Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

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Lecture No 15

The Ubiquity of Meromorphic Functions_ The Nerves of the Geometric Network Bridging Algebra, Analysis and Topology

Alright so we are discussing Meromorphic functions okay and we were looking at Meromorphic functions on the extended plane in the last class in the last lecture and we prove that a function which is Meromorphic on the extended plane is none other than quotient of polynomials okay namely a rational function okay, so what I need to what I want to tell now is about the collection of Meromorphic functions on a domain okay, you take a domain in the extended plane and look at the set of all Meromorphic functions defined on the domain okay then that it has a has a nice structure in fact algebraic structure it is a field okay and in fact it is an algebra over the complex numbers okay and so it is a field extension of the complex numbers and the properties of this field extension, algebraic properties of this field extension they have captured a lot of topological and geometric properties of the domain okay.

So this is how there is a link from the complex analysis side to the algebra side okay, so you know geometry involves an interplay of various ways of mathematics, so studying something Geometrically will in all studying it from the analysis viewpoint okay and studying it from the topological viewpoint also studying it from the algebraic viewpoint but when you looking at a particular nice object okay when yesterday the analytically it will have some properties okay, some special properties and then when you study it algebraically it will have some properties, special properties. When you study topologically it will have some special properties and the fact is that these properties are interrelated.

There is some beautiful relationship, hidden relationship between the analytic, the algebraic at the topological properties of a nice object and that relationship is what you may call as geometry okay. So if you want to really understand geometry of an object you have to analyse it using all the 3 viewpoints algebraic, analytic, topological okay, so in that sense know how do I do geometry on a domain in the complex plane or in the extended complex plane okay. What I can do is of course the analysis is there, the analysis will worry about what kind of functions you can define on the domain, what are the holomorphic functions or analytic functions on the domain? What are the Meromorphic functions on the domain? and so on that will be the viewpoint from analysis but then how do you go to algebra.

The point is that you take the set of Meromorphic functions that forms a field okay and that is the field extension of the complex number and you study the properties of this field extension okay, so in field theory you have lot of you would have come across in a course in the algebra in field theory that field extensions are of so many types okay there are algebraic extensions, there are transcendental extensions and then they were normal extension, there are splitting fields, there are (())(5:17) extension okay there are of course separable and non-separable extension and we study all these things and of course the most important thing here in general is the study of the nature of (())(5:29) extensions because that connects up group theory which is it connects up with the so called (())(5:37) groups. So you see the moment you look at the field of Meromorphic functions you get an extension of the complex numbers and then you can do algebra okay and somehow these things are all connected and I will try to give couple of examples. So 1^{st} of all let me begin by 1^{st} saying that if I take a domain in the extended plane then the set of all Meromorphic functions in the domain is actually a field okay.

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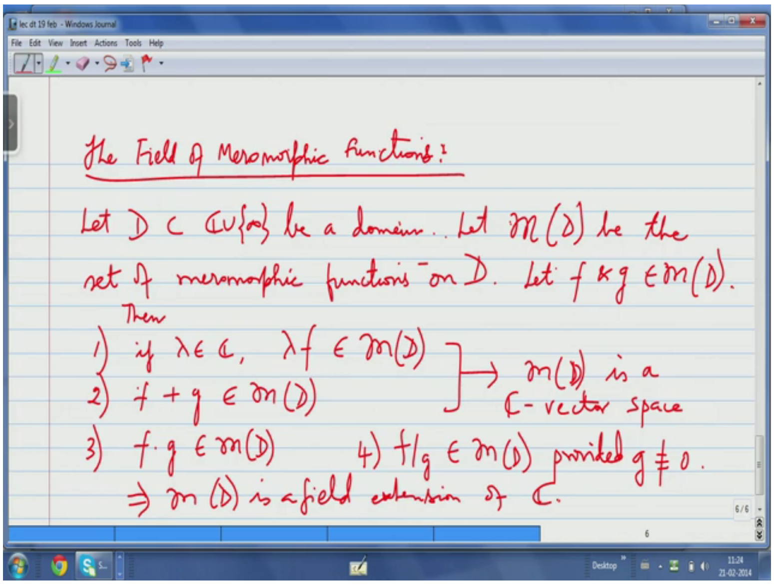

So let me write that down the field of Meromorphic functions, so let D inside C union infinity be a domain, so you are taking a domain in the extended complex plane C union infinity, so in particular mind you it is a nonempty open connected set okay and the advantage of taking a domain in the extended plane is that you can also look at a neighbourhood of infinity okay that is the advantage, so you are also including the point at infinity right, so let so here is the notation I will put script m of D be the set of Meromorphic functions on D okay.

So what is this script m of D? Script m of D is the collection of all Meromorphic function on (())(7:27) and you know what a Meromorphic function is? You have defined the Meromorphic function we are function which is analytic at all points but except or points on an isolated sets which have to be always singularities okay, so it is analytic except for poles and the moment you say analytic except for poles it means the singularities can be only poles and that in particular means that the singularities can only be isolated because poles are isolated singularities by definition okay.

