Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

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Lecture No 12

Proofs of Two Avatars of the Residue Theorem for the Extended Complex Plane and Applications of the Residue at Infinity

So let us continue we are looking at the key idea of residue at infinity okay, so you know basically the way we define residue at infinity is exactly the way that we define residue at a point in a complex plane namely you calculate the contour integral around the point going in a positive sense about that point of the function and divide by 2 pie i okay, so the only thing is that if the point is a point at infinity then you see the contour you should anyway be contour a simple closed contour on the plane but saying that it goes around infinity in the positive sense amounts to saying that you are choosing this simple closed contour in outside a circle of sufficiently large radius and the fact that it is going around infinity in the positive sense means that you will have to give it the clockwise orientation okay and we saw that using the stereographic projection that this is a this make sense okay and well hash.

Now what I want to say is that so there is a version of the residue theorem that we want which will also work for the point at infinity and so I was explaining last time that you see this is this version of residue theorem will be the statement of that will be the same as the statement of the usual version of the residue theorem but there is there are 2 significant differences the 1st thing is that when you talk about of the residue theorem and of course you are only worried about singular points and you are not worried about…

So the residue theorem says that if you integrate a function which is analytic except for isolated singularities around the simple closed contour then what you pick up is the integral gives you 2 pie i times the sum of the residue of the function at the at the finitely many points where the function has isolated singularities okay and the point is that you see you are really not worried about points where the function is analytic or for example isolated singularities which are actually removable because at such a point where if you have an isolated singularities a removable singularity the residue will turn out to be 0 okay.

So essentially what you will get is you will in the usual residue theorem what you when you refer to some of residue or will you refer to residues what really matters is the residues at the (())(4:32) singularities because residues at the removable singularities will always be 0 and that is because of Clausius theorem because if you integrate an analytic function then you are going to get 0 okay, so I mean of course over a closed contour.

Now the point is the point is that when you are doing this also to include the point at infinity you have to be careful, the fact is that infinity behaves very differently in the sense that function can have a residue at infinity even if it is analytic at infinity that is the big difference, so when you talk about residues and also want to include the point at infinity you compulsorily include the point at infinity okay irrespective of whether infinity is rarely (()) (5:23) singularity or not okay so even if infinity is a removable singularity have to include the residue at infinity is the big point okay and you would not have to do that for a point other than infinity in the complex plane because the residue will then turn out to be 0 for a point with the removable singularity but so the easiest simplest illustration of this is the function 1 by w okay f of w equal to 1 by w you know that function is continuous at infinity because at infinity in fact it is bounded and well you know at infinity function being bounded or continuous is good enough, it is the same as it having a limited infinity a finite limit.

A finite means a limit which is a complex number and you know all these 3 are equivalent because of Riemann's removable singularity theorem rather the inspiration given by a theorem which allowed us to cleverly define analyticity at infinity will be equal in to one of these things okay, so if you take one by w that is well-behaved at infinity okay in fact it is 0 at infinity okay limit goes to 0 as w tends to infinity, so it is analytic at infinity but if you integrate 1 by w around infinity around simple closed contour that goes in the positive sense with respect to the point at infinity then you are going to get minus 1 you are going to get minus 1 in fact you will get minus 1 into 2 pie i okay 2 pie i times minus 1 so the residue is minus 1 okay so the moral of the story is that here is a function 1 by w all negative powers of w are good functions at infinity okay.

So 1 by w is the simplest is good at infinity the residue at infinity is not 0 even though it is analytic at infinity the residue at infinity is minus 1 okay, so the big deal this is the big deal, the big deal is whenever you talk about residue for the extended complex plane then you must always compulsorily include the point at infinity even if the point at infinity is a point of analyticity that is very important, so let me and of course this example 1 by w f of w equal to 1 by w also illustrates another things, it illustrates the following fact that you know this 1 by w has only 2 singular points, one is at w equal to zero that is an isolated singularity because it is not defined at 0 of course you can define it as infinity at 0 if you want so that you get something continuous on the extended complex plane for example what you do with any (()) (8:09) transformation but the point is that there is also another singularity namely the singularity at infinity that is also an isolated singularity but it is a removable singularity and the singularity at infinity as a residue of minus 1.

