Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

Dr. Thiruvalloor Eesamaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology Madras

Lecture No 16

Continuity of Meromorphic Functions at Poles and Topologies of Spaces of Functions

So let us continue with our discussion see we were discussing Meromorphic functions okay and you know we have to as I told you we have to worry about Meromorphic functions because eventually our aim is to prove the Picard theorems and to prove the Picard theorems we need to actually do some topology on a space on a suitable space of Meromorphic functions okay. So you see doing topology on a space of function is kind of the background theme okay so you must understand what this is all about okay and for that I need to 1st of all tell you about spaces of holomorphic functions okay or analytic functions and of course you know analytic functions or holomorphic functions are also considered as Meromorphic functions okay.

So Meromorphic also includes analytic okay and of course the last time we saw that if D is a domain in the extended plane okay then the set of Meromorphic functions on D into those functions which are analytic on D except for poles okay that set is actually a field, it is a field extension of the complex numbers and I told you that the behaviour the algebraic properties of this field extension have got a lot to do with the geometry of the domain okay and this philosophy is in fact very useful and it is exploited when you replace the domain by in fact Riemann surface okay.

Now you see but whenever you are looking at a domain and of course let me also remind you when I say domain I am taking a domain in the extended plane which means that the domain can also include the point at infinity okay which means that I am also allowing study at infinity, so I am allowing neighbourhoods of infinity also to be domains okay and this is very important because of the following reason, you see you take a Meromorphic functions okay then on a domain in the extended plane okay, now of course you know it is a Meromorphic function so it can only have the only singularities it can have our poles, so in particular it can have only isolated singularities and they are poles okay and if you go to as you approach a poll the function values approach infinity okay.

So that the function values approach infinity is can be understood in 2 ways, one way is in a…that is the way in which would have learned when you did the 1st course in complex analysis that you see saying that f of Z tends to infinity is same as saying mod f Z tends to infinity okay that is it becomes a modulus of the function becomes larger and larger there is one way of saying it but then there you do not really think of infinity as a point but now you know we are used to thinking of infinity as a point namely a point in the extended plane and you physically see it as the North pole in the Riemann sphere when there is when you look at this identification of the Riemann sphere with the extended plane okay.

So you know you picture a domain in the extended plane as by thinking of its image on the Riemann sphere, so you imagine the Riemann sphere okay there is the North pole which corresponds to the point at infinity in the extended plane okay and mind you the Riemann sphere is homeomorphic to the extended complex plane okay it is holomorphic to the extended complex plane with the North pole corresponding the point at infinity, so you can make you can clearly see that you know a small disk like neighbourhood of the North pole that will correspond to the exterior of a circle of sufficiently large radius on the plane by the stereographic projection and when you think of a domain in the complex plane or in the extended complex plane you can think of its image on the Riemann sphere okay.

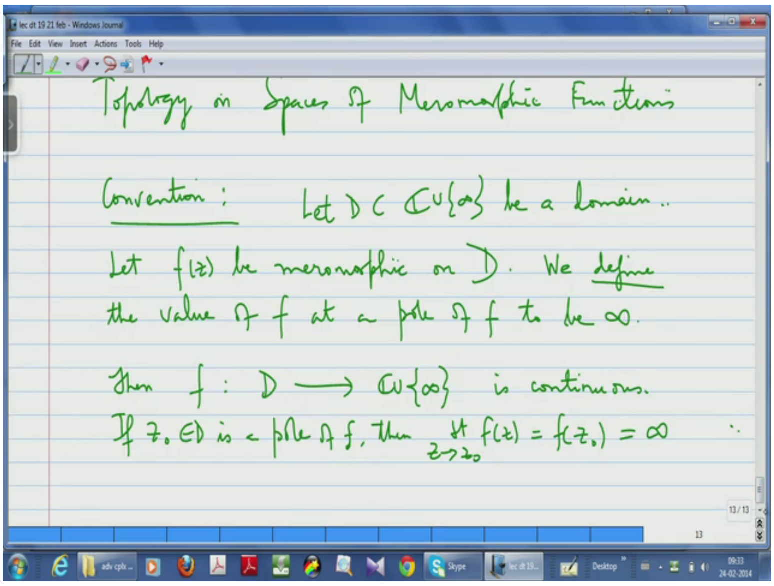
So you can think of a domain in the extended plane as some open set on the Riemann sphere that open set can include the point is can include the North pole or it need not include the North pole that depends on whether the original domain you are looking in the complex plane is actually in the complex plane or it is in the extended plane including the point at infinity okay, so you must imagine whenever you are thinking of a function defined on a domain in the extended plane you must always imagine an open set on the Riemann sphere and that open set can include the North pole in which case it corresponds to a domain on the extended plane which contains the point at infinity okay and therefore we…

