Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

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Lecture No 17

Why Normal Convergence, but Not Globally Uniform Convergence, is the Inevitable in Complex Analysis

Alright so we are continuing with our discussion about studying spaces of Meromorphic functions okay. You have to do topology on spaces of Meromorphic functions and then you know I was trying to explain last time that you know in general if you go and draw inspiration from topology okay the usually if you take the space of functions okay. Then what you do is that you put restrictions like you know and of course you know you assume that you are worried about functions which are taking real values or complex values okay.

So basically you start with a topological space if you want in general or you take a metric space and then you look at the functions on that which are either real valued or complex value okay stop of course you know even if the set you start with this is just a set, it is not even a topological space. Even then this set of all functions real valued functions or complex valued functions will form a real or complex algebra okay that is it will be a ring a commutative ring under point wise addition and multiplication and it will include the constants which are the real numbers or the complex numbers as constant functions and it will also have a vector space structure over the constants.

So it will be if you are considering real valued functions it is a real vector space if you are considering complex valued function it is a complex vector space and it is in fact it will be real algebra or complex algebra because it is also ring it is a commutative ring with the ring structure compatible with the vector space structure, okay the ring multiplication is compatible with the scalar multiplication okay. So but then this is for any set but then now you know if you make that set to also to have topological space structure in addition okay then what you can do is… well even before that suppose you still only have a set okay X and you are only looking at say real valued functions are complex valued functions on the set okay which is I told you is an algebra okay, what you can do is that if you are looking at bounded functions okay namely that you are looking at functions whose images in the real line or the complex line is they are bounded subsets okay.

If you are looking at such bounded functions that forms a subset okay and in fact it forms a sub algebra okay because the sum and product of bounded functions is again bounded alright and also you get a smaller algebra which consist of now which consist of only real or complex valued functions but which are bounded okay. Now the advantage of having this bounded nice is that you can now define a norm on this vector space structure and make it into normed vector space or normed linear space okay and the norm is just supremum norm, so what you do is that since you are looking at a function on a set which is taking real or complex values and since the values it is taking a bounded you can simply take the modulus of its values and take the supreme of all those values okay and you get what is called the supreme norm of a function okay.

Now this supreme norm will be a finite quantity okay it will be a finite positive real number and it will be and it will be (())(5:20) metric space structure, so what happens is that if you look at just the set of real valued or complex valued oceans which are bounded and you add this norm then you get a normed vector space, it is an norm linear space and it is in fact you know this any norm linear space has a metric which is just given by basically you know the distance between 2 elements in that vector space is just the norm of their difference okay that is a metric.

Now with respect to this metric it becomes metric space and then in fact with respect to this metric space it is actually it is actually a complete metric space okay, it becomes a complete metric space and the completeness is just because of you know completeness of the real line or the completeness of the complex plane or which is more generally completeness of the Euclidean spaces RM okay R1 is the real line, R2 is same as complex plane (())(6:19) topological okay. So therefore we get so much if you just have a set X if you just have a set X let me repeat and you are looking at the collection of all real valued or complex valued function on X which are bounded that is already a banach space, it is a complete norm linear space. It complex as a metric space for the metric induced by the norm okay, now you put the extra condition that the set X is also a topological space example it may be a metric space alright you put an extra condition now on the set X is no longer just a set but it is a topological space.

Now the advantage that you have these extra structures of topological space on X tells you that you can further restrict functions instead of looking arbitrary functions you can look at functions which are tenuous okay. So now what you can do is you can start looking at art only bounded real valued or complex valued functions on the set X but you can also insist that you are only going to look at those among these that are continuous with respect to the topology on X okay and that is how you get the space of a real valued or complex valued continuous bounded functions on a topological space X and both of these are banach algebra okay they are commutative banach algebra over the base field which is either complex numbers or the real numbers okay and but the point is that the topology on this Banach algebra, this topology has got to do with actually it is got to do with, it has got to do with point wise convergence okay.