The singularity at the origin has a residue of plus 1 okay and you know at then you get 0 and this is actually the version of the residue theorem for infinity says the total sum of residue is 0 okay that is the version of residue theorem, that is one version of the residue theorem at infinity for the for the extended complex plane okay and it tells you to it tells you also another important thing why is that the Clausius theorem fails at infinity, see if you try to integrate the function 1 by w which is analytic at infinity around a simple closed curve goes around the point at infinity you not going to get 0 you are going to get of course you are going to get minus 1 which is in fact you will get minus 2 pie i which is not 0 okay.

It is 2 pie i times minus 1 which is 2 pie i times residue at infinity you are not going to give 0 and that is the violation of Clausius theorem because Clausius theorem by Clausius theorem we expect if a function is analytic you expect the integral over closed curve to be 0 okay but the reason is it is not violating Clausius theorem nothing, what is actually happening is that is nonzero integral at infinity, the value at infinity is to compensate for the residue at 0 so that the total sum of residues is 0 okay, so when you integrate 1 by w at infinity around infinity you get minus 2 pie i and you do not get 0 even though it is analytic at infinity but then this it has to compensate for the plus 2 pie i that you will get if you integrate it around 0 okay and the sum so that the sum minus 2 pie i plus 2 pie i is 0 and that is the version of the so that is the version of residue theorem. So what you must understand is that the fact that the residue theorem works is actually equivalent to the fact that Clausius theorem does not work this case okay and now you see that the fact that Clausius theorem does not work is not really a sad thing because you are getting something in exchange for that you are getting nice version of residue theorem okay.

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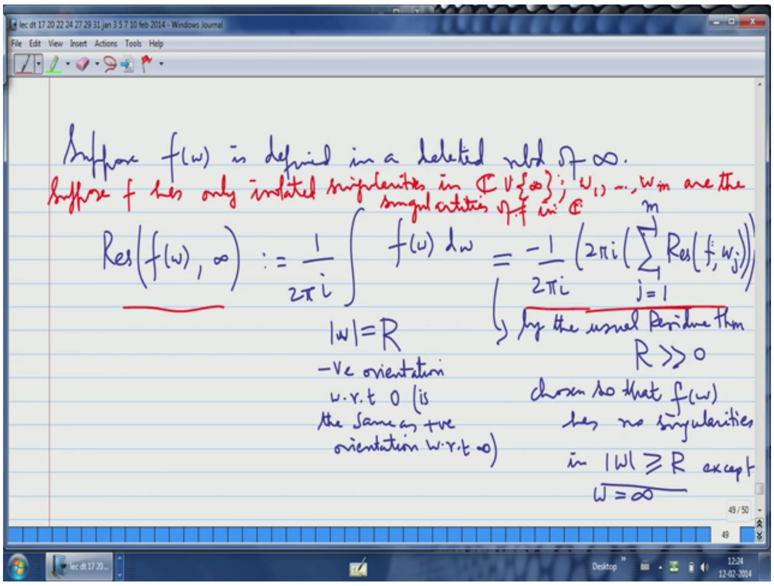

So what I will do now is let us prove these versions of residue theorems so let me start here suppose that suppose f of w is defined in a deleted neighbourhood of infinity okay which means it is defined or all w in the exterior of a sufficiently large circle of sufficiently large radius. Now what you can do is that so if you well so the residue of f of w at infinity if you calculate this quantity is by definition equal to you integrate over you for example integrate you can integrate over any you can integrate over any simple closed curve going around infinity in the positive sense.

So let us take a circle and you take R sufficiently large you take R sufficiently large so that you know there are no other singularities outside the circle of radius R except the point at infinity okay, so that is how large R should be chosen and of course the point is that you give negative orientation, so the reason you give his negative orientation this is negative orientation with respect to the usual conventions okay and mind you this is negative orientation, so let me write with respect to 0 is same as positive orientation with respect to infinity okay, so actually I am taking this integral over mod w equal to R around the I mean with the clockwise orientation okay, so I take this integral and I integrate what?

I of course integrate the function fw dw and well I divided by 2 pie i whatever I get and this is the residue at infinity alright and the point you have to remember is that this integral is anyway defined because I am taking am actually integrating the function over the circle and on the circle it is analytic in fact, in fact it is analytic in a neighbourhood of the circle okay. The only problem is the point inside the circle which is the point at infinity mind you the point at infinity is inside the circle you heard me right that is because the interior of the circle will be given the clockwise orientation will actually be the exterior of the circle in the usual sense okay.

