Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

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Lecture No 14

Meromorphic Functions on the Extended Complex Plane are Precisely Quotients of Polynomials

Okay so let me have your attention you see we are now more or less you know comfortable with thinking of the point at infinity of functions behaving at infinity, and analyticity at infinity, singularities at infinity, so you know we wanted to do that because we needed to look at functions on the extended plane okay and this viewpoint is very important because it is not…normally when you do complex analysis say for example in a 1st course you are worried about only questions on the plane okay on the complex plane you are not worried about the point at infinity okay and but if you include the point at infinity you get more information that is the point, so I explain this last time but let me again repeated.

So I am saying or example look at let us take an entire function okay let us take an entire function take a non-constant entire function okay then if you look at the little Picard theorem what it will tell you is that the image of the whole plane is going to be either the whole plane or the pole plane minus 1 point which is one value which it may not take and that is the best possible I mean it cannot mix 2 values, what it means is that if an entire function Mrs more than one value it has to be constant okay.

So you see and of course you have but then you know if I ask you now take this entire function and take the exterior of a circle of sufficiently large radius okay and take its image, what will the image look like? Okay then you know you do not have an answer, you do not have an answer, so whereas for example if you that the if you can think of infinity as a singular point and in fact if infinity is an essential singular point okay then you can apply the great Picard theorem which will tell you that the image of every deleted neighbourhood of infinity will be either the full plane or the plane minus a point okay.

So you can use that which is a more powerful theorem to say that if you have an entire function and suppose it is not algebraic namely that infinity it is a transcendental entire function that infinity is you know essential singularity then if you take the exterior of a circle of sufficiently large radius for that matter if you take exterior of any circle okay because exterior of any circle is going to be a deleted neighbourhood of the point at infinity in the extended plane and if you apply the big Picard theorem you get the information that the image of the exterior of any circle is always going to be the whole plane or the whole plane minus a point okay and it will also give you the additional information that all the values it takes it takes infinitely many times okay so you see the advantage of thinking of the point at infinity okay you get more information.

So along these lines you know that for example you can characterise entire functions as transcendental or algebraic based on whether infinity is an essential singularity or not if infinity is not an essential singularity then it is a pole or a removable singularity infinity is a removable singularity then by Liouville's theorem the function has to be constant if infinity is a pole then the function has to be a polynomial and otherwise it is a transcendental entire function, infinity is an essential singularity, okay.

Now what we need to do is see we are the whole aim of this all this is you trying to prove the big Picard theorem and as a corollary deduce the little Picard theorem okay and you know theory record or that is to do analysis on compact spaces of Meromorphic functions okay, so you have to study the apology and you have to understand…do some analysis using compact families of Meromorphic functions, so we need to worry about Meromorphic functions. So let me start by telling you what a Meromorphic function is? I have told this before but this is let us go into that this a little bit more detail because that is something that gives you a link to algebra okay.

So the reason why a lot of complex geometry is connected to algebra is because of the fact that all the Meromorphic functions they form a field naturally and that field is an extension field of the complex numbers because the complex number is always can be thought of as one functions. Every complex number corresponds to the constant function given by that number, so you know and constant functions are of course Meromorphic okay they are analytic, so you see that the field of Meromorphic function as the field extension of the complex field is...the properties of this field extension in algebra for example in (())(6:55) and things like that, that determining a lot about the geometry of the domain on which you are studying the Meromorphic functions okay and this is extensively used for example in classifying Riemann surfaces and more generally if you want to classify manifolds and so on complex manifolds you can use this theory.

It gives you an interface between complex analysis and topology and algebraic geometry, complex geometry okay and algebra in that sense okay. So what is the Meromorphic function? The definition is pretty simple, it is a function which has which is analytic on a domain okay but the only singularity it has are poles, okay that is the definition of a Meromorphic function, okay.