So if you have a sequence of functions which converges to a limit function in this Banach algebra okay then what it means is that it not only means if the functions are converging point wise okay it does not only mean point wise convergence it actually means uniform convergence okay that is the fact, so a sequence of so let me say it in simple words a sequence of…let X be topological space, let f n be a sequence of continuous bounded real valued or complex valued functions on this topological space X then f n converges to f for the topology or the metric space structure on the banach algebra of functions if and only if f n converges to f point wise with respect to X and the convergence is also uniform okay, so the moral of the story is that in topology on the space of functions that you are trying to study has certainly got to do with point wise convergence and it has in fact got to do with the uniform convergence okay.

Now what I want to tell you is that see this is the motivation from topology but with complex analysis in serve more complicated because the functions are not just continuous functions, your functions are going to be analytic functions whenever you consider them and more generally we want to actually worry about families of Meromorphic functions okay and you know Meromorphic functions are slightly more complicated because they are poles and at the poles they go to infinity because by definition at a pole the function goes to infinity. So it means that you have to do something the complication you have to worry about is that you cannot just consider real or complex valued functions okay for example if you are looking at Meromorphic functions you have to take functions with values in the extended complex plane. You have to include the value at infinity because that is the value you will assign to the function at a pole okay that is 1 point okay then the other thing is about.

So you know the modification as you come from the topological point of view which is the motivation of our situation which is we want to study Meromorphic functions, the 1st

modification is you have to include the value infinity okay, so you cannot just look at complex valued functions so you have to look at actions which values in the Riemann's sphere essentially being thought of as the extended complex plane, so you should not take functions values in C we should take function values in C union infinity okay.

Now and thankfully C union infinity is not very bad in fact it is very good because it is in fact a metric space it is a complete compact metric space okay and that is because of the stereographic projection which identifies it with the Riemann sphere okay that is one part of the story, the other part of the story is that I told you that from the topological point of view topology on the space of function has got to do with point wise convergence and in fact it has got to do with uniform convergence but you see again when you come to complex analysis when you are studying holomorphic functions or analytic functions you do not get uniform convergence in general what you get is only uniform convergence on compact sets you do not get uniform convergence just like that on whatever domain you are working with okay.

You only get normal convergence which is point wise convergence which is uniform only when restricted to compact subset, so you see therefore what I am trying to tell you is that say when you when we take inspiration or when we try to think of an analogy from topology and move it to complex analysis okay. In topology what you are thinking of is just space of continuous functions, bounded continuous functions okay and they are real valued or complex valued okay and the topology has got to do with exactly uniform convergence but when you come to complex analysis you are worried about space of analytic functions or holomorphic functions or you are more generally worried about space of Meromorphic functions okay.

Then the complication is that 1st of all is that you have to worry with values in the extended plane you have to add the value a point at infinity to get a value at infinity the other thing is that you will also have to worry about is uniform convergence is less because uniform convergence you do not get, so I will tell you why I will give you a very simple example. The simplest example of that is for example gotten by looking at the you know the geometric series which is you know the fundamental series that we all have been studying from high school and if you really if you go back to your 1st course in complex analysis you will see that almost everything that you have proved about power series started with an argument that has got to do with you know simple properties of geometric series okay.

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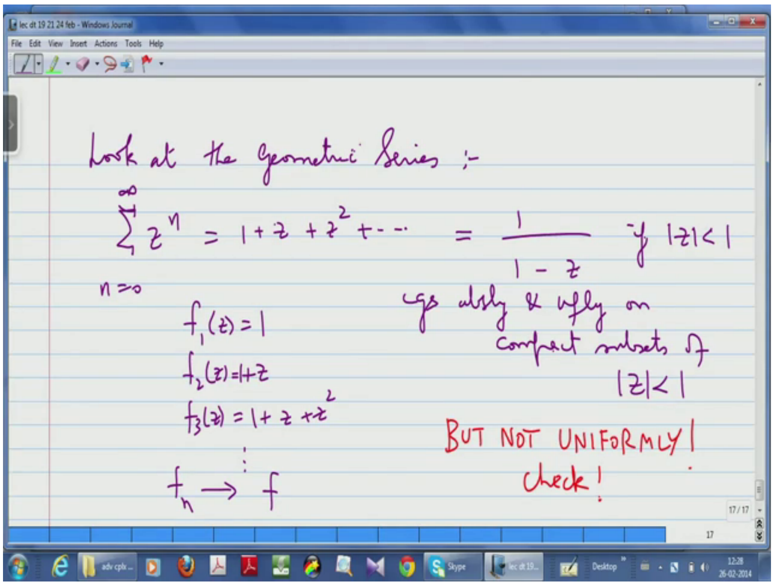