So the point is that if you are looking at a function which is the Meromorphic on a domain on such a domain in the extended plane then you know it will have only poles okay but if you now approach…you take the points corresponding to the poles on the Riemann sphere okay you take the points corresponding to the polls on the Riemann sphere okay outside those points the function will take values only in complex numbers okay whereas at the poles what happens is at the poles the function tends to infinity okay, so the moral of the story is what you can do is you can define the function value at each of the poles to be infinity okay and thereby the resulting function becomes continuous at those points okay.

Now there is a subtlety here, I am saying that the function is made a continuous at a pole I defining its value at the pole be infinity okay and when I am saying this I am not saying that the function is continuous in the usual sense okay a function is continuous in the usual sense is if it is continuous at a point it is supposed to have a finite value at that point okay but I want…so I am really thinking of infinity as an extra value okay there is some subtlety involved there is a little bit of confusion involved here because when you say a function is continuous at a point okay it means you know Riemann's removable similarity theorem says if an analytic function has an isolated singularity at a point and if the function is continuous at that point and it is analytic, the point is not actually a singularity what am not saying that I am saying what you do is you take a pole of the function and you make the function continues at that point at that pole in the following sense.

You declare the function value at that point to be infinity okay and regardless function as a mapping in not into C but regarded as a mapping into C union infinity the extended plane and then with respect to that now you on C union infinity is the extended plane there is a there is a topology okay that is exactly the topology which is topological space structure which is homeomorphic to the Riemann sphere okay and so I am thinking of the function as taking values in the Riemann sphere in some sense okay and I am saying that function is continuous at the pole by declaring its value at the pole will be the value infinity okay. Now this is a certain point of view that is very important okay.

(Refer Slide Time: 9:25)



Let me put this following convention, let D in the extended complex plane be a domain, so it is an open connected subset of the extended plane okay. So what you must remember is that D corresponds under the stereographic projection to a domain on the Riemann sphere, so it is an open so the image of D under the stereographic projection is an open connected set on the Riemann sphere okay that is what you must remember okay. So let D be a domain, let f of Z be Meromorphic on D okay that means f is analytic except for an isolated set of points of D at each of which f has a pole that is what it means okay that is the definition of what are Meromorphic functions? Okay so we define the value of f at pole of f to be infinity, see you make this definition okay f at every pole f of a pole is equal to infinity we make this definition.

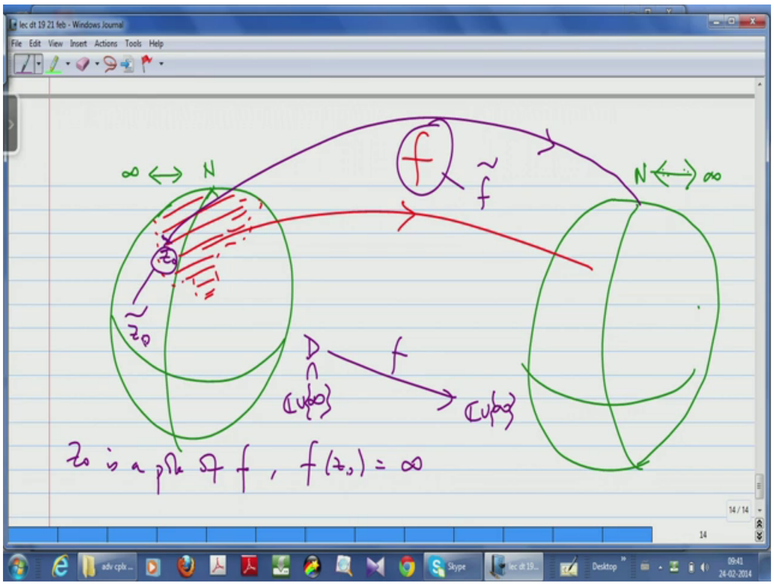
So then the beautiful thing is that f becomes a function from D to the extended plane okay. Now f is a function from D to the extended plane and the point is that as a function into the extended plane this is continuous okay. If you normally taking a 1st course in complex analysis and you are working with a function which has a singular point then what you do is that you consider the function to be defined on the complement of the singular points okay and it is supposed to take only complex values alright. Now what we are doing is we are having Meromorphic functions so that means you are restricting the singular points to be isolated and they must be poles okay, so in principle you should be worried only about the functions outside the poles.