So this is the residue at infinity and now the point is that say suppose you assume that f has of course you know I have taken R sufficiently large so that you know f has no singularity is in mod Z greater than mod w greater than R greater than or equal to R and you only singularities at infinity, so let me write that down, so that So R sufficiently large chosen so that fw has no singularity is in mod w greater than or equal to if you want R, okay there are no singularity is, right except so I should say except w equal to infinity because now I am also whenever I say mod w greater than or equal to R I am actually thinking of the extended complex plane so the point at infinity is also there it is an interior point mind you the point at infinity is an interior point for this for this reason mod w greater than or equal to R, okay and interestingly mind you if you look at this region on the complex plane it is unbounded okay it is unbounded and closed okay but if you look at the same region mod w greater than or equal to R in the extended complex plane it is compact it is bounded, okay.

So because you have added that one point at infinity to compactify it so it becomes bounded so this is so you know mod w greater than or equal to R whether you are looking at it in the complex plane or whether you are looking at it in the extended complex plane makes a lot of difference topologically. In the complex plane it is closed unbounded, in the extended complex plane it is closed and bounded it is compact okay. Anyway so fine now you see suppose you assume that function f has only isolated singularities on the whole plane wherever it has singularities suppose it has only isolated singularities okay then what happens is that you know you see all those singularities in the plane which means that I have left out the singularity at infinity all those singularities in the plane are going to lie inside this circle mod w equal to R even the usual positive anticlockwise orientation okay.

So which is a positive orientation about the origin a and then the usual residue theorem applies, the usual residue theorem will tell you that this this integral will give you minus of 1 by 2 pie i the integral over the same circle given there positive orientation okay and that will be minus 2 pie i times sum of the residues of the function at the isolated singular points inside the circle in the usual sense okay and then if you put these 2 together get the that the total sum of residues and the extended plane is 0 which is the 1st version of residue theorem okay, so let me write that down, so this is also equal to minus 1 by 2 pie i times 2 pie i times summation of the residues of f at w i, i equal to 1 to... or rather let me use not i let me use j wj, j equal to 1 to m.

So this is what I will get and here so let me say let me tell you that what I have done here is that I have simply put a minus sign because I am evaluating around the circle in the anticlockwise sense okay and then I am using the residue theorem okay so here is so here is by the usual so of course you know the big deal here is that I am making an assumption I am making this assumption that suppose f has only isolated singularities suppose f has only isolated singularities in the complex plane okay and so let me also say suppose it has isolated singularity in the extended plane because I want them to be only finitely many and therefore there are only finitely many singularities mind you an isolated set in the extended plane has to be finite because the extended plane is compact alright.

So suppose f has only isolated singularities the extended plane and then there are only finitely many isolated singularities for f and infinity may or may not be a point of singularity okay but in any case if you early without then you will get only finitely many isolated singularities in the plane and am calling those singularities as w 1 through w m okay, so let me write that here w 1…W m are the singularities of f in the complex plane okay, so you know the basically what I am saying is that if w 1 through w m are the only singular point for a function in the complex plane okay.

Then you then you can enclose them by a sufficiently large circle okay centred at the origin and then you integrate around that circle what you are going to get is 2 pie i times the sum of the residues of the function at those points that is all I have used here okay and now if you put all these things together if you look at the left side at the extreme left and you look at the extreme right and you put them together, what you will get is that for a function which has only isolated singularities on the extended plane (())(20:58) sum of residue is the same and that is the that is one version of the residue theorem for the extended plane, so let me write that down so okay.

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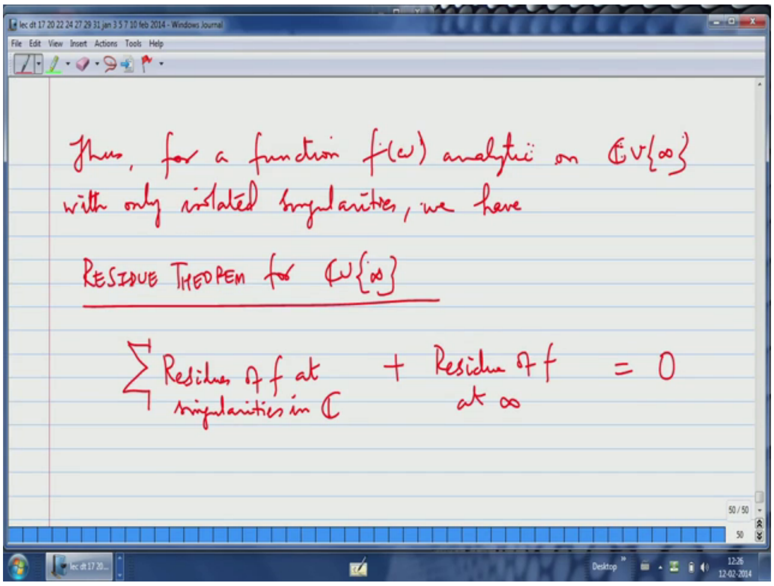