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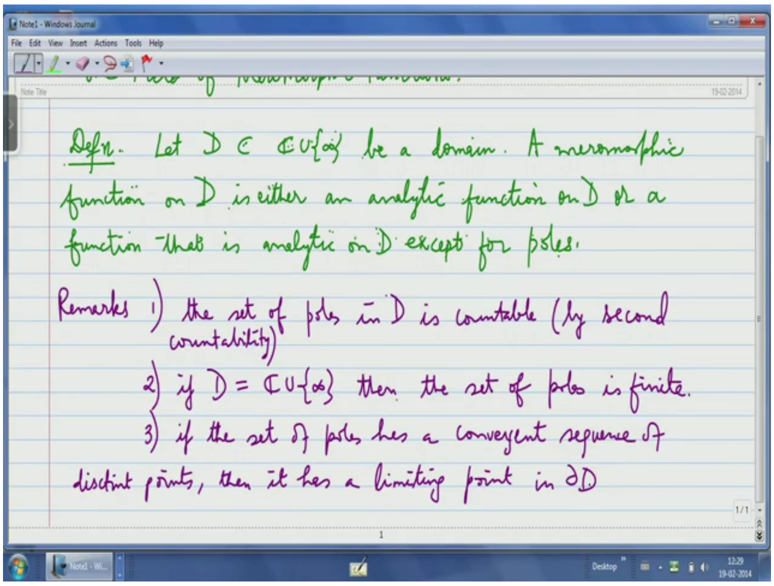

So let me write that down, so the field of Meromorphic functions so here is the definition. Let D in the extended complex plane be a domain okay, so I am looking at a domain in the extended complex plane by mind you it means that it is an open connected set and of course we always assume that it is nonempty okay. A Meromorphic function on D is either an analytic function on D or a function that is analytic on D except for poles, okay. So Meromorphic function by definition is a function that might that could be define on D or D minus points where those points will be of course singularities okay and the point is that they have to be isolated in fact they have to be poles, okay.

This is the this is the definition okay, so we say that we often use the abbreviation and I mean we use this short phrase analytic except for poles that is the that is what Meromorphic means okay and now we need to make a few observations, the 1st thing is that you see the number of poles okay the set of poles of the function it may be of course empty in which case the function is actually analytic or holomorphic but if it is not analytic then it has poles at least one pole and the fact is that the set of poles in D will be accountable set, so the 1st thing is that you cannot have an uncountable set of poles, so a Meromorphic function will have only accountably many poles and what is the reason for that, the reason for that is because of 2nd accountability okay.

The complex plane is just R2 and you know Euclidian spaces RN they are all 2^{nd} countable because they have accountable bases, so you can take all the open balls centered at points with rational coordinates and with radii given by rational numbers okay and use the fact that the rational number form a countable set okay and then you get this collection of open balls centred at points with rational coordinates and have a rational radii as accountable collection of open sets and they will form a basis or the topology okay in the sense that any open set is a union of such sets okay and the intersection of any 2 sets like that of this types again is union of sets of that form okay.

So it is a basis for topology so topology has countable bases this is 2^{nd} countability and now in particular For example the plane are 2 or the complex plane as topologically is 2^{nd} countable so you know what you can do is and of course any subspace of a 2^{nd} countable space is also 2^{nd} countable because all you have to do is that you have to take the countable base, countable base for the bigger space and intersect it with the subspace to get a countable base for the subspace okay.

So you take any domain in the complex plane or extended complex plane, it will be 2^{nd} countable okay and by the way if you are looking at the extended complex plane of course it is homeomorphic to the Riemann sphere and the Riemann's sphere is a subset of R3 real three-dimensional space and since real three-dimensional space is 2^{nd} countable the Riemann sphere being a subset is also 2^{nd} countable okay so the extended plane is also 2^{nd} countable okay after all you are just adding one more point at infinity and if you want you can take all the disk centred at infinity but of course you should not say finite radius okay unless you are looking at the images in the Riemann sphere with the point at infinity corresponding to the North pole okay but in any case if you are taking a domain in the extended plane is going to be 2^{nd} countable and now if you take a function which has poles in that domain then the set of poles poles by definition are isolated, poles are by definition isolated singularities.

So you have an isolated subset of a 2^{nd} countable set of a 2^{nd} countable space and isolated subset will always be countable okay. The reason is because since the points of the subsets are isolated you can cover each of those points by a member of the countable bases okay and you can choose different members from the countable basis because the points are separated from each other because they are isolated and in this way you get a mapping an injective mapping from the set of isolated points in this case these are the poles to the this countable base okay and the moment you get an injective map what it tells you is that whenever you get a injective map from a set to another set which is countable then your original set also countable because it is a subset of a countable set is countable okay.