So let me say that so look at the geometric series, let us look at this geometric series Sigma Z power n, n equal to 0 to infinity okay this is simply one plus Z plus Z square and so on this is a familiar series and you know you have all studies this, this is actually we write this as one by 1 minus Z if mod Z less than 1 okay. This is something that we have all seen from probably high school okay. Now but what is going on here, so the 1st thing is that one by one minus Z the function on the right is an analytic function in the domain mod Z less than 1 which is the unit disk okay and whatever we have written on the left side which is the geometric series is actually the Taylor representation. It is just the Taylor expansion over the function centred at the origin, so it is actually the MacLaurin expansion okay.

So this is just MacLaurin expansion, so all you are saying is that one by one minus Z has a MacLaurin expansion given by the geometric series okay. Now the fact is that if you were to go little back to your 1st course in complex analysis you would have noticed that you know the big deal is that this convergence is…the point is that this convergence is absolute okay it is uniform on compact subsets okay. These 2 come because of the Weierstrass m test okay with the Weierstrass m test mind you is a tool that will help you to decide whether a series converges uniformly and the way it works is that it will always give you absolute convergence because what the Weierstrass m test does is actually it tests the absolute series that is the series gotten by putting mod to the terms of the original series and it test that series for convergence and you know absolute convergence implies convergence it is stronger than convergence.

So that is what we use to okay that this series converges absolutely and uniformly on any closed subset of the disk but the point is that if you take the whole disk the convergence is not uniform okay it is this is something that you can work out okay. You assume that you can do it in 2 ways either you can directly write out estimates or you can For example assume it converges uniformly on the whole disk and you will get glaring contradiction okay stop so the point about so this is the point I am trying to tell you, see look at this fact that this series does not converges uniformly on the unit disk, so you see what look at what we are getting let me put F1 of Z equal to 1, so let me let me write this here about converges absolutely and uniformly.

So I am using absly for absolutely and ufly for uniformly as abbreviation okay on compact subsets of mod Z less than 1 and since any compact subset of mod Z less than 1 the unit disk is always contained in a closed disk centred at the origin you can actually verify this condition only for closed disk centred at the origin contained inside the unit disk okay and that is how it is done when you do the Weierstrass m test you take the radius of that closed disk as the…and its powers as the Weierstrass Constance the m n that I used in the m test okay, so the point is so let me write this in red because this is very important, so let me write here but not uniformly okay.

So this is something that I want you to check if you have not objected please check it, please check it because it is a lesson that you know it is not the way you always wanted to be. It is not uniform but not…so when I say but not uniformly I mean it is not uniform on the whole disk okay you get uniformness only on compact subsets of the disk, so what does it mean? Suppose I write the sequence of functions which are given by the partial sums of the series, so I write f 1 of Z is 1, f 2 of Z is 1 plus Z then I write f 3 of Z to be 1 plus Z plus Z square and so on then you see what we are saying is that f n tends to f okay because by definition that is what definition of conversion of a series means it means its convergence of the partial sums and f n is the nth partial okay, so f n converges to f and mind you how is this convergence, this convergence is only normal.