So you take all the Meromorphic functions on the domain okay now the fact is that this is a field okay, so let us see that let f and g be Meromorphic functions on D okay then you see then you can notice the following things number 1 is lambda is a complex number okay then lambda f is also a Meromorphic function okay multiplying a Meromorphic function by a constant is going to keep it Meromorphic okay of course if the constant is 0 you will get 0 and this 0 is a constant function. Of course when you say Meromorphic, analytic is also included okay, so the definition for Meromorphic is that it is analytic except for poles that does not mean it has to have a pole it can be it can have no poles and it can be analytic everywhere, so holomorphic functions also included in the set of Meromorphic functions okay.

So this statement is obvious if I take a Meromorphic function multiply by constant, if the constant is 0 of course I am going to get the 0 function which is holomorphic which is analytic because it is a constant function okay but if lambda is not 0, lambda times f will also be Meromorphic and it will have the same poles, okay by multiplying by lambda you not going to change the poles and you are not going to change the order of the poles. Essentially you just multiplying by a constant okay, so this is one obvious thing then the 2^{nd} thing is that if you take the sum of these 2 Meromorphic functions, this will also be a Meromorphic functions okay. The sum of f and g will also be Meromorphic why because you see the fact is that f is Meromorphic so it has some it has a collection of poles okay and an isolated set of points.

Then g is also Meromorphic so it has also poles in another isolated set of points and then you take the union of these 2 isolated set that is again an isolated set okay and these are the only points where f plus g will have problems okay so at a point where f does not have a problem and g does not have a problem, f plus g will not have a problem that is at a point where f is analytic and g is analytic, f plus g of course will be analytic okay, so the only problems for the function analyticity of the function f plus g will be at the points where f and g have problems okay and it is possible that some of…there could be some cancellations okay.

So for example f may be 1 by Z minus Z naught g may be minus 1 by Z minus Z naught, so if I take f plus g I will get 0 which does not have a pole at Z naught okay, so some poles can cancel out also and sometimes the order of a pole can come down okay when you add course when I say add it also includes a subtraction because subtraction is just adding with minus 1 multiplied by the 2nd function okay, so the moral of the story is that sum of 2 Meromorphic functions is again a Meromorphic function. It could very well be analytic okay some poles might cancel out all the poles may cancel out for example if you take the function and you take its negative and add it you will get 0 and that is clearly holomorphic is a constant function, so some is Meromorphic so you know the moment you look at the first 2 things is will tell you that you know m of D is a vector space or complex numbers see because it is you see so there is a scalar multiplication.

If you think of complex number a scalars then there is a scalar multiplication and there is addition, so this becomes a…so m of D is C vector space so you get that immediately okay. Now let us look at f times g look at f times G, see f multiplied by g will also be Meromorphic on D this is also very clear because just from the fact that you know what are the problem points on Mark the problem points are the points where f has problems and g has problems okay, so if you take out those problem points then f times g will be analytic, so at a point where f is analytic and g is analytic f times g will be analytic and the only place where f times g will fail to be analytic it is probably on the (())(12:51) of the set of poles of f and poles of g okay and so you see and of course if you want you can write out always the principal parts and see that you know if you functions have poles at the same point they have common pole then if you multiply the product function will have a pole with higher order in fact it will have order equal to sum of the orders that is obvious if you write out the principal parts okay, so in the Laurent expansion alright.

So I mean the point is that you know all these algebraic operations of adding, subtracting multiplying by a constant and just multiplying and of course we are going to see division all these things they do not change the Meromorphic nature okay, so by adding or subtracting or dividing or multiplying or multiplying by a constant you cannot change a Meromorphic function into a non-Meromorphic function. If you are only working with Meromorphic functions you will get back again Meromorphic function okay, so fine so you have f and g are the product f time g is also Meromorphic of course by product 1 means point wise product okay, so f g is a function which at each point Z is defined by f of Z times g of Z alright then of course I can say the same of f by g this is also a Meromorphic function why did g is not identically 0 okay of course I should not divided by 0, so the fact is that see when I take f by g okay what other problems points?

The problem points will be poles of F, poles of g and now you will have extra problem points at 0 so g because they are zeros of the denominator they become the zeros of g will become poles of f by G, they are likely to become poles of f by g and of course you know it might happen some zeros of g may cancel out with some zeros of f because the zeros of f on the numerator the zeros of g are on the denominator some zeros might cancel but a set of problem points are just the poles of F, the poles of g and the zeros of g and you know zeros of an analytic function also isolated, you know that, there is a theorem okay in fact that is another version of identity theorem if you have seen it in the 1st course in complex analysis, so therefore the set of points where f by g will have problems is still an isolated set of points okay and at each of those points you can only get poles you cannot get anything worse, so therefore f by g is also Meromorphic in particular I could have taken 1 by g I can put f equal to 1 I will get 1 by g is also Meromorphic and that means so what will I get I will get 1 by g is Meromorphic if g is not identically 0.

So that means every nonzero Meromorphic function, every Meromorphic function which is not identically 0 as an inverse okay and that is what you require for a field okay. A field should be basically a group under multiplication if you throughout the 0 element commonly set okay, so well if you if you look at all these things, these things will tell you that m of D is a field and you know you put it together with this fact that we saw earlier we have seen just above that m of the D is also a complex factor space of field extension with the field which is also a vector space is an algebra okay, so basically you can very will see that m of D contains complex numbers because the complex numbers sit as constant channels okay you take any complex number lambda you think of it as a constant function lambda. Constant function lambda is analytic is defined everywhere.

