Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

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Lecture No 13

Infinity as an Essential Singularity and Transcendental Entire Functions

You see we have been looking at a point at infinity and what we have seen is situation when infinity is an isolated singularity okay and I have already told you what happens when infinity is a removable singularity or infinity is a pole and even more generally we have also seen the so-called residue theorem for the extended complex plane okay, so the point that one has to remember is that the residue theorem for the extended complex plane which we discussed in the last lecture in that the contribution of the residue at infinity will always be there okay irrespective of whether the point at infinity is a removable singularity or a pole or an essential singularity.

So the issue is that you know even if infinity is a removable singularity okay for example for the function f of w equal to 1 by w okay infinity is w equal to infinity is a removable singularity okay because the function tends to 0 as w tends to infinity but still the residue at infinity is not 0 which does not happen for a point in the usual plane, in the complex plane at a point at which the function has a removable singularity is a point where the function can be redefined so that it becomes analytic and therefore if you calculate the residue at that point you will get 0 okay whereas this does not happen for the point at infinity that is 1 point that you should always keep in mind.

Now we are going to talk about the situation when infinity is an essential singularity okay, so what does it mean to say that infinity is an essential singularity, so basically you are looking at a function for which the point at infinity is a singular point and isolated singular point that means it is defined in a neighbourhood of infinity in a deleted neighbourhood of infinity and it is analytic in a deleted neighbourhood of infinity which means that basically you are looking at an analytic function which is analytic outside a circle of sufficiently large radius okay and because that is what neighbourhood of infinity is okay you know that the stereographic projection.

Now what does it mean to say that the function has infinity as an essential singularity, it is the same as saying…there are 2 ways of saying it of course in fact 3 ways of saying it and all the

3 ways of saying it are correct, so there is one way which is using the Laurent expansion at infinity, so one way of saying that infinity is an essential singularity is by saying that if you take Laurent expansion at infinity then the singular part has infinitely many terms okay and there what you must remember is that the singular part is actually a polynomial part I mean it is the…if you expand it in powers of the variable okay if w is the variable then you are going to expand the function in positive and negative powers of w okay and you should look at an expansion which is valid outside a circle of a sufficiently large radius mind you that is the Laurent expansion in a neighbourhood of infinity okay and because you are looking at a point at infinity the positive powers, the terms involving positive powers of w that is a singular part okay that is a singular part and the term involving the constant and the negative powers of w is the analytic part because negative powers of w behave well at infinity okay.

So the condition for the function to have infinity as an essential singular point is that if you write out its Laurent expansion in a which is valid outside at all points outside a sufficiently circle of sufficiently large radius centred at the origin okay then you must get you must get the positive powers of the variable that occur, they must be there must be infinitely many terms which is the same as saying that the singular part at infinity has infinitely many terms, so what should not happen is that the singular part has only finitely many terms at infinity which means the singular part is actually a polynomial okay. So in other words what we are saying is that you know how do you recognize that action has infinity as an essential singular point, you write out the Laurent expansion at infinity valid in a neighbourhood of infinity then what happens is that the singular part is not just a polynomial okay.

In particular what this means is that if you take a polynomial functions they are certainly not going to have infinity as an essential singularity and you know that you have seen it last time a polynomial of degree polynomial is in fact have infinity as a pole and the order of the pole is actually the degree of the polynomial okay, so this is one way of defining infinity as an essential similar point of course the other way is that the limit of the function as you approach infinity does not exist okay that is also correct okay.

So usually there are 3 characterisations one of any singular point there are 3 characterisations one is you look at the Laurent series and look at the condition of the Laurent series whether it has no terms at all in the singular part or whether it has finitely many terms in the singular part or whether it has infinitely many terms in the singular part and the 2nd condition is on the limit of the function as you approach that point okay. Of course the 3rd condition is about the bounded mass of the function okay or the unboundedness of the function but of course the boundedness is equivalent to the point being a removable singularity okay and the unboundedness means that it is either it could either be a pole or an essential singularity okay.