It is absolute of course on the whole disk it is absolute on the whole unit disk it is absolute but it is uniform only on compact sets that is it is normal but it is not uniform on the whole disk. So therefore you see you now have a problem in adapting in trying to think off or in trying to compare with the situation in topology. See in topology if a sequence of function converges to a function f if you wanted in the space of functions then it should be point wise convergence and uniform convergence okay but that is not happening here that is not happening here. What is happening here is that you are getting point wise convergence no doubt f n convergence to f point wise is always there the problem is that it is not uniform, it is not uniform on the whole domain which is a unit disk, it is uniform only on compact subsets you are getting only normal convergence you see that is a technical point.

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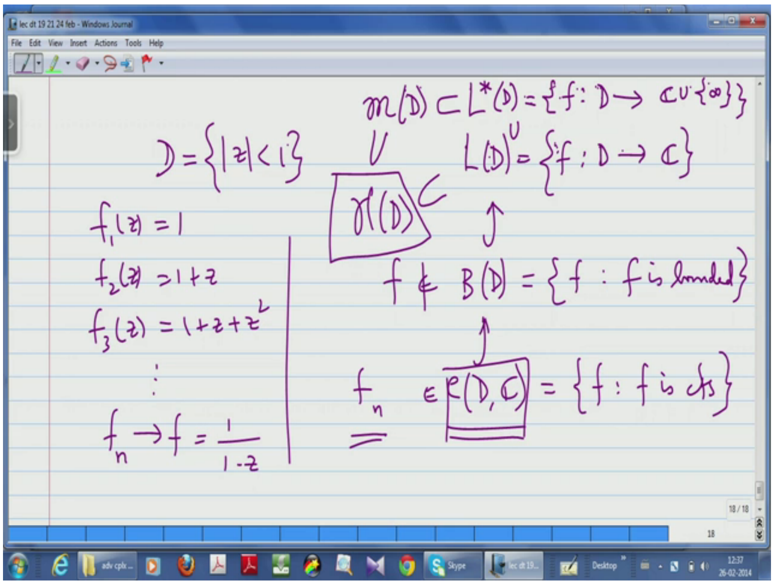

So you know so what I am trying to say is the following thing say if you take the domain D to be mod Z less than 1 the unit disk okay, the set of all Z such that mod Z less than 1 okay and then you know I can do this I can take L D which is the set of all maps from D to C, this is a set of all set theoretic maps D to C and this is an algebra this is C algebra okay one would take B of D to be the set of all among all the functions you take the bounded functions okay such that you take the set of all f such that f is bounded okay and then inside this you take the set of all continuous functions and you know that you know this is an inclusion of these are all inclusion of banach algebra okay.

Now here you see I am looking at this last set below this is the set of all continuous bounded complex valued function on the desk alright. Now you see what happens is that if you watch I have all these functions f 1 of Z is 1, f 2 of Z is 1 plus Z, f is 3 of Z is 1 plus Z plus Z square I mean these are just the partial sums of the geometric series, now you see and you know f n converges to f which is one by one minus Z okay each of it is of course bounded because if you can use a triangle inequality for example f 1 is of course constant so it is bounded and f 2 is 1 plus Z and mod f 2 is mod of 1 plus Z which is less than or equal to 1 plus mod Z by the triangle inequality that is less than or equal to 2.

Similarly f 3 is (())(21:50) equal to 3 okay you see that all the f n are bounded suddenly, bounded continuous functions and the limit function f which is one by one minus Z which is the limit of the geometric series that function is also continuous, holomorphic…f does not even belong here f is not even here okay, so f is not bounded because values of f can become very large if Z tends to 1 or example on the real axis if I approach from the left okay, so in any case what I want you to understand is that you see if you have f n tending to f of course you do not have it in this space okay but you have f n tending to f in it is rather funny you have this space H of D okay you have this space H of D, this H of D is the set of all holomorphic functions on D, analytic functions on D okay and the point is that there are analytic functions like one by one minus Z which are not bounded okay that is because they have a singularity on the boundary of D okay.