So it is analytic on every domain and it is Meromorphic because mind you when I say Meromorphic I am allowing also analytic or holomorphic. Meromorphic means that it can either be analytic and if it is not analytic that is if it has singularities, the singularities must be only poles that is what it says. So Meromorphic does not say that it should not be analytic, so in particular m of D contains the complex numbers as a subfield you know the complex numbers of those form a field and therefore m of D is a field extension of the field of complex number, so m of D is a…

So let me write that m of D is a field extension of the field C of complex and the beautiful thing is that the geometry on the domain D is done by a lot of topology of the domain D is connected to… and a lot of analysis on the domain D namely the behaviour of the existence and behaviour of Meromorphic functions on D is connected with the algebraic properties of this field extension okay that is the geometric content okay. So if this goes back to the work of classical giants like Riemann and Clifford and Weierstrass and Able and Yakobi you know all these people who developed theory of Riemann surfaces okay of course principally from Riemann, so let me write that down.

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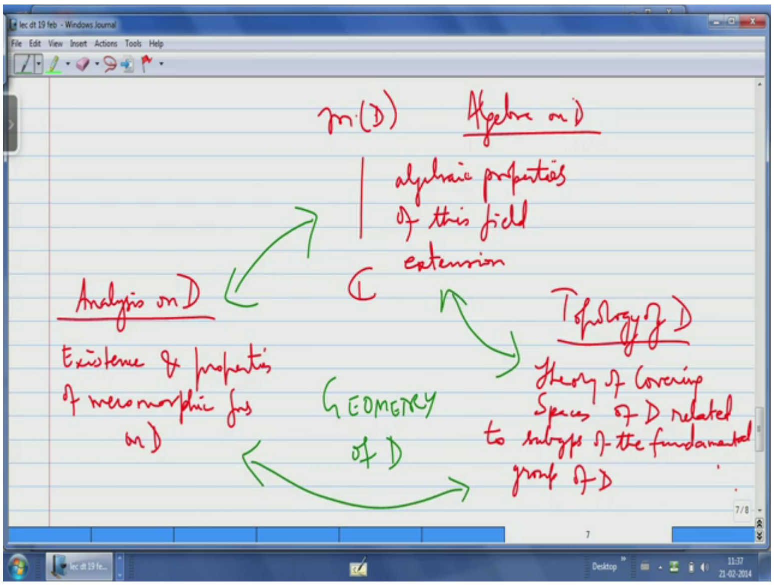

So let me write it out as a diagram so you have m of D and this is over C, so I am using this field theory notation you write a field, a bigger field on top and you put a smaller field below and then you put a vertical line thing that the thing that comes above is a field extension and things that come below are some fields okay, so the algebraic properties of this field extension they are connected so there is a, so the analysis and D which is existence of a special existence and properties of Meromorphic functions on D that is analysis on D okay and this part is algebra only, the algebra on the domain is actually studying you might think of it as studying this field extension and then there is the topology of D, topology of the domain D.

So I am not trying to be very particular or trying to go in detail but topology and the minimum for example D is of course connected but one of the simplest thing that you can look at is whether D is simply connected or not okay and then or if it is not simply connected you can see if it is multiply connected and how many holes it has and so on and so forth okay after all D could be something like an amoeba with some holes okay after all an open set can look like that and then the topology worries about whether if it is simply connected, if it is not simply connected how many holes are there and so on and so forth okay, so all this topological information for example is encoded in the fundamental group of D and so on and so forth and in fact more precisely I should say that you have to study the theory of covering spaces of the domain D that is what is the topology of the means.

So let me write this here the theory of covering spaces of D related which is actually related over the fundamental group, in fact subgroups of the fundamental group of D, of course you know the so you know at this point let me tell you that if you have done decent (())(22:07) in topology see there is something called covering space theory which takes the topological space with decent properties or example something that is house of locally house of connected locally path connected, locally simply connected and then you study what are called as coverings of topological covering and (())(22:25) there is a Galva theorem of covering which says that you know there is a Galva correspondence between the coverings and subgroups of the fundamental group of your topological space and in fact under this Galva correspondence so this Galva correspondence is an analog of the Galva correspondence that you have in field theory.

See the Galva correspondence in field theory is you correspondence between field extensions of a given field and subgroup of the Galva group okay and there is an analog, so the Galva correspondence in field a really is a correspondence on the one side between field extensions and on the other side between subgroups of a group and in this case is the Galva group okay, so it is a connection between field theory and group theory okay and it is very useful because a lot of field theory problems can be translated to group theory problems and lot of group theory problems can be translated to field theory problems. In the same way (())(23:32) space theory is very similar, what it does is it translates topological coverings that is topological data into subgroups of fundamental group, so it also connects to topological side, the topological side to group theory side okay so that you can use some algebra in your topology okay.

So that is where usually this is a part of usually 1st course in algebraic topology okay, so of course all this is very uninteresting if D is simply connected because if it D is simply connected then the fundamental group is (())(24:04) okay but then it is still not so easy in fact there is I will explain why. Whatever I have written here, the algebra, the analysis and the topology of D I have written it for a domain D in the extended plane okay but what the philosophy is that this holds for any Riemann surface okay. Now there is something called Riemann surface, the Riemann surfaces something at locally looks like a plane okay but globally it may be different surface, so for example it may be a cylinder in 3 space okay it may look like a torus alright or it may look like n torus, so it might look like several tori which are stuck together by removing disk and open disk and pasting the boundaries of the open disk okay.