Therefore you see the 1st observation is that a Meromorphic function will have only countably many poles and then if you are looking at the Meromorphic function on the whole Riemann sphere okay which means you are taking the domain in the extended complex plane often I keep saying Riemann sphere because I think of the extended complex plane as Riemann sphere they are one and the same, so your domain is the whole extended complex plane and you are looking at a Meromorphic function of the whole extended complex plane you get more, what you get is not only is a set of poles countable it is actually finite because an isolated set of points in a compact set 2nd countable set is going to be only finite okay because you know one property of compactness is that if you had in finite subset it should have an accumulation point okay therefore the moral of the story is that if you are looking at a Meromorphic function on the whole extended plane then it should have only finitely many poles, okay.

So let me so let me write that down so here are remarks, so remark number 1 the set of poles in D is countable this by 2nd countability okay, so I am just saying that am just using the fact that the poles are of course isolated and I am using 2nd accountability okay and the 2nd remark is that if D is the whole extended complex plane then the set of poles is finite and that is because of compactness of the extended plane okay and then there is yet another fact if you see the set of poles they form they are set of isolated points because poles are by definition isolated but on the whole some of them might converge. See they may be isolated points but they make it closer and closer and closer okay.

So like the sequence of real numbers 1 by n okay the sequence of real numbers1 by n consist of you know it is a set of isolated points but it converges, it converges to 0 okay so you could have set of poles okay you could have a convergent sequence of poles and then the fact is that this convergence sequence of poles as to converge only on the boundary it cannot converge in the interior stop the reason is that if it converge in the interior you are going to get a point which has to be in the domain of F.

So it has to be either a singularity or it has to be a point of non-singularity but then you know it cannot be kind of non-singularity it cannot be a point of analyticity because it is approached very as close as you want I poles and if a point is non-singular point there should be a disk around that point where there are no singularities that since this limit is being approached by poles this cannot be a non-singular point it cannot be a holomorphic point or analytic point it has to be therefore necessarily a singular point but then you assume that the singular points are all going to be poles and here you have gotten a non-isolated singular point that is a contradiction therefore if the set of poles of the Meromorphic functions has a convergent subset then it will converge only on the boundary okay and of course this can happen if in the case when the domain is the whole extended plane because when the domain is the whole extended plane okay the boundary is empty alright and there are only finitely many points okay.

So well so let me write this down if the set of poles has convergence sequence of distinct points then it has a limiting point in the boundary of D okay that point cannot it cannot converge inside D okay, so you know so the moral of the story is following, the moral of the story is when you are looking at Meromorphic functions of course if you are looking at Meromorphic functions on the whole extended plane then there is not anything we are going to see that these are exactly the rational functions okay.

We are going to prove that rational functions which are given by you know quotient of polynomials they are exactly the same as a Meromorphic functions on the extended plane is nothing more okay but if your domain is not the extended plane okay then the Meromorphic functions if you look at for example if you are looking at Meromorphic functions on the unit disk okay it is possible that you may have a sequence of poles is converging and it is converge to a point in the boundary of the unit disk okay and behaviour at that point of this of this Meromorphic functions or this family of Meromorphic functions is very important topologically okay, so this is very important that you need to study the boundary points also okay especially when you have sequence of poles tending to the boundary okay and of course what the remark says is that if there is a sequence of poles which is converging it has to go to only to the boundary it cannot come (())(21:11) interior point okay.

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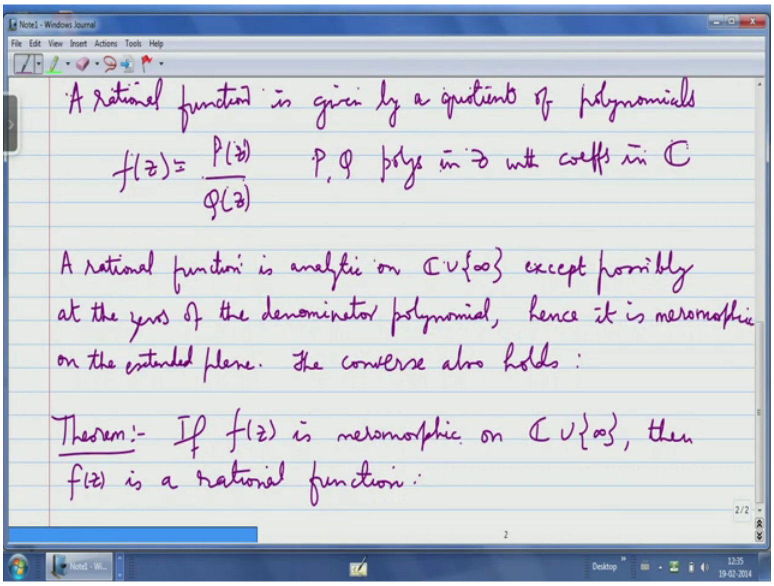