So what is happening is that this this convergence is happening here in the space of holomorphic functions and you see that you know it certainly it is point wise convergence that is for sure it is point wise convergence okay but it is not uniform okay it is only uniform on compact sets, so you see why I am saying all this is that I want you to I want you to see that when you go to complex analysis things are very different I mean in normal topology you look at bounded functions and then you look at among those bounded functions you look at continuous functions okay but then look at what is happening in the context of complex analysis even in the case of a very simple domain which is a unit disk it is a bounded domain.

Even in that case you have a sequence of functions which are very much in the space but they converges to a point which is outside okay and the convergence is of course point wise but it is not uniform is only uniform on compact subsets okay, so this should tell you that you know that at something has to be done if you want to do topology on the space of holomorphic function something more serious has to be done okay and of course you know let me tell you that if you if you take the Meromorphic functions on D, if you take the space of Meromorphic function on D then things are you know slightly more complicated I cannot think of them as functions from D to C okay. I can think of them as only as actions from D minus the poles of that Meromorphic functions to C and if I want to think of them as functions on D I will have to include the point at infinity.

So what I will have to do is I have to take this set, set of all f from D to C union infinity I will take this C union the point at infinity, I will take this guy I will take this and let me call this as something else star or L star of D this is a bigger set than this, so this is subset of this and this is here and this is (())(25:38) here okay so the moral of the story is that you have to worry about this value at infinity and you will have to worry about the fact that life if you are looking at analytic functions you are not going to get uniform convergence, you are going to get uniform convergence only on compact sets okay and the simplest example namely geometric series tells you that okay. Of course there are situations where you can get uniform convergence on an unbounded set that can happen but they are special cases okay.

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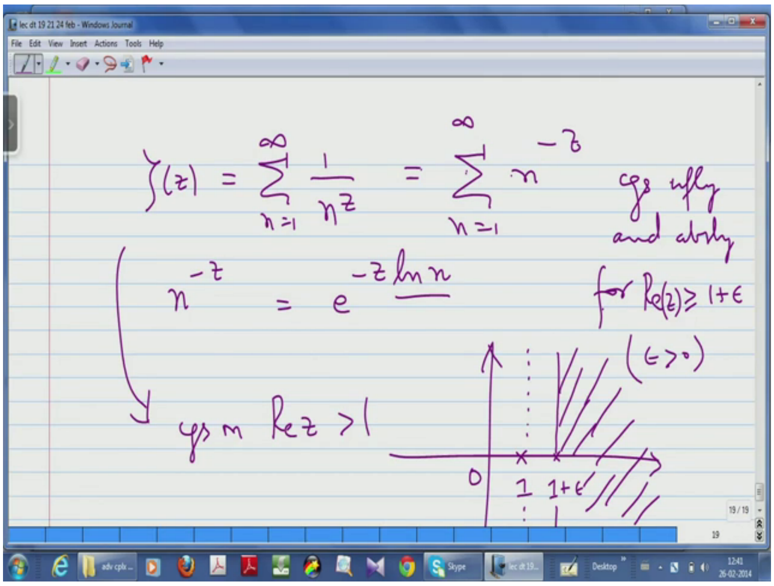

For example you know let me give you one example take the zeta function, take Zeta of Z be you know this Sigma 1 pi n power Z, n equal to 1 to infinity okay and this is by definition well this is Sigma n equal to 1 to infinity n power minus Z and the way you define n power minus Z is you define it as E power minus Z (())(26:43) n okay using properties of logarithms and this (())(26:49) n is actually the real logarithm okay it is real logarithm and the fact is that this function here is Riemann's zeta function and of course you know it is very famous it is the most famous function introduced of course by Riemann 157 years ago okay and it is the most mysterious function in all of mathematics and the Riemann hypothesis which is a conjecture of both that function is one of the most famous unsolved problems and the effect of solving that is like you would have solved thousands of theorems in number theory okay.