So these are called Riemann surfaces and these were studied by Riemann and Riemann was fascinated to know that on these Riemann surfaces you can put many complex structures there you can put non-isomorphic complex structures and you must think of a complex structure as a structure which allows you to decide whether a function on that surfaces is holomorphic on not okay, so Riemann found that you know, see Riemann try to do what we do in complex analysis on the plane. On the plane what we do, we take a domain and ask when a function is analytic at the point okay and if it is analytic then of course you know if it is not analytic then you see whether that point is an isolated singularity and so on that is how you do the analysis.

Now what Riemann wanted to do was he wanted to do it on the surface, so he wanted to say that suppose I have now a function on a torus okay or say even an open subset of the torus alright, when I say open subset you take the induced topology from R3 in which the torus sits okay and then suppose I have function which is complex valued defined on an open subset of the torus, when can I say it is holomorphic, when can I say it is analytic? So you are trying to study when a function defined on an open subset of a surface is analytic, the answer to this is that we should define what is called Riemann surface okay and there are different Riemann surfaces structures you can put and Riemann found that…he was fascinated by these different Riemann surfaces structures and the most beautiful theorem in Modley theory is that you actually take the set of all these Riemann surface structures that itself become a nice object.

It becomes an analog at least on an open set, it becomes an analog higher dimensional analog of Riemann surface which is called a complex manifold and of course it could have boundary which could have some singular points but it is a very beautiful object okay, so the moral story is that I am trying to say that whatever I am writing here for D, D at domain in the extended plane it also works for a Riemann surface okay and so for example in that context it is really amazing that you get a lot of… so you know let me ask you a fundamental question, the fundamental question suppose if you have a simply connected Riemann surface okay, so the moment I say simply connected the topology seems to be very trivial because in the sense that the fundamental group is trivial so you do not expect anything special but then you can ask how many simply connected Riemann surfaces are there which are not isomorphic to each other okay.

Now you more or less know the answer partially because the Riemann mapping theorem tells you if we have seen it in the 1st course in complex analysis which you should have done at you know any simply connected open subset of the complex plane which is not the whole plane has to be holomorphically isomorphic that is by holomorphic to the unit test okay, so if you take domains in the complex plane okay, simply connected domain is in the complex plane there are only 2 types up to holomorphic isomorphism, one is the whole plane the other one is the unit disk okay, so now it is an amazing fact that even before that let me say look at the Riemann sphere okay which is you know we use that study the point at infinity as the stereographic projection, the Riemann sphere is also a nice surface of course and is compact okay and you can actually make it into a compact Riemann surface okay. Now the fact is that there is also simply connected, Spear is simply connected so that is also another simply connected Riemann surface.

Now it is a very deep theorem that you take any simply connected Riemann surface it has to be isomorphic, holomorphically isomorphic to one of these 3. Any simply connected Riemann surface has to be either it should either look like the whole plane okay or it should look like the unit disk or it should look like a Riemann sphere there are no other possibilities. It is a very deep theorem there is called uniformization, so even the simply connected case you get a very deep theorem and the theorem is very hard to prove because you have to use a lot of techniques from analysis to prove it okay, so it involve a lot of analysis, it involves study of harmonic functions, Meromorphic functions, et cetera and it involves a reasonable amount of function analysis and measured theory you have to do all this to get that theorem okay. So anyway so the fact I want to say was at now given all these 3 aspects of (())(29:41) putting them together is what geometry is all about, okay. So let me wright here so geometry of D is the interplay between these 3.

The geometry of a domain is actually the interplay between the analysis on the domain, algebra on the domain and the topology on the domain and I have given you a rough idea, the analysis on the domain is the complex analysis part okay. The algebra on the domain is to really study the field of Meromorphic functions, the algebraic properties of the field extension given by the field of Meromorphic function the domain and the topological part is to study the curving space theory of the domain okay and it so happens I mean as the great classical giants like Riemann and Clifford and Able and Yakobi and Weierstrass have found and Clifford for example at you know all these properties, all these various points of view, they are all interrelated okay so it is an amazing fact and discovering that is what doing geometry is all about okay.

So you should not think that high school level the geometry is just about drawing triangle encircles and you know measuring of angles and arcs and things like that but it is really higher geometry in the higher sense is actually looking at the interplay of all these things okay. So I well now you see I want to give you a couple of examples, so here is the 1st example so here is an example.

