense I am saying this convergence is also correct with respect to the spherical metric.

The reason is because the spherical metric when you restrict it to the usual complex plane it is equivalent to the usual Euclidean metrics, so you do not lose anything. So what I am saying is that this definition of sequence of functions converging to another function normally on a domain that works irrespective of whether you are using the Euclidean metric or whether you are using the spherical metric but the point is it helps you when infinity values are taken it helps you because when infinity values are taken you cannot use Euclidean metric you can use only the spherical metric okay. So the normal convergence under the spherical metric is just an extension of the normal convergence under the Euclidean metric as far as subset of the complex plane is concerned convergence under this spherical metric is same as convergence under the Euclidean metric. Normal convergence under the spherical metric is same as normal convergence under the Euclidean metric because they are equivalent okay, so this is 1 point that you need to understand.

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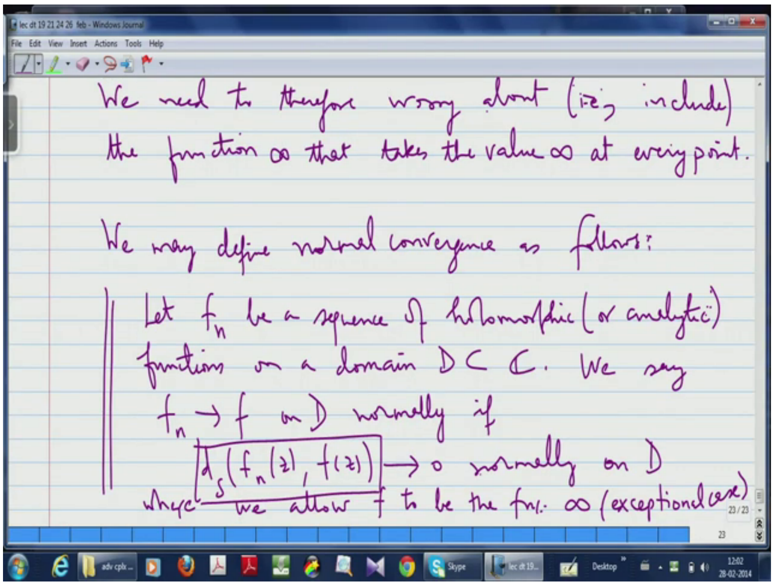


So you know so let me say this so let me write this specifically we need to therefore worry about by the time it include we need to worry about that this include the function infinity that takes the value infinity at every point of your domain. Then we may define normal convergence as follows; let f n be a sequence of holomorphic functions or analytic holomorphic is same as analytic functions on a domain D in the complex plane. We say f n converges to f on D normally okay. If the spherical distance between f n of Z and f of Z goes to 0 normally which means uniformly on compact sets on D where we allow f to be the function infinity this is an exceptional case okay, so you define normal convergence in the following way.

So what is the normal convergence you have sequence of functions on a domain in the complex plane you say f n converges to f on the domain okay. If the spherical distance okay that converges to 0 okay and you this spherical distance is a function of Z, so Z is varying on the domain, so I wanted to understand this Z is varying on the domain. This quantity here is also a function of Z it measures for each Z it measures the spherical distance between f n of Z and f of Z okay and what I want is that a function of Z should go to 0 the constant function 0 uniformly in Z on compact subset of set of the domain, so I wanted to go to 0 normally on the (())(45:09) that is my definition and now the beautiful thing is you know we need this definition because if you take the domain as I told you if you take the domain to be mod Z greater than 1 and you take f n of Z to be Z power n such a definition is necessary.

So what it tells you is that now you have to also worry you are not worried only about functions which takes complex values you have also allow functions in the value infinity but then notice if you take the value infinity if you allow function take the value infinity then you can include Meromorphic functions because you can define the value or the Meromorphic functions at a pole to be infinity and the beautiful thing is the very same definition this very same definition works absolutely well if you change holomorphic by Meromorphic. So that is because of a circle and cemetery rotational symmetry that is the about for this spherical metric that I will explain in the next lecture at the point is so the important observation is that this same definition works with holomorphic replaced by Meromorphic and that is all that we need to do all the analysis we want okay. So I will stop here.