Outside the poles the function should be analytic and it should have complex values okay but what we are doing is… we are also including the poles they are defining the function value at each of the poles to be infinity therefore the function becomes a function into the extended plane okay although it also takes the value infinity and the point is after you do this continuity is not lost and see the reason why the continuity is not lost at a pole is that as if Z naught is a point of the where f has a pole then limit Z tends to Z naught f of Z is infinity okay but infinity is also f of Z naught by definition because Z naught is a pole we have defined the value of f at a pole to be infinity okay, so f of Z naught is infinity and there is also equal to limit Z tends to Z naught f Z therefore you know that is the definition of continuity okay and Z naught, so the function become continuous at Z naught okay and again let me warn you you should not confuse this with continuity in the usual sense.

I am not saying that as an analytic function it is continuous at a pole that is not correct okay it certainly is discontinuous because when I talk about analytic function am only worried about functions which are taking complex values not the value at infinity okay. So when I say continuous here you must take it as a pinch of salt you have to be careful okay, so let me write that down if Z naught belonging to D is a pole of f then limit Z tends to Z naught of f of Z is equal to f of Z naught is equal to infinity, so this is the extra definition okay that we make. So actually you know sometimes in the language of Riemann surfaces we say that Meromorphic function on a Riemann surface is the same as a holomorphic function into the Riemann sphere okay we say this often and it really make sense in this sense okay.

So the point is that by including the value at infinity you are able to give a function value at a pole namely the value infinity okay and that makes the function continues on the domain okay the point is that the target domain becomes the extended plane and the topology on that is he you know the topology of the one-point compactification which is holomorphic to Riemann sphere okay, so for all purposes you know the picture that you imagine is the following, so let me draw this picture it is very important.

(Refer Slide Time: 15:16)



So whenever you know for all practical purposes this is the best thing to imagine on one hand you have the Riemann sphere okay, so on the other hand you also have the Riemann sphere and you know well this point is the North pole okay this corresponds to the point at infinity and this is again the North pole which corresponds to the point at infinity okay and whenever I put this double sided arrow I actually refer to the stereographic projection which compares the Riemann sphere as topologically isomorphic with the extended plane with the point at infinity corresponding to the North pole N and what we are doing is…see basically you are looking at you are actually looking at a domain in the extended plane, so you know it is going to be some domain like this okay and in fact I should not put I should not put the boundary.

So it is something like this so this is my domain, so the way I have drawn it now it contains the North pole, so that means that when you take its image under the stereographic projection it actually corresponds to a neighbourhood of infinity okay and this is an open connected set on the North pole okay and I have this function f I have the function f which is defined on that and it is taking values again in you must think of it as taking values again on the Riemann's sphere okay and whenever you think of Riemann sphere see you picture the Riemann sphere what always think of C union infinity okay the Riemann sphere minus the North pole is the complex plane by the stereographic projection and the North pole corresponds to the point at infinity, so what will happen is that the function f is defined you must picture it as being defined on a subset of the Riemann sphere and the image it take the image of the function is also a subset to the Riemann sphere okay and the point is that if you take a point which is a pole of f okay then the function has a value there that is what we have done.