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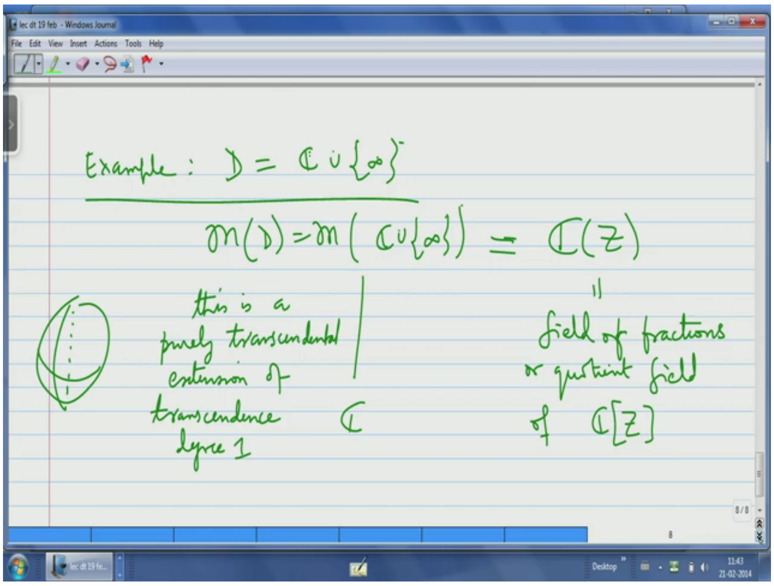

You take the domain to be the extended plane itself okay see after all we are studying domains in the extended plane so take the whole extended plane that is also domain. In fact it is simply connected because you know there is homeomorphic the Riemann sphere and the

Riemann sphere is simply connected, so this simply connected, it is compact okay is a very nice thing. Now what are the what are the field of Meromorphic functions on D okay, what are the fields of Meromorphic functions on the extended plane, so you know this is an extension of the complex numbers as we have seen this is an extended (())(32:05) of complex numbers but you know what is it that we proved last time.

A function which is Meromorphic on the extended plane is none other than a quotient of polynomials, it is a rational function okay and therefore this is exactly equal to the, this is exactly equal to C round bracket Z this is the algebraic notation C round bracket Z and the C round bracket Z is actually the field of fractions or quotient field of C of square brackets and C of square bracket Z is standard notation is the ring of polynomials in the variable Z with complex coefficients okay and C of Z is the field of fractions which is quotient of such polynomials, so you take quotient of polynomials but of course you do not put in the denominator 0 anything other than 0 you put okay, so the moral of the story is that you have very nice description of this field extension in the case of the C union infinity which is the extended plane and usually you know extended plane is thought of as Riemann sphere you know they are isomorphic but you can make them also isomorphic in a holomorphic sense by giving the Riemann sphere a Riemann surface structure okay.

So often people do not use if you see the literature you will see that people often use C union infinity instead of C union infinity they keep saying Riemann sphere all the time. So now you can see that what are the properties of this field extensions you see this field extension is actually is purely transcendental and has transcendence degree 1 okay. It is purely transcendental and has transcendence degree 1. Well the transcendence degree is actually the number of algebraically independent variable is that it generate the bigger extension okay, so the bigger extension C of Z is generated by a single variable Z and that is the only algebraically independent variable okay that one variable is enough, so the transcendence degree is actually one okay and it is purely transcendental because there is no element in C Z which is algebraic there is no elements in C Z which is not in C and which is algebraic (()) (34:33) and that is you know why that is because Comcast numbers are algebraically closed they are all algebraically closed.

So field theoretically so this is what is called as a function this is the simplest example of what is called a function field in one variable okay and the beautiful thing is that now if you take any compact Riemann surface okay then if you take the field of Meromorphic functions on that compact Riemann surface what you will get is a function field in one variable but the only thing is that it may not be purely transcendental about the there may be an algebraic part okay so it will be 1st of transcendental extension, purely transcendental extension of degree 1 just like this and then about that you will have an algebraic extension and which will be a finite extension okay. So that is how it looks in general okay and well I will give you another example for that, so let me write this here this is a purely transcendental extension of transcendence degree 1 okay. So the picture that is associated with this is the Riemann sphere okay, so this is the picture that is associated with this and for all practical purposes you think of the extended plane as the Riemann sphere okay.

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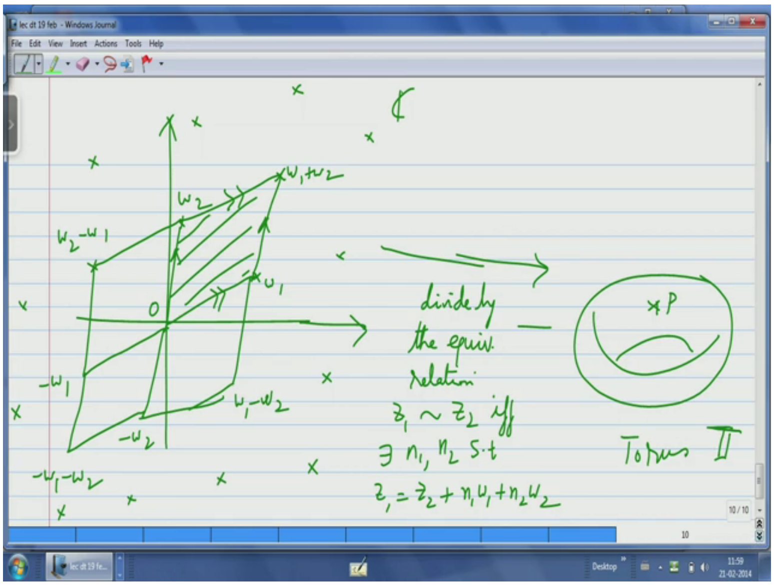

Now let me tell you more generally what is it that happens with something… so let me give you an example of a more complicated case is the case of so-called elliptic functions or doubly periodic function okay, so here is what I am going to do? What am going to do is I am going to take, so I am going to define what doubly periodic function is? A doubly periodic function so this is the topic of what are known as elliptic functions this is also an example okay w periodic function is a function f of Z with f of Z plus Omega 1 is equal to f of Z and f of Z plus Omega 2 is equal to f of Z where Omega 1 by Omega 2 is not real and of course omega 1 omega 2 are nonzero okay so I am just defining what a doubly periodic function is?