Thus, for a function f w is analytic on the extended plane with only isolated singularities we have residue theorem for the extended plane summation of residues of f at the singularities in the plane plus the residue of f at infinity is equal to 0, the total sum of residue is 0 okay, this is one version of residue theorem at infinity, right. Now and what about the so this looks slightly different from the usual version of the residue theorem, the usual version of the residue theorem says that you integrate around a curve function then you are supposed to get 2 pie i times some of the residues of the function at the single points inside the curve okay in the interior of the curve, does this also work for the extended complex plane? It does, okay so that is actually equivalent to as right.

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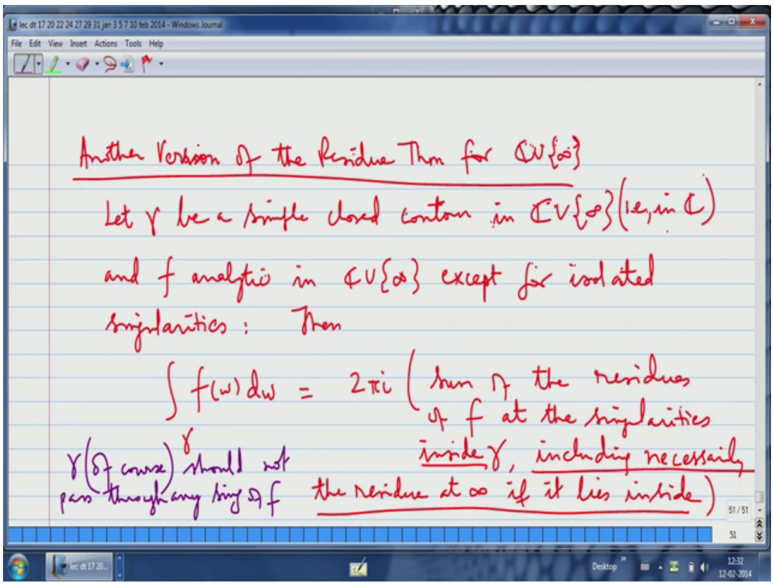

So let me write that down so here is another version another version of the residue theorem for the extended plane, let gamma be a simple closed contour in the extended plane by which we mean actually a simple closed contour in the complex plane, okay. Whenever we talk about simple closed contour or contour of integration you never think of a curve going to passing through the point at infinity because it really is something that you cannot see on the plane okay maybe you can think of that on the Riemann's sphere and work with that but the problem is to do that you will have 2 go to the language of Riemann surfaces you have to convert the Riemann's sphere into a…you must think of it as a Riemann surface and do integration.

You can actually do integration on the Riemann sphere along a for example a circle on the sphere which passes through the North pole you can really do that okay but to do that you will have to really use a language of Riemann surfaces okay and for more details about that you can look at my video course on Riemann surfaces the same NPTEL series but we are not going to do that, so I am just going to look at the simple closed contour only in the plane okay and that is what it will mean also for a simple closed contour in the extended plane okay in C and f analytic in the extended plane except for isolated singularities. Then integral over gamma fw dw is equal to 2 pie i times sum of the residues of f at the singularities inside gamma including necessarily the residue at infinity if it lies inside, okay.

So this is the so this is the extra thing okay, so it is again the residue theorem but it works also for the extended plane the only thing is you the integral of the function is 2 pie i use sum of the residues only thing is you have to be careful if infinity is inside gamma okay then you have to include also the residue at infinity irrespective of whether infinity is a point of analyticity of f or not okay that is the big deal. The big deal is you have to include infinity absolutely necessary you cannot omit it okay if it is a usual wind on the complex plane and if it is a removable singularity at that point you need not included because the residue at that point would be 0 because the function is analytic at a point where the function is analytic, the residue is 0 okay but it is not this is not true for the point at infinity as we have seen okay so you have to necessarily include the residue at infinity okay.

So this is the in this sense you know the residue theorem works alright and of course it is very important that gamma does not pass to any of these singularities the singularities in the finite plane that is of course always assumed okay. So let me write that here probably so you know always you have to keep remembering gamma of course should not pass through. Any singularity of f this is of course this is always there okay I mean if the curve which you are trying to integrate passes through a singular point of the integrant then you are in trouble because of course you know if that singular point is a removable singularity is not you really do not have to worry about it okay and of course when we write things like this we are really not worried about points on the plane which are removable singularity is okay, we simply assume that they are points where the function is analytic okay.