Well now so let me go ahead with this fact that I told you that you know a rational function of Meromorphic and converse holds a rational function is given by a quotient of polynomials f of Z is equal to P of Z by Q of Z okay, so you know now I want to tell you something I am now all this time you know I was using the variable w I was writing f of W. The reason why I was using the variable w is because I wanted a study the point at w equal to infinity and the way I would do that is by studying g of Z equal to f of 1 by Z which is f of w by making the transformation w equal to 1 by Z okay but now we have by now we understand how to deal with the point at infinity, so I am switching back to the variable set okay.

So Z will be our complex variable from now on, so rational function is just quotient of polynomials, so P Q polynomial in Z with coefficients in complex numbers of course you know coefficient of polynomial of course include even constants okay you can take Q to be 1 okay constants are also treated as polynomials of the be 0 okay. In fact in particular 0 is also considered as a polynomial, constant polynomial and therefore you must understand that constants are also Meromorphic functions okay and of course you know of force a function rational function like this is certainly a Meromorphic function on the extended plane that is very clear because if you take a function of this form if you take a function which is coefficient of polynomial is okay it is going to…where is it going to have problems with analyticity is going to have problem with analyticity where the denominator vanishes.

So you take those…and where the denominator vanishes is just the set of zeros over polynomial that is only finitely many points okay and maybe some of these may be the numerator polynomial so has zeros at some of these points, so some of the zeros they cancel out okay but then the fact is that this function cannot have poles at winds more than the set of zeros of the denominator polynomial okay and at all other points it is analytic, so it is an analytic function defined on the whole you know in the extended plane with having only you know finitely many points which are going to be poles.

So it satisfies the definition of Meromorphic functions, so you see there for a rational function is certainly a Meromorphic function okay and the fact the theorem is that the converse is also true okay, so that is what I am going to about. A rational function is analytic on C union infinity except possibly at the zeros of denominator polynomial, hence it is Meromorphic on the extended plane, so the theorem is that the converse holds, the converse holds so here is the so essentially here is the theorem. If f of Z is Meromorphic on the extended plane then f of Z is a rational function okay, so there is no difference between Meromorphic functions and quotient of polynomials okay there is absolutely no difference and you know now this should you should now think of it like this.

If you look at the polynomials in one variable say in the variable Z with the complex coefficients that gives you the polynomial ring in one variable over the complex numbers C square bracket Z okay and you know that is an integral domain if we have studied that in algebra and then this integral domain has what is known as a field of fractions, a quotient yield this is just like you get the rational number as a field of fractions when you look at the integral domain consisting of the integers, so if you look at the field of fractions of the polynomial ring in one variable over complex numbers you are going to get just quotient of polynomial is and these are precisely the Meromorphic functions on the extended plane. So the moral of the story is that if you look at the extended plane okay the set of Meromorphic functions automatically is a field it is none other than the field of fractions of the polynomial ring in one variable over C okay, so you can see that.

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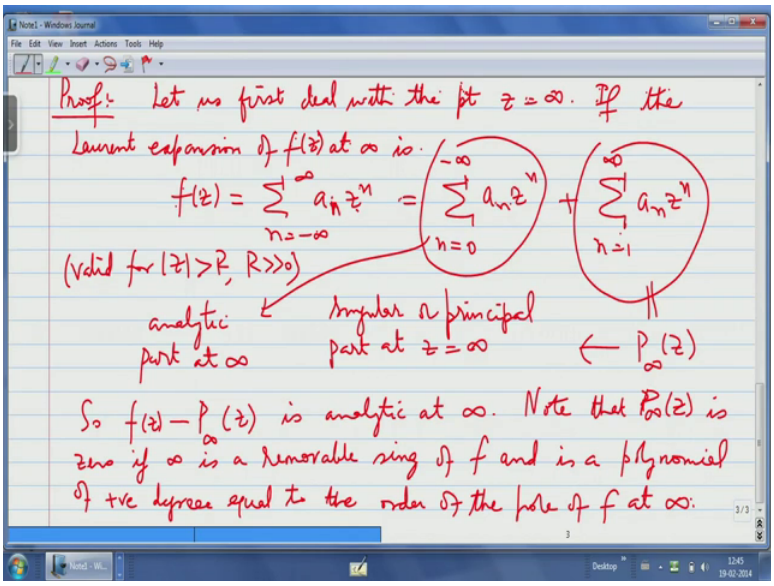