So see the definition is very simple for example sign beta will you know is periodic with 2 pi because sin of theta plus 2 pi is the same as sin theta, so the idea is that to the variable of the function or the argument of the function you add the period the function value should not change okay, so what the 1st equation says is that f of Z plus w 1 is equal to f of Z actually tells you that w 1 is a period and the 2nd equation f of... The 2nd requirement f of Z plus w 2 equal to f of Z tells you that w 2 is also a period, so w 1 and w 2 are periods and the fact is that we want these periods to be linearly independent over R in other words what you want is that if you take these 2 complex numbers w 1 and w 2 then you join the origin to them okay namely you take the vectors that they represent in the plane then these should be different the vector should be linearly independent they should be in different directions you know they will be linearly dependent if and only if the quotient w 2 by w 1 or w 1 by w 2 is a real number okay.

So this condition that w 1 by w 2 is not real it is just to tell you that these 2 vectors are 2 different vectors. They will form therefore a basis of C over R, the complex numbers over R is a 2 dimensional vector space and (())(39:23) will from a basis so it is equivalent to saying that w 1 w 2 form the basis for (())(39:27) okay and you are putting this condition in order to make sure that essentially have 2 distinct periods which are in 2 so it is periodic into directions okay. The fact that f of Z plus w 1 is equal to f of Z tells you that you know if you translate along the direction of w 1 by integer multiples of w 1 the function value does not change okay, so you must remember that when I say f of Z plus w 1 is f of Z it follows that f of Z plus n 1 w 1 is also f of Z for all integers n 1 okay because I can just use induction f of Z plus w 1 is equal to f of Z, so f of Z plus 2 w 1 is f of Z plus w 1 plus w 1 which is f of Z plus w 1 which is f of Z and so on and so forth okay.

So what it tells you is that the moment something is a period then all its integer multiples are also periods okay and similarly you also have for the other but what is adding W1, see addition of a complex number is just translation along the direction along the vector that is represented by a complex number okay, so you know basically if I have a point Z, what is Z plus W1? It is actually this vector, so this will be Z plus W1. I am just a translating Z by the vector w 1 okay and then similarly what is Z plus w 2 I am just translating Z by the vector w 2 alright and that is if I add w 1 but if I add minus w 1 you know I am translating in the other direction. If I add minus 2 w 1 I am translating in the direction opposite to w 1 2 times and so on and so forth okay.

So the moral of the story is that you know basically the function value do not change if you translate along 2 different directions okay that is why this called w periodic, it is periodic and the period there are 2 different periods okay and such functions are called actually now you have to put some more condition on these functions, the condition you put on these functions is that you know to make them very interesting these points w 1 these points which are given by integer multiples of w 1 added to integer multiples of w 2 okay that they will form a lattice, a grid in the plane okay and the function becomes very interesting if the function is Meromorphic exactly at those points okay and such functions are called elliptic functions and believe it or not they are exactly the functions which are the function is Meromorphic on a torus at a single point okay and this is the beginning of the so-called Weierstrass phi theory there is something called as Weierstrass phi function which is fundamental model of this kind of function and the beautiful thing is that every torus the complex structure on any torus can be controlled by prescribing such a function okay and so the Weierstrass phi functions completely give you.

So if you want to study the various complex structures you can put on a torus what you will have to do is you have to study various doubly periodic Meromorphic functions which are otherwise called elliptic functions, the reason they are called elliptic is because this is beautiful the moment you put a complex structure on the torus it becomes believe it or not it becomes a cubic curve, it becomes a cubic curve and therefore it becomes an algebraic geometric object okay, so algebraic geometry also comes in, geometry also comes in, algebra comes in in a beautiful way okay and this is also a part of a very deep theorem which says that you know you take any compact Riemann surface it is algebraic it is just given by a common 0 set of a bunch of polynomials okay and that is an amazing theorem okay.

So what I want to tell you is that I have given an NPTEL video course on Riemann surfaces and all these things are explained in detail throughout the course when you find time you can have a look at that and the other thing that I want to tell you is that there is this book that I have written and it reads “An introduction to families deformation and moduli. This book is basically available as a freely downloadable copy in the form of a navigable PDF file and it contains a lot about the geometry of Riemann surfaces, so at least the 1st chapter so that is also something that can be advanced reading material for people who are interested in pursuing this. So let me continue so I have also f of Z plus n 2 w 2 is equal to f of Z or all n 2 in Z, so in totality what I will get is I will get f of Z plus n 1 w 1 plus n 2 w 2 is equal to f of Z for all n 1, n 2 in Z if I put both these together and what are these points n 1 w 1 plus n 2 w 2? They are the vertices of a grid of parallelogram okay in fact if you draw this if I draw a diagram it is going to look like this.