If you take a pole of f the function values infinity, so you know if I draw a pole so suppose if I had Z naught, Z naught is a pole okay then f of Z naught is equal to infinity okay and so what will happen is that this f of Z naught you must picture it is being mapped down to the North pole okay. Mind you in this the way I have drawn it is not entirely accurate because that is an identification on the left and on the right with C union infinity, so what is happening is that there is this thing on the left is C union infinity okay this thing on the right is also the C union infinity, the C union the point at infinity, so both of them are Riemann sphere and then I have a domain the inside C union infinity and I have this function f this is the picture.

So D is the domain in C union infinity, f is taking values in C union infinity you allow the value infinity and that is a value you assigned to a function at a point which is a pole okay and in principle this f that I have written in red is actually is gotten from the f at I have written in purple after identification with the stereographic projection, so in principle for this f here I should use some other symbol I should use f tilda if you want and furthers point Z naught that I have written in purple is not exactly Z naught it is actually stereographic projection Z naught, the point Z naught is actually as a point in D okay and after I identified the C union infinity with the Riemann stereographic projection than the point Z naught in D corresponds to this point Z naught on the Riemann sphere okay.

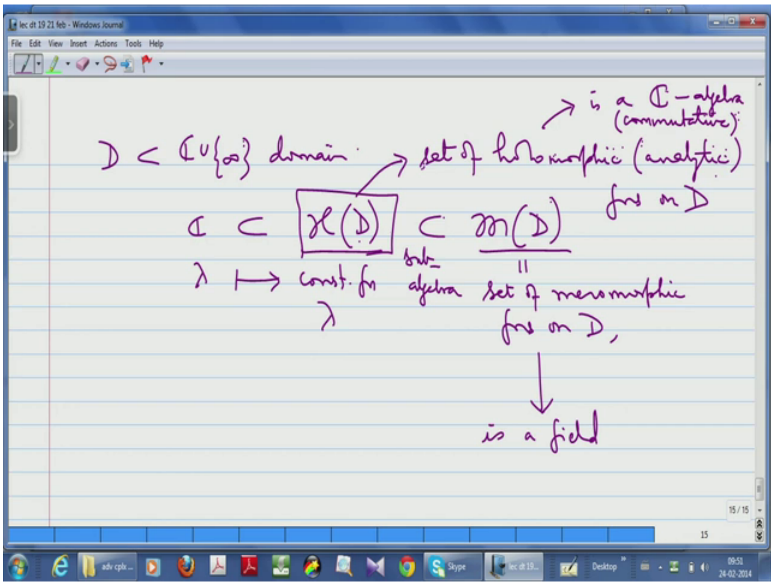
So you know so forgetting this forgetting the notational complication that is involved in all the time carrying the stereographic projection okay you always imagining that things are happening on the Riemann sphere okay and reason why this is very useful is because you see if you are looking at only the extended plane the point at infinity is something that you cannot see okay you cannot see it as a point and more importantly you cannot for example define a distance between the point at infinity and some other point in the plane okay is no way of doing it in the usual sense at if you think of the extended complex plane as a Riemann sphere and the point at infinity is just the North pole and now you know if you now you can define a distance between any 2 points in the extended plane by simply taking the distance on the Riemann sphere of the images of these points under the stereographic projection, so you know for example therefore if you are imagining all the time the Riemann sphere you can even talk about neighbourhood of infinity which has certain radius.

You can talk about a circle centred at infinity with some finite radius okay and that is something that you can picture as a circle as an interior of a circle as a circle surrounding the point N on the stereographic projection with a certain radius, finite radius okay and that radius can be made smaller and smaller and you can imagine that as I make that radius smaller and smaller under the stereographic projection I get a larger and larger circle on the complex plane okay, so see this is a certain point of view that you should always remember. You always think about a domain in the extended plane as actually a domain on the Riemann sphere and open connected set of the Riemann sphere okay and open connected set in the on the Riemann sphere topology is mind you is just the induced topology R3, Riemann sphere is just the sphere surface of the sphere centred at the origin radius one unit in real 3 space and the real 3 space is Euclidean space you know it has a standard topology it is a metric space it is a complete metric space.