So it is a very deep function, very mysterious function at the fact is that this converges uniformly for any closed right half plane to the right of real part of Z equal to 1, so let me write this converges uniformly and absolutely 4 mod Z greater than equal to 1 plus Epsilon, Epsilon positive so I should not say mod Z I mean real part of Z, so this is you know this region is something like this, so you have so this is 1, 1 plus Epsilon is somewhere the right of that and in fact if you take this shaded region along with the boundary, this is a shaded region where you get uniform convergence okay it is a half plane, it is the half plane along with that vertical line which passes through x equal to 1 plus Epsilon okay and that is a close set mind you because the boundary is included it is a close set and this is an unbounded close set, it is not a compact set but inspired of this being unbounded you get actually uniform convergence okay and in fact therefore you will get convergence on you call right half plane, so what you will get is that it converges on the real part of Z greater than 1 okay.

So, order of the story is that there are special cases where you can get uniform convergence on a huge set even on an unbounded set okay but that is not but that will happen only in special cases, you cannot expect it to happen always alright, so there is an issue the issues that whenever you are studying spaces of analytic functions or more generally Meromorphic functions you can expect only normal convergence okay that is one issue. Then see now what I want to say is that in any case, so to sum up what I want to say is that we will have to worry about when we are worrying about analytic functions or Meromorphic functions you have to worry about 2 things, one thing is that is not uniform convergence it is normal convergence is what you will get.

So whenever so you must keep this in mind as a philosophy, whenever you are worrying about sequences of functions converging in theorems in complex analysis, mind you this is equivalent to doing topology on the space of such actions and the topology is being done essentially by looking at point wise convergence which is normal okay this is one fact that you should always remember and the other thing that you will have to remember is that you have to worry about the value infinity when you are worried about Meromorphic actions okay and the way you deal with infinity is by always appealing the stereographic projection and comparing infinity the North pole and comparing neighbourhood of infinity with the neighbourhood of the North pole via the stenographic projection, so these are the techniques that we should use okay.

Now what I will tell you is that at least this is again something that you should have seen in 1st course in complex analysis but let me recall. Suppose you have a domain in the complex plane suppose you have a sequence of functions analytic functions defined on the domain okay, so these are honest analytic functions they are holomorphic functions. Suppose the sequence converges to a function normally on the domain then the limit function is actually analytic, it is holomorphic okay, so it is not that it will go out of the space of analytic functions okay, so I thing you should have seen the proof of that but in any case let me recall it.

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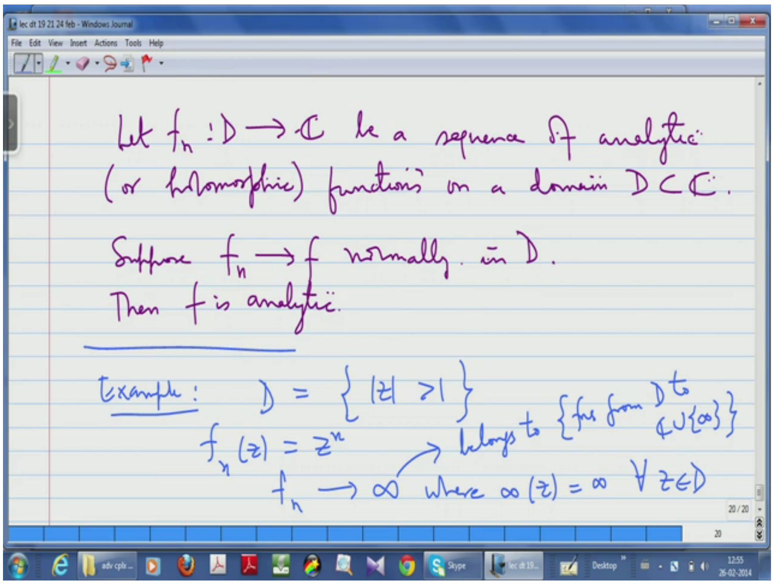

So let so the 1st thing is you take a limit of analytic function you should get an analytic function okay at least that you must have otherwise you cannot go ahead but the point is we have…the reason why it works out well is because we have agreed that whatever we do it will be point wise convergence which is normal and this normal convergence is the technical thing with which we can do a lot of things because actually it is a local version of uniform convergence. It is stronger than a local version of uniform convergence okay, see when you say there is a normal convergence you are saying it is normal the convergence is uniform in compact sets okay.