So I have this so this is my complex plane and you see I have w 1 here I have w 2 here okay and you know then if I draw this parallelogram then you know pretty well that this is w 1 is w 2 okay by the parallelogram law of additional vectors if you want and then you know if I extend this parallelogram below then I am going to get this point is going to be you know it is going to be w 1 minus w 2 okay and this point is going to be minus w 2 and if I extend it like this, this point is going to be w 2 minus w 1 and this is going to be minus w 1 minus w 2 and this is going to be minus w 1 okay and more generally if I draw…

If I look at all these points that go on that I get as the vertices of the parallelograms that I get by simply starting with this parallelogram and simply displacing it by either plus or minus w 1 or plus or minus w 2 okay that is by translating it with plus minus w 1 of plus or minus w 2 I will get so many…the whole plane is covered by this parallelograms okay and the vertices of the parallelogram are precisely the points which are of the form n 1 w 1 plus n 2 w 2 okay and that is called the lattice okay and the fact is that you see…just to give you an idea of what is going on where is the topology coming in India, so the fact is that what you do is you divide by the equivalence relation Z 1 is equal in to Z 2 if and only if there exist n 1 n 2 such that Z 1 is equal to Z 2 plus n 1 w 1 plus n 2 w 2 okay.

So see this is the plane this is the complex plane and I am defining an equivalence relation on the plane, the equivalence relation is 2 points or equivalent one of them is a translate of the other by one of these grid points okay and what is the advantage of this? The advantage of this is that if 2 points were related like that then the w periodic function will have the same value at both points because f of Z 1 will be equal to f of Z 2 plus n 1 w 1 plus n 2 w 2 but that is also equal to f of Z 2, so f of Z 1 will be equal to f of Z 2 okay because when I apply f to this equation okay on the right side I will get f of Z 2 because of periodic city of f so I will get f of Z 1 equal to f of Z okay. What it means is that the value of the function is not change if you change the point by a translate by a vector which belongs to one of the grid points okay.

So if you divide by this equivalence relation what you will get is you will get the torus, you will get a beautiful torus and you can see at very easily you just take this fundamental parallelogram okay, this fundamental parallelogram if you take the interior every point in the interior will be a unique representative in its equivalence class but for points but then you will have to only identify the boundaries, see if you identify the top boundary with the bottom boundary you will get a cylinder okay and then you will have to identify these 2 which will in the cylinder look like circles, so if you identify them you will get a torus okay.

So the moral of the story is you take the plane divided by a lattice like this you will get the torus okay and all the points in the grade including the origin, the origin is here they all will go to a particular special point on the torus okay and the beautiful thing is that the function at you defined on the complex plane will go down to a function on the torus okay and if the function is Meromorphic exactly at all these grid points it will go down to a Meromorphic function on the torus and you know you may wonder why should I worry about why should I not consider mophet functions in the torus and you very well know the answer will not be any non-constant holomorphic function on the torus because torus is compact. Since the torus is compact if you define a holomorphic function on the torus okay you are going to get something that is holomorphic you can use Liouville's theorem if you have a holomorphic function on the torus if you composite with this map that is that goes from the complex plane to the torus you will get a holomorphic function on the plane but since it is defined on the torus which is compact its images compact therefore the image is bounded.

So I get an entire function which is bounded and that is going to be constant by Liouville's theorem okay and this picture also explains why the only functions on the torus are exactly the functions on the plane which are doubly periodic with respect to the periods w 1 and w 2 and if you take this beginning point p which is the image of the grid the Meromorphic… since you know holomorphic function is not available they have constants, the only things that are available are the Meromorphic functions and then if you look at Meromorphic functions on the torus at the point p they will be the same as doubly periodic functions okay and therefore the moral of the story is that you know if you look at the field of Meromorphic functions on the torus okay we are the same as the collection of Meromorphic functions on the collection of doubly purity functions with these 2 periods okay and that is the field of course. Mind you that what is the domain now?

The domain is the whole plane okay and I am looking at functions which are Meromorphic with poles at points of the grid okay possibly at points of the grid alright and then what I get is I get field and what is that field? That will is nothing but the field of Meromorphic functions on the torus which are Meromorphic at a given point okay and so let me call this torus as T okay mind you this torus depends on the choice of w 1 and w 2 okay and it is a different story that there is a lot of geometry there but what I want to tell you is that well I want to tell you the following things.

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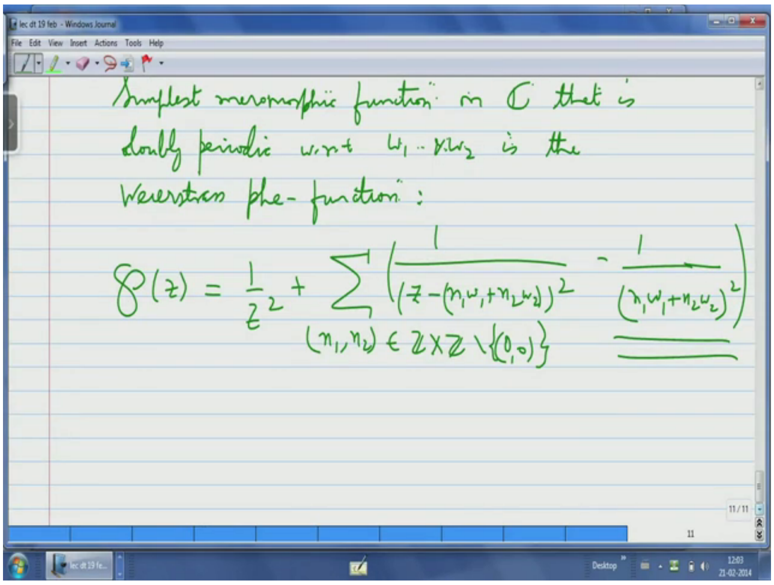