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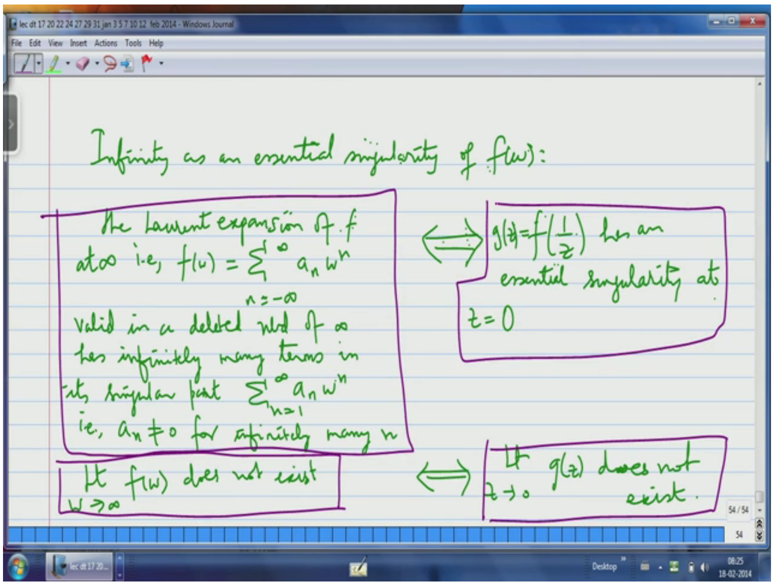

So let me write these points down infinity as an essential singularity of f of W, so you see so f of w is assumed to be analytic in a deleted neighbourhood of infinity okay that is a standard assumption blanket assumption and well so, so what are the conditions well you write a Laurent expansion of f at infinity that is f of w equal to Sigma and equal is to minus infinity to infinity a n w power n valid in a deleted neighbourhood of infinity has in finitely many terms in its singular part and what is the singular part?

The singular part is Sigma, the singular part is the positive powers of (())(9:47) so it is Sigma n equal to 1 to infinity a n w power n and that is the same as saying that a n not equal to 0 for many in finitely many n for infinitely many n, so and you see this definition is just the same as usual definition that in order to say that a certain point for example in the complex plane and isolated singularity is an essential singularity all you have to do is write out the Laurent expansion at that point then you find that the Laurent expansion has in finite singular point okay and there it will be in finitely many negative powers okay.

So of course I should tell you that this is equivalent to g of Z is equal to f of 1 by Z has an essential singularity at 0 okay, so g of Z f of 1 by Z has an essential singularity at 0 at Z equal to 0 okay and this is of course you know this is of course based on this philosophy that the behaviour of f of w at infinity is the same as the behaviour of f of 1 by Z at 0 okay you put w equal to 1 by Z we already know that this is a homeomorphism okay and it is a holomorphic isomorphism of the punctured plane, the plane minus the origin onto itself okay and so this is one thing. Then the other thing is so this is one condition the other condition is limit w tends to infinity f of w does not exist this is this is the other condition and of course this is the same as saying that limit Z tends to infinity g of Z does not exist.

So this is also something that you know and so well of course the usually we will have another conditions which will be on the behaviour of the function in the neighbourhood of that point and well certainly you cannot expect the function to be bounded in a neighbourhood of that point okay because that is equivalent to the function being actually analytic at that point and that is Riemann's removable singularity is theorem right. Anyway so let me go ahead and say some other things, so maybe I will put some I will change color and put a few boxes here, so this is equivalent to this and this part is the same as this okay.

Alright so now what I want to say next is that you know let us analyse this condition that infinity is an essential singularity okay, now the…so for example what are the functions? What are the entire functions which have infinity as an essential singularity, you can ask this question? Okay so we have already answered a similar question for poles okay and for removable singularities. See you take an entire function mind you an entire function means a function which is analytic on the whole plane okay and the moment it is analytic on the whole plane the whole plane is also mind you a deleted neighbourhood of the point at infinity you must not forget that there for a function is analytic on the whole plane is also having infinity as an isolated singular point automatically okay.

If you think of the Riemann's stereographic projection you see that the infinity corresponds to the North pole and the whole plane corresponds to the remaining part of the Riemann sphere which is the Riemann sphere minus the North pole okay by this geographic projection, so the plane itself is a neighbourhood of the point at infinity and therefore an entire function will always have the point at infinity as an isolated singularity. Now what happens if that isolated singularity is a removable singularity? Well then you are saying that the entire function as infinity as a removable singularity okay which is the same as saying that at infinity it is bounded or it has a limit okay and then by Liouville's theorem it will reduce to a constant okay.