So what really matters the points in the plane at really matter for this statement are the points which are (())(28:34) singularities which are either poles or essential singularity is it is not a removable singularities okay so and I am saying gamma should not passed through any singularity of f of course it should not passed through a (())(28:43) of f it should not passed through a pole or an essential singularity of f and the reason is because at each of these (()) (28:50) singularities the function is not a continuous okay function is not continuous at those points and you know when a function is not continuous at a point it is in general you do not try to define its integral at that point unless you know for example is bounded okay you know the complex contour integration is also defined by integrating by taking part integrals of the real and imaginary parts which are real valued functions okay and so basically it reduces to real integration and you can use Riemann integration if you want.

You can use Riemann integral at the point is that and of course you know the Riemann integral is a limit of Riemann sums okay and if the function become unbounded at a point then at that point you really cannot expect Riemann sum to converge properly okay in a neighbourhood of that point, so you would never try to in general define the integral at a point of discontinuity especially when the discontinuity is of is not of a jump type if it is a very bad discontinuity not removable kind of discontinuity and you do not try to integrate the function at that point, okay. So the moral of the story is that this is usual version and how does one prove it? The usual version of the residue theorem just follows from the other version of residue theorem the extended plane which says that the total sum of residues including the residue at infinity is 0.

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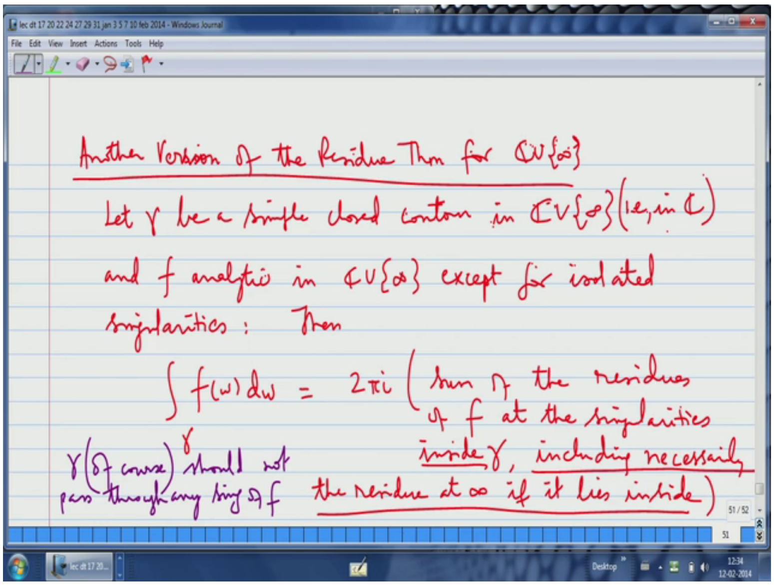