Now there are 2 types of compact sets that you can think of which are important for complex analysis, one kind of compact set is a canny point, take a small disk around that point okay and then take its closure that is a compact set but the beautiful thing is it is a compact set which contains an open set around that point, so if you have a normal convergence then you will have certainly a uniform convergence and therefore on sufficiently small neighbourhood of every point that is because given any point all you have to do is you have to choose a sufficiently small neighbourhood whose closure is also in your set okay. Basically you will have to choose a sufficiently small circle surrounded by that points such that the circle and the interior of the circle is in your set which will happen because you are only working with open sets and points inside open sets.

So therefore what you will get is you will get uniform convergence on sufficiently small disk, so you must remember normal convergence does not give you uniform convergence on the whole set but locally it gives you convergence, uniform convergence it gives you uniform convergence on small disk okay, sufficiently small disk that is one fact then the other fact is what is another kind of compact sets that you are always interested in you see as you know one of the most important techniques in the theory of complex analysis is integration, (()) (33:38) integration theory and what do you integrate on you integrate on contours and what are contours? They are of course compact sets because they are curves basically there are just continuous images of a closed and bounded interval compact interval on the real line, so they are compact.

So the advantage of that is that the other important type of compact set that you are always interested in are contours and then uniform convergence on the contour will tell you that integrating the limit function on the contour is the same as integrating each of the functions on the contour and then taking the limit, so you can interchange the limit and the integral that is what compactness, uniform convergence for contours tells you, so you see that… therefore moral of the story is that while normal convergence is not uniform convergence on the whole set, it still gives you everything that you need locally for the differentiation theory.

It gives you everything that it gives you normal… It gives you actually uniform convergence on sufficiently small disk surrounding every point and further integration theory it does give you uniform convergence on every contour therefore everything goes through nicely okay, so what this tells you is you really do not need uniform convergence on a whole open set and the example of the geometric series tells you that that is what is expected and that is what you will get you will not get anything better than that okay, so that is good enough okay.

So let me write this down let f n from D to C be sequence of analytic functions, analytic or holomorphic functions on a domain D inside the complex plane. Suppose f n converges to f normally in D then f is analytic, so the moral of the story is that you know under normal convergence, normal convergence preserve analyticity. When a normal limit of analytic function is analytic that is all I am saying okay and so let me tell you what is the way you prove this, the way you prove this is 1st of all because each f n is continuous okay f will become continuous and why is that true because continuity is a local property show that f is continuous on the domain it is enough to show that f is continuous on sufficiently small disk surrounded at every point but if I take a sufficiently small disk okay such that its closure itself is contained in the domain then I will in fact get uniform convergence and you know uniform limit of continuous function is continuous therefore locally I will get continuity but continuity is a local property if it is true locally it is true globally.

So I get continuity okay. Same argument applies to analyticity, how do I check a function is analytic if it is locally analytic it is analytic because analyticity is also a local property, so what I have to do is that I have to take any point in the domain I have to show that in a small disk surrounding that point the function is analytic, how do I do that? It is very simple, if I take a small disk surrounding that point of course I have uniform convergence okay because you have taken a sufficiently small disk, the closure of that disk is a compact subset of the domain okay. Now how do I get analyticity, it is very simple I simply use Moreira's theorem. What I do is that I take a point, I take a sufficiently small disk surrounding that point open disk okay.