So well so let us try to prove that theorem let me use another color so proof, so you see so I am given a function f Z the function is supposed to be considered on the extended plane okay which means you are also considered the point at infinity okay and then the only information is given to you is that a function is Meromorphic it means you know that it has only poles and since the set in consideration with the extended plane which is compact you know I have already told you there are going to be only finitely many poles okay and it is possible that infinity may be a pole or not okay.

So 1st let deal with infinity first okay and so you look at the Laurent expansion of the function at infinity alright and you know that the Laurent expansion of the function at infinity will consist of both the positive and negative powers of Z okay and of course the constant term and you know the singular part at infinity is the part that consist of positive powers of Z okay and you know since, so now you see infinity there are 3 choices for infinity, see infinity can be either the removable singularity okay or it could be a pole okay and of course it cannot be an essential singularity because we assume the function is Meromorphic so it cannot have any singularities other than poles okay.

Now if infinity is a removable singularity okay it means that if you take the Laurent expansion at infinity okay the principal part which consist of positive powers of Z there is a singular part has no terms okay and if you assume, so it consist of only the constant part and the negative powers of Z okay that is if you assume infinity is a removable singularity. If you assume infinity is a pole which is the only other possibility and you know that the principal part of the singular part consisting of positive powers of Z is to be finite, so it has to be polynomial of positive degree okay without a constant term and of course the degree will be the order of the pole at infinity okay.

So in any case you are going to get the similar part as either 0 or a polynomial that is what I am trying to say at infinity and if you take the function and remove that singular part whatever is left is going to be analytic at infinity that is what you must understand. So this is something that we use repeatedly, what is the point about the Laurent expansion at a point of a function. The Laurent expansion consist of a singular part or principal part and an analytic part and if you take the function and subtract the singular part what you will get is only a Taylor series which will be actually the Taylor series of the analytic function which is given by the difference of the function and its singular point.

The moment you take away the singular part, the principal part the function becomes analytic okay, so this is a trek that you always use if you want to extract the analytic part what do you do? You take the function and subtract the singular part or the principal part okay, so let me write that down so let us 1st deal with the point at infinity the point Z equal to infinity if the Laurent expansion of f of Z at infinity is f of Z is equal to Sigma n equal is to minus infinity to infinity a n Z power n that is Sigma so let me write it like this Sigma n equal to 0 to minus infinity a n Z power n plus Sigma n equal to 1 to infinity a n Z power n. If Z is this and now let me call this fellow as p infinity of Z okay, this p infinity of Z is what this is the singular or principal part at infinity okay.

So this is the singular or sensible part at Z equal to infinity okay and of course whatever is left out here this is the analytic part okay, so you see of course you should take this Laurent expansion to be valid for mod Z greater than R for R sufficiently large, so valid for mod Z greater than R, R sufficiently large, so this is very important okay and so if you take, so the point is that so you should take f of Z minus P infinity of Z okay. This is going to be this is analytic at infinity okay this is going to be analytic at infinity okay because f of Z minus p infinity of Z will only consist of a constant term a naught and negative powers of terms involving negative powers of Z and negative powers of Z behave well at infinity okay and of course should tell you something if you look at the point at infinity since we have assumed that f is Meromorphic this p infinity of Z is not actually a power series is only a polynomial okay p infinity of Z will be 0 if infinity is a removable singularity at is a point of analyticity or f and it will be a polynomial of positive degree equal to the order of the pole of f at infinity if infinity is a pole okay.

So let me write that down, note that p infinity of Z is 0 if infinity is a removable singularity of f and is a polynomial of positive degree equal to the order of f the order of the pole of f at infinity. It is no other possibility you do not have the situation when infinity is an essential singular point because we have assumed f is Meromorphic the only singularity that are allowed are poles okay, fine.