So the moral of the story is that a non-constant entire function cannot have infinity as a removable singularity we have already seen this okay and then you can ask the question is when will an entire function have infinity as a pole okay and have seen that that will happen if and only if the entire function is well is a polynomial okay and the degree of the polynomial will be the order of the pole okay. So now we ask the question, when will an entire function have infinity as an essential singularity? And the answer to that will be that it should not be a polynomial okay basically if you write out the if you write out its Taylor expansion at any point, the Taylor expansion of course you will get you have to ride only a Taylor expansion because it is an entire function okay.

There is no question of Laurent expansion okay, so at any point you choose any point in the plane it is analytic everywhere, the plane so you choose any point and write the Taylor expansion at that point. That Taylor series will have in finite radius of convergence because this function is entire okay and the point is that the Taylor series should be a power series, it should not be a polynomial okay, so the moral of the story is an entire function as infinity as an essential singularity if and only if its Taylor series is not a polynomial okay.

So and the Taylor series of an entire function being a polynomial is the same as you entire function itself being polynomial okay therefore all you are saying is that you know an entire function if you wanted to have an essential singularity at infinity okay then it should not be a polynomial alright and for this reason we call such entire function as transcendental okay so usually a polynomial function and the Meromorphic functions which are given by quotients of polynomials they are called algebraic and everything that is not algebraic is called transcendental, so such entire functions which have infinity as an essential singular point are called transcendental entire function okay.

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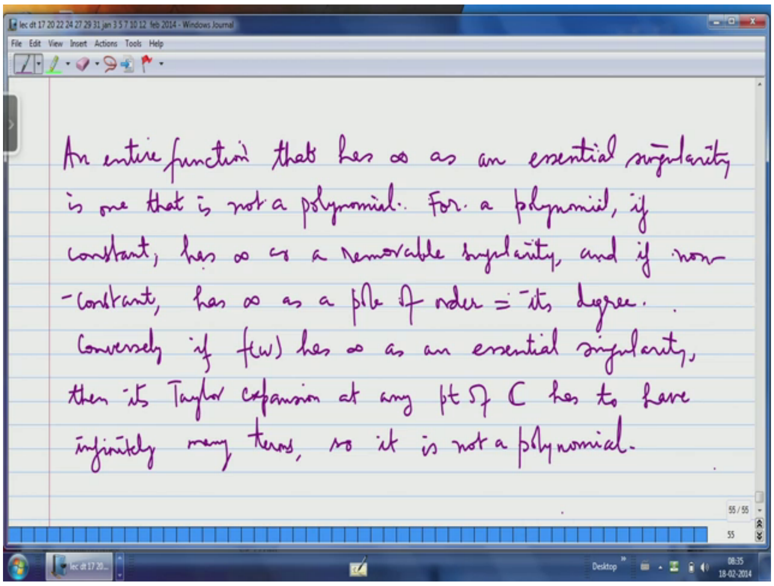

So let me write that down an entire function that has infinity as an essential singularity is one that is not a polynomial okay, so and you know what will happen if it is a polynomial, if it is a polynomial infinity is a pole okay, if it is a polynomial of positive degree of course if you… we also consider conscience as polynomials, polynomial is of degree 0 and that is the case of a constant function, so if you are looking at a non-constant entire function okay then it is… the only way infinity is an essential singularity is that it is not a polynomial of positive degree okay.

So or a polynomial, if constant has infinity has a removable singularity and if non-constant has infinity as a pole of order equal to its degree okay, so a polynomial is certainly not an entire function that has an essential singularity at infinity and conversely if an entire function as an essential singularity at infinity because you take any point and you write out the retailers expansion at that point you will see that if infinity is a is a pole then the Taylor expansion should terminate and it has to be a polynomial, so if infinity is not a pole then your Taylor series will have infinitely many terms okay which is the same as saying that there are in finitely many terms in the Laurent expansion at infinity okay, mind you if you take a polynomial the polymer normal itself is the Taylor expansion of the function that it represents at the origin okay and sense that expansion is valid on the whole plane it is also a Lauren expansion at infinity.

The expression for the polynomial itself is a Lauren expansion at infinity and it is except for the constant part the positive part terms involving positive part of the variable, that is automatically the singular part at infinity and that it has only finitely many terms tells you that infinity is a pole okay. So let me write that down, conversely if f of w has infinity as an essential singularity then its Taylor expansion at any point of C has to have, at any point of C has to have in finitely many terms, so does not a polynomial okay, so say things in short a transcendental entire function is something that is different from a polynomial.