So you take their induced topology on the subset so there is a nice topology and it is in fact even a matrix space okay therefore you think of it like that okay, so this is something that you must so you know if you want I should do the following thing you know I should actually relabelled this as f tilda you know and I should actually relabelled this point as Z naught tilda where this tilda means you translate everything to the Riemann sphere and you will have 2 therefore bring in the stereographic projection but most of the time you do not worry about it okay.

See what we are going to do is as I told you we are going to do we have to worry about space of you have to look at the space of Meromorphic functions on a domain okay and you have to do topology on it and more generally the simpler cases trying to do topology on a space of analytic conscience on a domain. So see what is this topology all about? See the most important that we will be worried about in the topology of a space of functions is compactness okay, so you see compactness is the somehow the most important property that we have study, so essentially what we have to study is compactness of spaces of functions. What kind of functions? The functions essentially we have to study Meromorphic functions but then before that you will have to also worry about holomorphic functions or analytic functions because they are special cases of Meromorphic functions stop Meromorphic functions which has no singularities essentially holomorphic or analytic functions.

(Refer Slide Time: 23:55)



So now let me do the following thing, so let D you take D to be a domain in the extended complex plane okay then what you have is…so domain, domain means it is nonempty it is an open nonempty connected subsets okay and then you have this so you have the complex numbers sitting inside as constant holomorphic functions in H of D, so let me put this notation and this is m of D okay. So what is this m of D this is something that I told you last time this is the set of a Meromorphic functions on D okay, so let me write this here set of Meromorphic functions on the okay. This is the field as we have seen last time okay and at this point you only you do not worry about the value at infinity okay at a pole okay this is completely algebraic so you are really not worried about looking at the function which looking at infinity as one of the values of the function okay.

So this is a field we have seen this and you have and therefore this is a field extension of the complex numbers, see this is this map C sitting inside this is you are thinking of lambda as a constant function lambda okay. If lambda is a complex number you identify it with the constant function which is equal to lambda and what is this H of D this is actually the set of holomorphic functions on D. Holomorphic is same as analytic okay this is the set of holomorphic function on the right and the fact is that you know the set of holomorphic functions will only be it will be a ring in fact it will be C algebra it will not be a field because you know I cannot invert a holomorphic function together holomorphic function unless I know that it is a non-zero that is not ever going to be 0 because wherever it is going to be 0I cannot invert it okay because for the inverted function the 0 of the original function will be a pole of the inverted function okay.

So and the moment a function has a pole it is not holomorphic okay it is not analytic. So the point is that you see this is so this is C algebra and in fact it is commutative it is commutative algebra mind you whenever we talked about algebra, the algebra is a ring which is also a vector space over a field and in general it could be the multiplication of the algebra need not be commutative for example the simplest example are of course the matrix algebra you know matrix multiplication is not commutative, so of course algebra's functions are commutative okay because they are taking values in C and C is a commutative field multiplication in C is commutative alright, so the fact is that if you take a holomorphic function in D it is certainly a Meromorphic function on D okay.

If you take a function which is holomorphic on D I have to only worry about to invert it I have to only worry about points where the function has zeros but then you know the zeros of a holomorphic function are isolated therefore at each of those zeros of the inverted function will only have a pole therefore there should be no problems, so if you take any holomorphic function on D you may not be able to inverted to get a holomorphic function on D but if you inverted will certainly get a Meromorphic function on D, so in particular you see the Meromorphic functions on D will contain inverses for all holomorphic functions okay.