So now what you do is now let us look at the other singular point, see I have assumed f is Meromorphic so it has only finitely many poles because it is Meromorphic on the extended plane which is compact that is very important okay there are only finitely many poles in the usual complex plane and forget the point at infinity because I have already dealt with it. I now want to keep track of the points on the plane where f has poles there are only finitely many let me call those points Z 1 through let us say Z n okay, so let us write that down.

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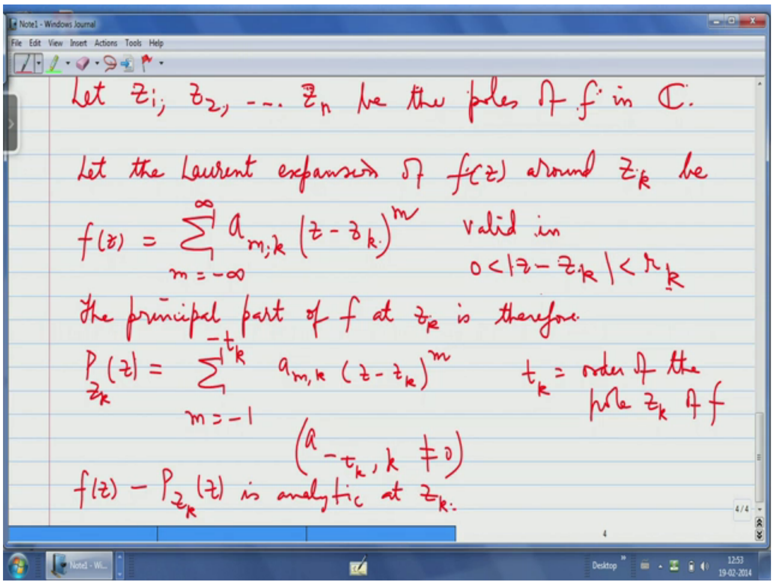

Let Z 1, Z 2 and so on Z n be the poles of f of Z in the complex plane this okay and again let me stress this is very important that you getting finitely only finitely many poles because you have assume that f is Meromorphic on the extended plane. The compactness of the extended plane is doing a big job here otherwise you need not get finitely many poles okay. So you take these holes now whatever you did at infinity you do at each of those poles okay, so take any of those poles then in a deleted neighbourhood of those poles each of those poles f that means a Laurent expansion and you know if you take the Laurent expansion you take the singular part it is going to have only finitely many terms okay and of course the highest negative power occurring will be equal to the order of the pole okay.

So let a Laurent expansion of f of Z around Z k be f of Z is equal to Sigma m, m equal to minus infinity to infinity and I will call the coefficients as a m k and mind you it will be Z minus Z k to the power of m okay this will be the Laurent expansion, this is the Laurent expansion centred at Z k okay and valid in a deleted neighbourhood of Z k and of course you know that it will be a actually disk centred at Z k and the radius will be equal to the distance from Z k to the nearest of the other Z j, so valid in 0 less than mod Z minus k there of course r k I am not writing it r k is a distance of Z k to the nearest of the other Z j, J naught equal to k okay and what is the principal part at Z k it is going to have only finitely many terms okay and the highest negative power of Z minus Z k you are going to get that is going to be the pole of f at Z k.

So the principal part of f at Z k is therefore p Z k of f which is going to be let me write this as p Z k of Z that is going to be Sigma it is going to be m equal to minus 1 to minus let me put t k, a m k into Z minus Z k to the power of M okay and mind you where t k is equal to order of the pole Z k okay of f mind you these are all the powers of Z minus Z k r in the denominator starting with m equals to minus 1 and going all the way up to minus t k and of course a minus t k, k is not 0, a minus sub t k, k is not 0 I mean this is the coefficient of the highest negative power of Z minus Z k, right. Well so this is the principal part at Z okay and now what you do is that you do the same trick as before you use the fact that if you take the function and subtract away, take away the principal part whatever you going to be left with is going to be analytic because it is going to be a power series.