Now I want to make the following statement that take any entire function okay exponentiated a, so you take f of w to be an entire function okay, f of w may have infinity as you know either a pole or may be an essential but if you take E power f of w okay if you take E power f of W, I claimed that it will always be transcendental okay, so that means I am saying that or the power f of W, w equal to infinity will always be an essential singularity okay that is also very easy to see and how do you see it let me tell you the argument and words you see so I know that f of w is entire and am looking at E power f of w okay.

Of course E power f of w is also entire because it is a composition of entire functions, exponential function is of course entire okay and E power f of w is f of w allowed by the exponential function it is composition of entire function, so it is entire okay and what other possibilities of E power f of w at infinity? Infinity can either be removable singularity or it can be a pole or it can be an essential singularity. If infinity is a removable singularity you know then the power f of w must be a constant because of the Liouville's theorem and if E power f of w is a constant then f has to be a constant because f is if E power f of w is a constant that constant is a complex number which are not be 0 because (())(23:45) exponential function never takes the value 0 and f has to be log of that okay.

F can be one of the logarithms of that constant, nonzero constant okay and so in that case you power f so what am trying to say is that if you are looking at an entire function a non-constant entire function okay the power have has to be transcendental okay job the only case you will have to worry about is the constant function. When of course E power a constant is also a constant, so if you have a non-constant entire function okay the power f also will be non-constant right, so infinity or the power f if infinity is a removable singularity then f is a constant, so if you are looking at a non-constant f then he power f will not have infinity as a removable singularity.

So it can be a pole now if E power f is a pole has a pole at infinity, infinity is a pole for E power f then E power f has to be a polynomial because we have seen that the only entire functions which have pole at infinity are the polynomials so E power f has to be a polynomial but that is not possible because you see if you take a polynomial of constant a non-constant polynomial of positive degree the fundamental theorem of algebra says that it will have zeros, so there are…it will assume the value 0 but that is equal to E power f and E power anything can never be 0 okay therefore E power f cannot be a polynomial alright at means that infinity cannot be a pole for E power f so therefore the only thing that is left is E power f should have infinity as an essential singularity, so the moral of the story is that you take any non-constant entire function and you take E power that, the resulting function will certainly have infinity as an essential singularity in other words I am saying that the resulting function it will always be transcendental okay.

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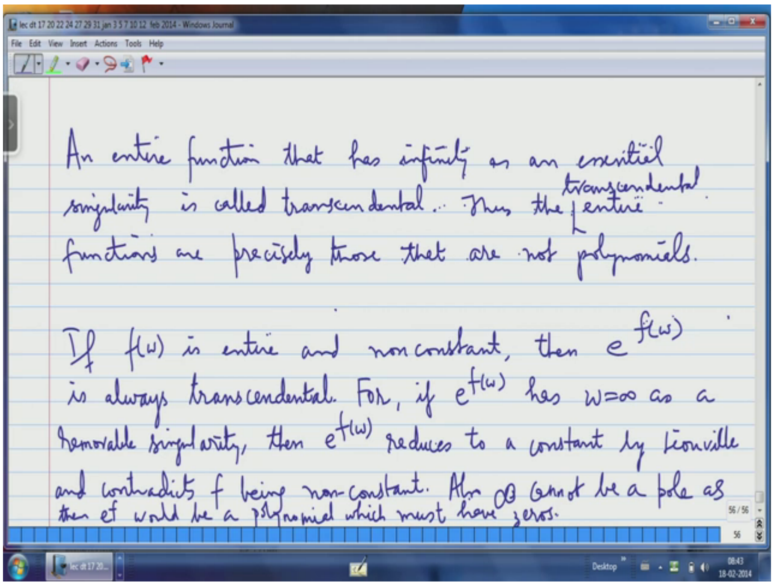

So let me write that down okay so let me use a different color an entire function that has infinity as an essential singularity is called okay, an entire function that has infinity as an essential singularity is called transcendental and so all entire functions which are not polynomials are transcendental okay. Thus the entire functions, the transcendental entire functions are precisely the non-polynomials, precisely those that are not polynomials okay if f of w is entire and non-constant then he power f of w is always transcendental. For if E power f w has w equal to infinity as a removable singularity then E power f w reduces to a constant by Liouville and contradicts f being non-constant also E power f also infinity cannot be a pole as then he power f would be a polynomial which must have zeros okay.