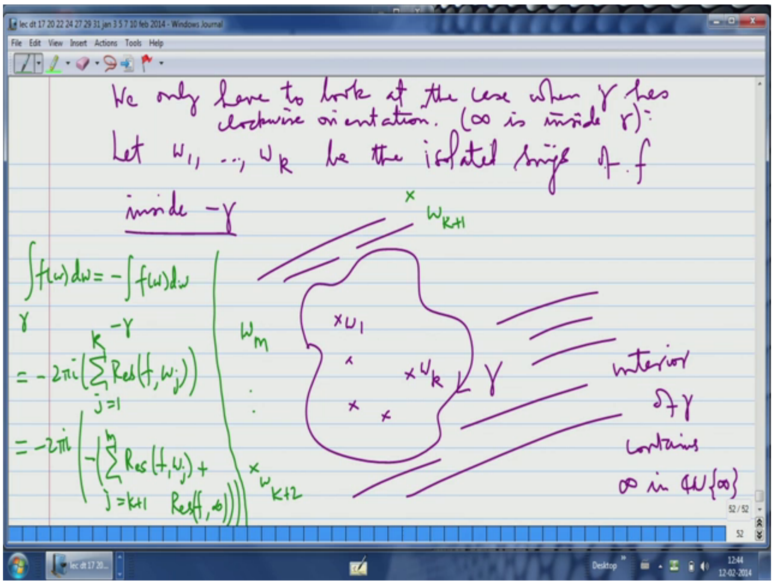

So I will I will just write it down it is pretty easy, so let w 1 et cetera w k be the isolated singularities of f inside, so you know okay so let me say something here, so let me go back to the statement and tell you that you know if you are really looking at gamma being you know positively oriented in the usual sense okay if you are looking at gamma a simple closed contour on the plane which is positively oriented going a that is going in the anticlockwise direction then you know the interior of the gamma is going to be a bounded domain in the plane and you know you are only worried about the integral there and that integral will give you 2 pie i times sum of the residues of f inside that a bounded domain then there will be only finitely (())(31:26) okay and that is usual residue theorem okay and you do not include of course you do not include the residue at infinity because infinity is not there okay because we have taken gamma to be clockwise I mean anticlockwise okay.

So actually there is nothing to prove in this statement unless you are looking at gamma which is going clockwise so that the interior is actually unbounded in the usual plane but of course bounded in the extended plane with infinity as an interior point okay, so really the version of this theorem that you have to prove is only for the case when gamma is having a clockwise orientation okay and that is the version that is the part that I will prove. If it is anticlockwise if you taking gamma with anticlockwise orientation then it is usual residue theorem there is nothing to prove okay, so that is the reduction I am making obvious reduction, we only have to look at the case when gamma has clockwise orientation because this is the only case when infinity is inside gamma okay.

So let me write this here infinity is inside gamma so well let w 1 w k through w k be the isolated singularities of f inside minus gamma okay so you see gamma is taken in the clock wise sense, so minus gamma is taken in the anticlockwise sense the usual positive sense and will contain only a piece it will only contain a portion of the usual complex plane and it will have some singularities there, so the picture is something like this, so here is your so here is your gamma and mind you the orientation is clockwise okay and the reason for that is that the exterior of gamma well the interior of gamma really the interior of gamma is actually the exterior of gamma in the usual in the common sense.

So this is the interior of gamma what I have shaded is because it has clockwise orientation and it contains infinity in the extended plane if you are considering this in the extended… mind you if whatever I have shaded along with the boundary gamma if you consider it in the extended plane that is a compact set okay it is closed end bounded so because you are actually looking at its image on the Riemann's sphere okay which will be like a polarized cap alright so you should remember that because sometimes it is very difficult for people to think that this is bounded because on the plane it is unbounded but it is but topologically you should think of it as bounded okay so when you are thinking of Riemann sphere of the extended plane.

So well so here I have w 1… w k these are the fellows and mind you these are inside minus gamma okay minus gamma is well then have the usual anticlockwise sense orientation okay, positive orientation with respect to the usual plane and if you calculate the integral over gamma of f w if you calculate the integral over gamma of f w this is going to be the same as minus of the integral over minus gamma of f w by the very definition of the Riemann integral okay you change the orientation of the path the sign of the integral changes but if you calculate the integral of this over minus gamma you can apply usual residue Irom and you get that b equal to 2 pie i times sum of the residues of f at these w i or omega i from i equal to 1 to k okay that is what you will get okay.

So on the one hand you will get a minus 2 pie i times sum of the residues and w 1 through w k alright and by the residue theorem the other version of the residue theorem which we saw or the extended plane which says that the total sum of residues is 0 namely sum of the residues at all finite point is the plane does the residue at infinity that sum as 0. In this if you look at the if you look at that you will get that this integral is also equal to 2 pie i times are some of the residues of f outside gamma okay in the shaded region which is the diversion of residue theorem at we want, so that gives you the proof, so let me write that here so let me just use a different color, so let me write it here in the margin maybe I can remove some of this and make myself a little bit more space.

So integral over gamma fw dw is minus of integral over minus gamma fw dw and this is minus of 2 pie i times sum of the residues of f at w j, j equal to 1 to k this is usual residue theorem working alright and let us assume that of course you know I am looking at a function which has only isolated singularities in the extended planes so there are some more singularities and they are of course lying outside gamma. I of course as a told you I avoid the situation when gamma passes through one of the (())(37:26) singularities that is not allowed so there are these remaining w k plus 1, so w k plus 1 w k plus 2 and so on and there is a w let us say w m there are m finite points in the complex plane where f has isolated singularities and then there is a point at infinity.

So this is also equal to minus 2 pie i times mind you now I will get you see minus of summation j equal to k plus 1 to m the residue of f at w j plus well I will also get residue of f at infinity this is what I will get okay I will get this and of course this minus sign is common to both okay and the reason why I get this is because of the earlier version of the residue theorem which says a total sum of residues is 0 okay, so I get this now you see this minus inside the minus outside they cancel out and what you get is 2 pie i times sum of the residues of f inside, and mind you what are the residues of f inside gamma what are the points inside gamma? The points inside gamma are w k plus 1, w k plus 2 through w m and the point at infinity which you have to necessarily include as a singular point okay.