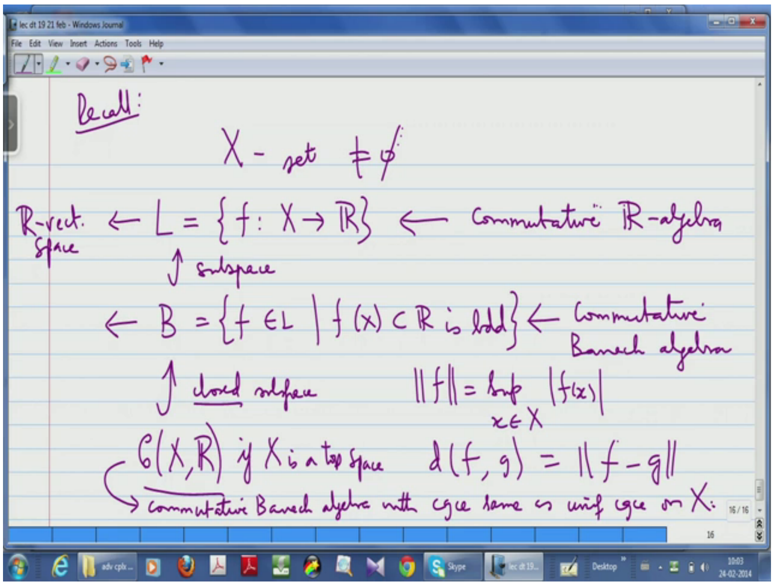
Now fine so m of D contains also inverses of nonzero elements in H of D okay and H of D is mind you it is a subring it is a sub algebra okay this is the sub algebra and H of D is not a field but m of D is a field this is what it is algebraic okay and I told you that there is a lot of geometry in studying this field extension m of D over C it is a field extension studying the algebraic properties where extension has got to do with a lot with the geometry of D okay. Now well keep these things aside, I want to go away the algebra and I want to go do topology, so you see let us look at just H of D just look at H of D and try to worry about topology on this on the H of D okay, so how do you do topology you think of H of D as a set and you would like to give some policy on it okay.

Now and you know is not that you do not give an [inaudible 29:36] topology, you give a topology each makes sense with the functions with the domain of the functions okay, so what is the idea? The idea is your trying to worry about a topology of a space of functions okay, now what I want to do is that you know I want to tell you that this topology has got to do with it has got to do with point wise convergence of functions okay with respect to points of the domain on which the function is defined which is D in this case okay, so I want to say that the topology on H of D or if you want to even worry about more generally topology on m of D that has to do with versions with the point wise convergence with respect to points of D okay.

So this is what am trying to emphasise when I say I want to do topology on this space of functions I want to worry about the topology on the set of holomorphic functions or I want to worry about ology on the space of Meromorphic functions, set of Meromorphic functions okay. Then the topology has to do with point wise convergence, now so why is that true? You need some motivation for that and in fact you know let me be even more accurate we have not really worried only about point wise convergence because point wise convergence is a very weak is something which is very weak you know the reason why it is week is you suppose you have family of functions or a sequence of functions which is converging point wise to limit function then the limit function need not be continuous even if the original members of the family of functions each of the members are continuous okay you know at you need a uniform limit or the limit function to be continuous.

So it is not only point wise convergence you are worried about, we need something stronger that is uniform but what happens in comics analysis you do not get uniform always on your domain. What you get is uniform on compact subsets of the domain example you will get uniform on closed disk inside the domain and that is a special kind of uniform convergence which is uniform convergence on compact sets that is called normal convergence and see what I want to impress upon you is that topology on the space of holomorphic functions or on the space of Meromorphic functions on a domain has to be studied with the viewpoint of normal convergence okay it is point wise convergence and it is not totally uniform convergence but it is uniform convergence on compact sets okay and you know in some sense why is all this important? Because all the you see let me recall a little bit of basic topology to help you to see the following thing.

(Refer Slide Time: 32:47)



Let X be any set so this is some motivation, X is any set take any set not even a topological space of course assumed is not empty okay then what you do is you take L to be the set of all functions X to R okay you take the set of all real valued function on X okay, take the set of all real valued functions on X, so these are just functions on a set okay the only thing is that these functions have real values. Now the point is that is L becomes immediately this becomes a real vector space, L becomes the real vector space alright and you know how because point wise addition and point wise (())(33:43) multiplication make sense, so if f is a function on X with real values and g is another function on X with real values, f plus Z can be defined as f of X, f plus g of X is just FX plus GX for each small x and capital X and similarly if lambda a real number, lambda f can be defined as lambda f value at X is a lambda times FX okay.