So that also cannot happen so as a result you take any entire function and you which is not constant and the exponentiated what you get is a transcendental entire function okay. So in this context let me also say that you know we entire function the functions which are either polynomials or quotient of polynomials okay they are called Meromorphic functions they are all called algebraic okay and the non-algebraic functions are the transcendental ones okay.

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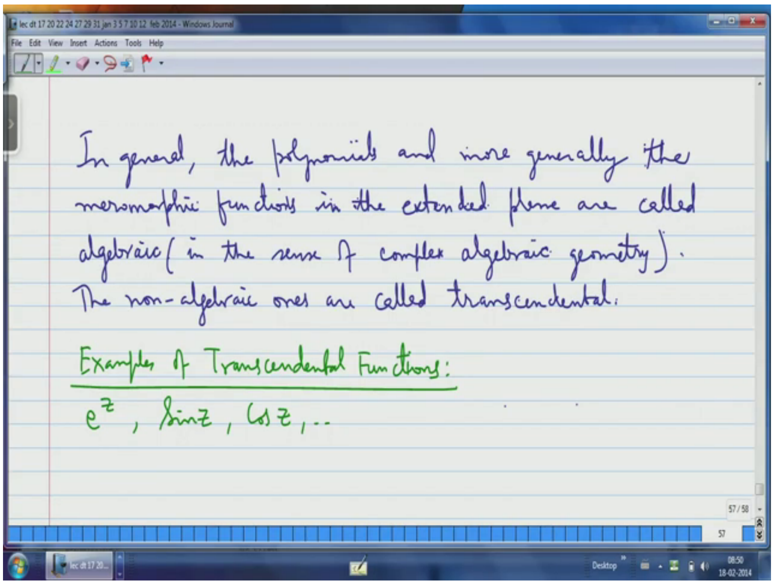

So let me write that in general, the polynomials and more generally the Meromorphic functions in the extended plane are called algebraic okay at least in the sense of algebraic geometry okay in the sense of complex algebraic the non-algebraic ones are called transcendental, okay. Examples, examples of transcendental functions, this is something that one should look ahead, so you take for example E power Z this is transcendental because Z equal to infinity is an essential singular point and why is that so because basically if you write the expansion for E power Z Taylor expansion or MacLaurin expansion which is a Taylor expansion in the origin you write the usual expansion that we all know and you know it has infinitely many terms okay.

So E power Z and then similarly you can take the trigonometric functions you can take sin Z you can take cos Z and so on okay, so these are all transcendental functions and the way of course you know you can also see that infinity is an essential singular point because if you change if you invert the variable you will get origin as the essential singular points, so if you take E power 1 by Z origin will be an essential singular point, if you take sin 1 by Z origin will be an essential singular point, if you take cos 1 by Z origin will be an essential singular point okay, so these are examples of transcendental functions and the other thing that I want to tell you is that I want to also recall the big Picard theorem in this connection okay take now you see you know take an entire function, take an entire function which is not constant okay and of course if infinity is an infinity cannot be a removable okay because if infinity is a removable singularity then it will reduce to a constant. Since I have taken a non-constant entire function, infinity is not a removable singularity.

The next possibility is infinity is a pole in which case the entire function is a polynomial and you know the image of a polynomial map is the whole plane okay because a polynomial can assume we will assume all values okay you take any value you can equate to the polynomial and you can solve for it and you will get solutions that is because of the fundamental theorem of algebra, so if you take a polynomial mapping it will be the image of the whole plane under a polynomial mapping will be the whole plane then you look at the 3rd case namely when infinity is an essential singular point okay. Now if infinity is an essential singular point okay you see what does the big Picard theorem say?

The big Picard theorem says that you take any neighbourhood of an essential singular point no matter how small, the image will be the whole plane and or it may be a punctured plane it might miss at one point okay. Now you see this is the so what you are saying is that if I take a transcendental entire function okay For example E power Z or sin Z or cos Z okay then the big Picard theorem tell you that even if you take the image of not the whole plane but the exterior of a circle no matter of how large radius you will still get the whole plane or the punctured plane and you can compare it to the little Picard theorem which tells you that the image of whole plane is either the whole plane or the plane minus a point, so I want you to understand the significance.

See if I take for example take E power Z okay the little Picard theorem will tell you that the image of E power Z is either the whole plane or it is the whole plane minus a point because you know it is E power Z we know it is the plane minus the origin okay but the little Picard theorem never tells you what is the image of anything other than the whole plane okay but if you apply the big Picard theorem to E power Z okay you are applying mind you when I apply the big Picard theorem to E power Z I am trying to apply I have to apply it only to an essential singularity and where does E power Z have an essential singularity at infinity.