Simplest Meromorphic function on C that is doubly periodic with respect to w 1 and w 2 is the Weierstrass phi function and this is the phi function so there is pretty symbol for that very special symbol so phi of Z is so there is a formula for that basically it is a formula that will tell you that it is a Meromorphic function which has a double pole with the residue 0 at each of those points of the grid okay, so you know what you are going to get let me write that down here you can find this in any standard book example (())(53:20) book on complex analysis which is a classic so it is 1 by Z Square summation over n 1, n 2 belonging to Z minus 0, 0 (())(53:40) in fact, so I should write it carefully n 1, n 2 ordered pair belonging to Z across Z minus 0, 0 it is 1 by Z minus n 1 w 1 minus n 2 w 2 the whole square minus 1 by n 1 w 1 plus n 2 w 2 the whole square.

So this is the expression for the Weierstrass phi function which was discovered by Weierstrass and of course if you go through in detail the lectures of my video course you will see that how this comes about but you can see something immediately you see that this 1 by Z square is the principal part at the origin and that will tell you that you know origin is a double pole and residue is 0 because there is no 1 by Z term and then you look at each of these other terms 1 by Z minus n 1 w 1 plus n 2 w the whole square tells you that n 1 w 1 plus n 2 w 2 is a point of the grid is a general point of the grid okay and if you when I write 1 by Z minus that point the whole square actually I am looking at a pole of order 2 at that point and again the residue there is 0 alright.

So as a result this already gives you were you know it gives you a Meromorphic channel which is having a double pole at each of these grid points okay and this extra term that is added here is for convergence because you know I have added infinitely many poles okay I have added poles at every point of the grid I have made every point of the grid into a double pole and I am getting a huge series I want it to converge and it is only for this convergence that this extra constant term is being added okay and therefore I get this phi function and here comes the amazing theorem.

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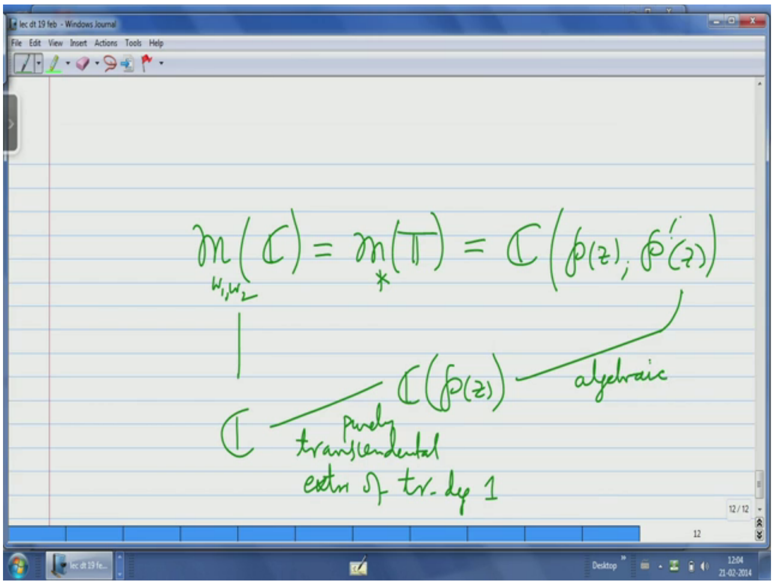

The amazing theorem is the following that if you take the complex numbers and you take the Meromorphic functions on D on the complex plane respect to w 1 and w 2 okay you look at the Meromorphic functions which are doubly periodic periods at w 1 and w 2 okay and mind you this is the same as the Meromorphic functions on the torus okay which are Meromorphic at that unique point which I will call at star which is the image of the grid the whole grid, the whole grid goes to a single point on the torus because all the points on the grid are equivalent to each other okay and they all define a single equivalence class so it is a single point on the torus, so mind you the torus is set of equivalence classes okay topologically and you give it the quotient topology alright.

Now on the beautiful thing is that what is this set of Meromorphic functions? You know it is a field, what is that field? You know what that field is, that field is just the field of fractions of phi of Z and its derivatives it is beautiful and how does this extension breakup, it breaks up as the 1^{st} phi of Z this is again a transcendental extension, it is purely transcendental extension of transcendence degree 1 and then from here to here this is an algebraic extension. This is an algebraic extension because then the derivatives of the phi functions free prime satisfies the polynomial relation with respect to phi and that is expresses a differential equation.

It is a very famous differential equation and that differential equation interestingly it comes from analysis what it tells you that the torus is algebraic, it tells you that the torus is nothing but a cubic curve okay which is an amazing illustration or the fact that in general a compact Riemann's surface is given is algebraic it is given by algebraic equations okay. So all these details you can have a look at it in more detail in my video lecture course but this is to tell you that a lot of geometry is involved by looking at the field extension given by the field of Meromorphic functions, okay. So I will stop here.