So it is going to represent an analytic action whose Taylor expansion at Z k is exactly the power series which is the analytic part of the Laurent expansion. See you must always remember this analytic part of the Laurent expansion is actually the Taylor series of what analytic function? It is the Taylor series of the analytic function which is given by a result function minus the principal part okay. So let me write that down we need to use it f of Z minus p Z k of Z is analytic at Z k okay so you see so now look at the scenario, the scenario is have there is Meromorphic function f there are these poles, finitely many poles that one through Z n okay and at each of these poles there is a principal part okay and if you take the principal part away, what you get is something that is analytic at that point and of course there is also in a principal part at infinity okay, now what I do is I take the function and remove all principal part is okay.

You take the function subtract the principal part at Z 1, subtract the principal part at Z 2 and so on and you subtract the principal part at infinity you subtract all the principal part and what are you going to get? You are going to get an entire function on the whole Riemann sphere and what is going to be? It is going to be a constant because of Liouville's theorem. An entire function which is… If you are going to get an analytic function on the extended plane which means you are getting an entire function which is analytic at infinity, an entire function which is analytic at infinity by Liouville's has to be a constant it is a bounded entire function, analytic at infinity means bounded at infinity, bounded at infinity means is a bounded entire function and it is constant.

So you moral of the story is you take this Meromorphic function and subtract all the principle arts you are going to get a constant and now you push all the principal part to the other side you will get that the Meromorphic functions is a constant plus all these principal parts but each of these principal parts are rational functions and a constant is also a rational function, so you have express the Meromorphic function as sum of finitely many rational functions therefore it is rational and that proofs the theorem okay, so that is all, so let me write that down.

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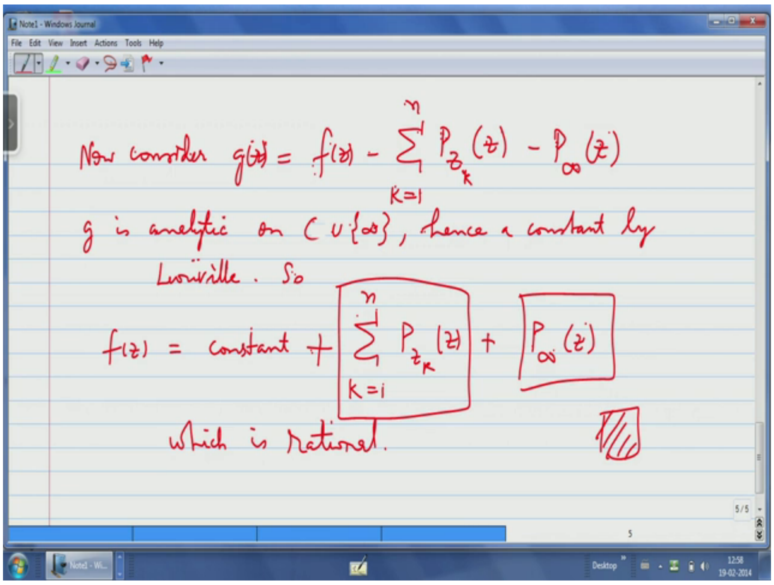

Now consider g of Z is equal to f of Z minus Sigma k equal to 1 to n, p Z k of Z so these are the principal parts at those end poles and then also take away p infinity of Z okay. Now what you get is g of Z is analytic on the whole extended plane and by Liouville therefore it is a constant. g is analytic on C union infinity hence a constant by Liouville, so f of Z is equal to constant plus Sigma k equal to 1 to n, p Z k of said plus p infinity of Z which is Meromorphic on the extended plane and that is the proof of that theorem that a function which is Meromorphic and the extended plane is just in fact what I want to say is not… of course it is Meromorphic and the point is its rational okay that is the point, we want to show that Meromorphic function on the extended plane is a rational function okay.

So what we have got as you prove that Meromorphic function f is constant plus these principal parts, finitely many principal parts, so this part p infinity of Z is a polynomial okay it could be 0 and these are all involving negative powers of Z minus k finitely many for each k okay and this is of course a rational function. If you take LCM you will see that you will get a quotient of polynomials therefore it is a rational function and the beauty of this proof is that this proof also tells you that you get for every Meromorphic functions you get the partial fraction decomposition, the each p Z k they are all the various terms of the partial fraction decomposition. So this proof in one stroke tells you that the Meromorphic function as partial fraction decomposition and is actually a rational function okay that is the advantage of this proof, okay. So I will stop with that.