So this becomes an R vector space alright and well what you can do is among these functions you can look at those functions which are whose images are bounded okay you can look at bounded functions, so in this L I can have this subset B okay which is the set of all f of belonging to L is such that f of x is bounded, the image is bounded so these are bounded real valued functions okay. The set of all real valued functions L is a vector space it is a real vector space and now you take the subset which consist of bounded real valued functions okay. Then it is also subspace because if f is bounded and g is bounded and f plus g is bounded, if f is bounded then lambda time f is also bounded okay, so the point is that this is a subspace this is the real subspace of this vector space alright.

Now the beautiful thing about this subspaces that because your images of functions are bounded you can make this into a banach space, you can make it into complete norm linear space you can define a norm on it the supremum norm that makes it into a topological space in fact it makes it into a metric space, the metric being induced by the norm and with respect to that metric it becomes complete and this is essentially because of the completeness of the real line okay so this object here is actually and in fact a beautiful thing is that I can in all these spaces I can even multiply to real valued function to get a real valued function and the multiplication is commutative, so these are not just spaces they are in fact commutative algebras. So L is actually in fact the commutative R algebra, B is also a commutative R algebra at B is in fact the commutative banach R algebra okay it is a banach space it is a banach algebra alright.

So let me write this so this fellow here is actually commutative R algebra and this fellow here is actually commutative banach algebra, it is a commutative banach algebra and the point is that you define norm f be equal to supremum small x and capital X of mod Fx and the supremum will be a finite quantity because I am looking at f in BF is the image is bounded and you know that one of the properties of real line is that you take a bounded set its supremum exist and it is a finite real number okay in fact you know that is equivalent to complete (())(37:17). The real line that every (())(37:19) sequence of real number converges is as strong as requiring that any subset of the real line which is bounded above has a supreme okay and that is equivalent to also saying that any subset of the real line that is bounded below has an (())(37:35) okay.

So if you take this norm this is a sup norm, it makes be into normed vector space and the norm induces a metric in the following way d of F, g is just norm of f minus g so with this D function, distance function this B becomes a metric space okay is said to be the metric induced by the norm and with respect to this metric B is complete, so it is a banach space and in fact it is an algebra so it is banach algebra okay. Now what you do is all this is correct if you are taking x to be just a set okay, now you put the extra condition that X is a topological space okay if you put the condition that X is a topological space then instead of simply looking at the functions X to R I can look at continuous functions from X to R because now the source X is a topological space it has a topology okay.

So I look at functions which are continuous also and when I do that then I get this familiar object that you should have seen in course in analysis namely the space of all continuous bounded real valued functions on the topological space, so this is if X is a topological space okay and the fact is that this is a closed subspace, so mind you B is already a topological space and this is the closed subspace you can show that this is the closed subspace namely you can show that if you have, if you have set I mean if you have if you take a point which is a limit point of continuous bounded real valued functions than the limit function is also continuous bounded real value okay.

Therefore this becomes…so in particular is space CXR and also becomes a commutative banach R algebra okay and the beautiful thing is the following, the beautiful thing is that or function for a sequence of…so you know now in CXR mind you CXR is also a metric space okay because that is a metric induced by the norm, so I and make sense of the sequence of functions converging to a function, what is that convergence? If you check it out that convergence is nothing but uniform convergence okay therefore the moral of the story is that uniform convergence of real valued functions sequence of real valued functions on a topological space is the same as the convergence in the norm so with respect to the sup norm on the banach algebra of a bounded real valued functions.

So this is to give you the motivation that whenever you trying to get hold of a topology on a space of functions okay the convergence should have to do with point wise convergence, in fact it has to do with normal convergence okay or at least to begin with it has to go to do with uniform convergence but normally what happens is you cannot get uniform convergence on a whole space in complex analysis for example in the case of holomorphic functions you can get only uniform convergence with respect to compact subsets, so the topology of a space of Meromorphic functions or holomorphic functions needs to be studied with respect to normal convergence okay that is the motivation okay. So we will continue in the next lecture, so let me write this down this is also commutative banach algebra with convergence same as uniform convergence on X okay.