So if I apply the big Picard theorem to E power Z at infinity okay then you see that you take any neighbourhood of infinity okay which is you take the outside the reason exterior to a circle of any radius, no matter how large, the image of that itself will be the whole plane or a punctured plane okay and in fact every value is taken in finitely many times, so you see you see in that sense how the big Picard theorem is far stronger than the little Picard theorem in the case of entire functions you can see that of course for polynomials there is nothing special because your fundamental theorem of algebra which will tell you things very clearly but if you are looking at non-polynomials.

If you are looking at transcendental entire functions you see you can really see the amount of difference in the conclusion and you can see the strength of the theorem, you see the great Picard theorem is you much more than the little Picard theorem okay, so if you take E power Z and you take the image of the exterior of a circle matter how large radius it will still be the punctured plane okay that is what the big Picard theorem applied to E power Z will tell you okay whereas if you try to apply the little Picard theorem you will not get anything it will give you only the image of the whole plane but it will not tell you anything about the image of the exterior of a circle okay, so in that sense you see how powerful the big Picard theorem is okay.

Now what I want to do next is tell you about tell you about Meromorphic functions okay, so you see so I want to concentrate on Meromorphic functions and you know we need to study Meromorphic functions in order to get to our main aim which is the proof of the Picard theorems alright, so the 1st fact about Meromorphic functions is that you see if you take domain either the domain may be in the plane or it may be a domain in the extended plane which means it could include the point at infinity as an interior point. On a domain if you look at the set of Meromorphic functions on the domain that is automatically a field okay, so you get an algebraic structure called a field and it will be an extension field of the field of complex numbers okay. So and this field extension its algebraic properties are deeply connected with a geometric properties of the domain you are studying okay.

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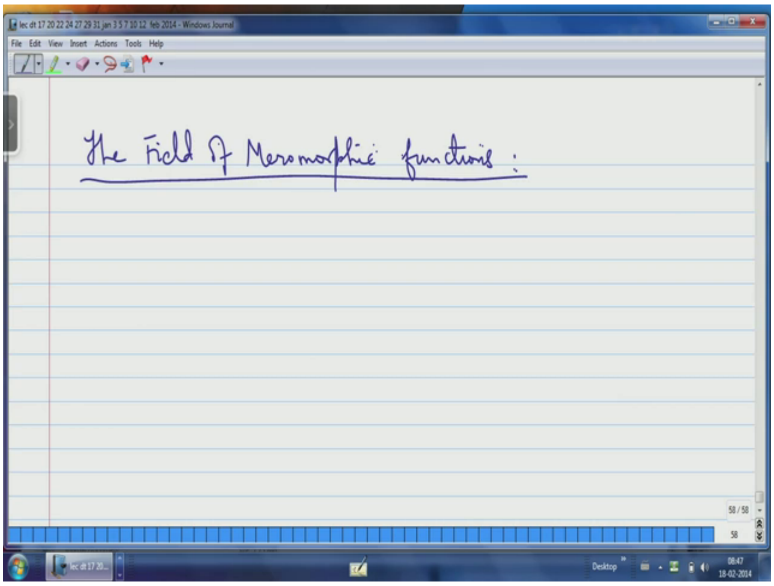

So let me make that statement so let me go to the next thing the field of the Meromorphic functions, so let me recall what is a Meromorphic function? So it is basically a function which is analytic and has only isolated singularities which can be at most (())(39:36) okay and of course they could be removable singularities but if they are removable singularity is you really do not consider them because they are actually points where the function is analytic okay otherwise the only (())(39:50) singularities that it has are poles okay, so the moment am talking about a Meromorphic function I have to remember that 1st of all what is allowed is … what kind of singularities are allowed us are only poles which means there are only isolated singularities there are no non-isolated singularities okay.

There are only isolated singularities and they these isolated singularities are actually only poles okay and the point is that you see if you look at a Meromorphic function in the extended plane okay then you see that since an isolated set in the extended plane has to have only finitely many points okay because the extended plane is compact okay therefore there will be only finitely many poles okay, so the moral of the story is that if you are looking at Meromorphic functions on the extended plane it will have only finitely many poles okay. I think probably I will stop here.