So it is correct so this is the proof of the statement I gave you, so the residue theorem works finely well even for any curve, any simple closed curve in the complex plane so long as you only thing is that the function should be should have only finite singularity, finitely many it should have only isolated singularities in the extended plane and your contour should be simple closed the function should not vanish it should not have any singularities at any point on the contour okay that is all you need okay so you can see there is a…so behaviour at infinity you can see is quite nice I mean you get also nice version of the residue theorem okay.

Now there is one more thing I want to tell you so let me reiterate one of the reasons well just in case this confuses, so one of the reasons the Clausius theorem fails for a function which is analytic at infinity because of this residue theorem okay because you get this in exchange which is good and of course the advantage of this theorem is that you can do some difficult computations okay. Some otherwise difficult computation be done using this okay is like the usual residue theorem allows you to include things compute integrals is extended version of the residue theorem will help you.

So for example I will give you I will give you philosophically an example suppose you have an analytic function okay which has isolated singularities and suppose there are lot of them okay but anyway I am only looking at chance which have only you know isolated singularities in the extended planes so there are only finitely many at the point is that this finite number may be huge suppose I am looking at an analytic function which has 1 million singularities okay 1 million isolated singularities and suppose there are 1 million poles at different points alright.

Now I can of course choose a huge circle which encloses all of them because they are anyway they are finite so there is going to be a sufficiently large circle where the which can enclosed one of them and what am I going to get if I integrate the function around that circle? Well the usual residue theorem will tell me that you can get the answer by you know taking a pie i times sum of residues at each of these million poles okay so that there are millions of them you have to compute 1 million residues and that is practically you can imagine how difficult this but then the extended version of the residue theorem says do not do all that, simply compute the residue at infinity and put minus sign that is it okay and multiply by 2 pie i okay, so in that way residue at infinity is very useful okay, so it is a very useful theorem okay it allows you to compute residues when there are huge number of singularities okay that is the advantage of this, so whenever we do something we should see some advantage in that, so in that sense that is the advantage of this okay.

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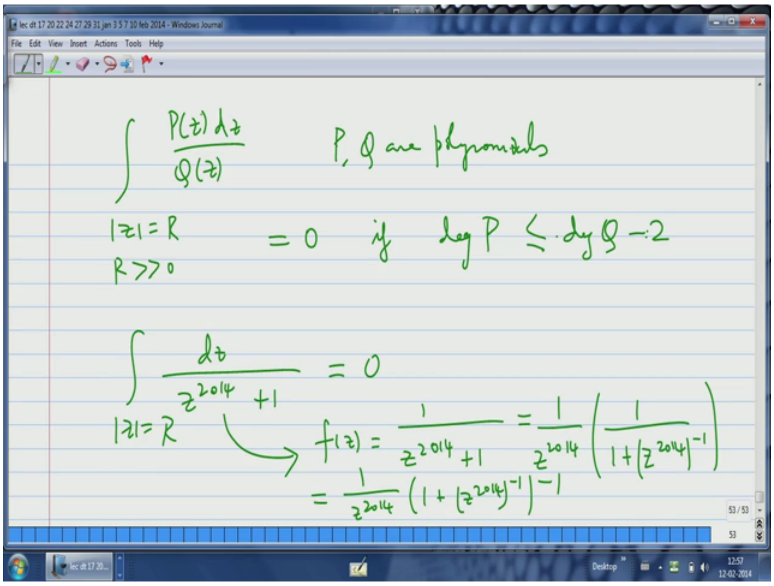

So for example you know well I will give you couple of illustrations see suppose I write integrals model Z is equal to R, R sufficiently large and if I ride P of Z dz by Q of Z where P and Q are polynomial okay so suppose I do this see you would have seen in the 1^{st} course in complex analysis that you know if the degree of P is is less than degree Q minus less than or equal to degree Q minus 2 okay that is the degree of the numerator is lesser than the degree of the denominator by least 2 powers of the variable okay and this integral is actually 0 okay so actually this is equal to 0 if degree of P is less than or equal to degree of Q minus 2 okay.