If I take the closure of the disk then the convergence is actually uniform because it is a compact subset, so the moral of the story is that to check that the function is analytic inside that disk I just have to… I know already it is continuous so I have to only check that the integral over any closed path is 0, the integral over any simple closed contour is 0 is what I have to check but if I want to, so I try to integrate the function around a simple closed contour inside that disk but then integrating the function is the same as integrating the sequence, each member of the sequence of functions and then taking the limit, I can change the integration and the limit okay, so but if I integrate each of those functions in the original sequence around a simple closed contour I am going to get 0 because they are analytic that is because of Clausius theorem.

So what I get is that the integral over this limit function over any simple closed contour in a sufficiently small disk surrounding a point is always 0 and Moreira's theorem will tell me therefore that it is analytic and therefore it is locally analytic and therefore it is globally analytic that is it okay, so this is how you do it with sequence of analytic functions but the problem is that when you start worrying about Meromorphic functions things becomes complicated, so you know so let me look at the following example, so let me take another example and see the points about these examples is that you should know what kind of things happens, so here is another example. What you do is you take the domain D to be the exterior of the unit disk okay you just take the set of all mod Z greater than 1. Mind you this is an deleted neighbourhood of the point at infinity you know the exterior of a disk is always a deleted neighbourhood of the point at infinity.

If you add the point at infinity this becomes actually a neighbourhood of infinity in the extended plane okay and this is the deleted neighbourhood of infinity and what you do is you put f n of Z you put it as Z power n. Take this (())(40:05) sequence of functions okay, now what will happen is you can see if you fix…mind you if you take any value of Z greater than one with modulus greater than 1 that is it is basically a point lying outside the unit circle okay in the complex plane okay and if I plug it in this sequence f n I will get higher powers of that point and that is going to go to infinity in modulus as n tends to infinity because mod Z is greater than 1, mod Z power n is also greater than 1 and mod Z power n will tend to infinity as n tends to infinity. So what happens is that at every point this sequence convergence to what?

The value it converges to is the point at infinity okay it converges to the point at infinity alright. Therefore the limit function is what? The limit function is a function which maps the whole exterior of the unit disk the point at infinity, it is a constant function with the value infinity okay that is what you get in the limit. Now this is the kind of thing that this is the kind of pathology that happens and we have to take care of this okay. So mind you in this example all these f n are in fact holomorphic functions f n is a sequence of holomorphic functions, f n is in fact a sequence of holomorphic functions and f n converges to the function which is constantly equal to the value infinity at every point and of course you know if you check very carefully is convergence is even uniform on compact subsets okay, so here is something that you do not expect that is happening.

A nice sequence of functions even holomorphic functions, even analytic functions on a domain can actually converge uniformly to the function which is constantly equal to infinity at all points okay and that is no function by any of our standards okay. It is certainly not Meromorphic function there is no set on which it is analytic, it is basically a constant function, it is the function that maps the whole domain onto the point at infinity, so let me write that down so f n converges to infinity and what is this infinity? Where infinity of Z is the point at infinity for all Z in D okay and this infinity this belongs to the set of all functions from D to the extended plane okay.

In fact it is that unique function which maps the whole domain to the point at infinity a, so this is one standard pathology, so you see what is happening even if sequence of holomorphic functions even a sequence of good analytic function is going uniformly to infinity alright this is happening, so the moral of the story is that when this happens for good holomorphic functions it will also happen for Meromorphic questions.

So the moral of the story is that whenever you study sequence of Meromorphic functions okay and particularly we will be interested only in normal convergence we will have to define what is meant by normal convergence of sequence of Meromorphic functions that has to be done carefully because we will have to worry at the value about the value at infinity and we will also have to worry about the point at infinity if your domain also includes the point at infinity you have in the domain also if you have the point at infinity and in the range also if you have the point at infinity then it is a double complication you have to worry about it okay but in any case we are going to be worried about sequences of Meromorphic actions which are converging normally and the 1st pathology you should expect and this is the only pathology that you have to really worry about is that that sequence may converge uniformly to the function which is infinity everywhere, so you will have to do this you have to add this function artificially okay as a function in your list of functions okay and deal with it, so this is something that one needs to worry about okay, so let me stop.