Now you see so it is the way you do this in the 1^{st} course in complex analysis there are 2 ways of doing it one way is well I mean the easiest way which is what people normally do is use the ML formula, we use the ML inequality which says that the integral of the modulus of an integral is less than or equal to the integral of the modulus and that is lesser or equal to the maximum value of the integrant M on the contour times L which is the length of the contour okay this is the ML inequality and what you do is that you do this ML inequality estimation okay and you know if it will it will give you immediately that this integral will go to 0 as you increase R if you make R of course R has to be large enough so that you do not allow any zeros of Q outside R okay they should all come inside alright.

So you make R big enough so that R includes all the zeros of Q and mind you zeros of Q are poles of the integrant okay and you make R sufficiently large to include all the poles of the integrant then you get a quantity which will go to 0 as R tends to infinity okay and sense in fact this quantity is independent of R because of Clausius theorem, it is independent of R because of Clausius theorem you can let R tent to infinity and you can (())(45:23) that this integral is 0 this is what people normally do okay and well now you know try to do this using the residue at infinity.

So you know integral over R integral over mod Z equal to R, R sufficiently large is going to give you 2 pie i times minus the residue at infinity okay and you try to calculate the residue at infinity for this meaning that you write this out in positive and negative powers of the variable Z if you want in this case and you look at the coefficient of 1 by Z and you will see that since the numerator degree is less than the at least 2 less than the denominator degree you will never get a 1 by Z term and what therefore it will tell you is that, that infinity okay at infinity you are not going to get 1 by Z term okay Z being the variable and therefore the residue is 0 and therefore the answer 0 okay so you can see this is 0 is like that okay, you can see this is 0 just like that and much harder thing is supposed degree of P is actually degree of Q minus 1 or if it is equal to degree of Q, how do you make these computations?

Okay your computations will be you will see that the computations are very easy if you really use residue at infinity, so residue at infinity is very useful to do these kind of calculations okay, so that is something that you should understand, so you know for example if I write integral over mod Z equal to R and I write d Z by Z to the 2014 plus 1 okay so this is a huge polynomial in the denominator all its zeros are simple zeros they are the 2014 roots of unit of minus 1 anyway they all lie on the unit circles, so I do not have to take R very large I just have to take R greater than 1 alright but the point is that if you now use the usual residue theorem and try to compute it, it is not all that easy okay you have to calculate the residue of this at each of those simple poles and there are 2014 of them and you will have to add all of them and then multiply by 2 pie i you would certainly not do that okay rather what you do is take minus 2 pie i times residue at infinity and you see that, that is 0 so you can easily see that this this is actually going to be 0 of course if you apply the previous criterion it is 0 but even you do not have to do that okay.

So let me illustrate what you would for example do with this case okay, so here the function is f of Z I have taken the variable as Z, so it is 1 by Z to the 2014 plus 1 okay and mind you I want to look at it at infinity okay which means I want the Laurent expansion at infinity and mind you that should be thought of as a Laurent expansion that is valid outside sufficiently large a circle of sufficiently large radius. Mind you this function has all simple poles which are zeros of the denominator and they all lie on the unit circle okay they all lie on the unit circle and therefore you know if you calculate the Laurent expansion about the origin you will get 2 Laurent expansions, there is one Laurent expansion that to be valid in the unit disk okay and it will actually turn out to be a Taylor expansion, the reason is because the function is actually analytic in the unit disk, in the unit disk there are no the denominator does not vanish right.

So if you write the 1st Laurent expansion centred at the at Z equal to 0 what you will get is it will be valid in the unit disk and it will actually be a Taylor expansion. Then you will get another Laurent expansion which is valid outside the unit disk okay and that is a Lauren expansion at infinity okay, so but if I go outside the unit disk what happens? Mod Z is greater than 1 so I should write an expansion in terms of for the situation and mod Z is greater than 1 and you know if I am trying to use geometric series I always look at the situation when the variable has modulus less than 1 so this tells you that I will have to write it in terms of 1 by (())() okay, so basically what I will do is I will take this Z to the 2014 out and then I will write it as one by one plus Z to the 2014 to the minus 1, so I write it like this and now you know.

So this is going to give me one by Z to the 2014 and here I am going to get one plus Z to the 2014 to the minus 1 whole to the minus 1, now if I expanded using a geometric series you will see that the coefficient of 1 by Z is 0 that will tell you the residue at infinity is 0 okay and you see immediately that this integral is 0 okay, so I am just trying to tell you that whenever you see problems go back to those problems that you did when you were when you took the 1st course in complex analysis try to apply residue at infinity and you see many of the problems are easier okay, so I will stop here.