Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

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Lecture No 11

Residue at Infinity and Introduction to the Residue Theorem for the Extended Complex Plane_ Residue Theorem for the Point at Infinity

Alright so what we are going to discuss now is about dealing with the residue at infinity which will make sense since we have been studying about a point at infinity okay and so you know let me tell you at the outset something that you have to remember which are distinguishes between the residue at infinity and residue at a finite point in the complex plane. The point is that the residue at infinity for a function which is analytic at infinity can be nonzero whereas the residue at a finite point in the complex plane there is a point in the usual complex plane or an analytic function the residue is 0 okay.

So this is a very important okay. Of course if the residue is 0 at a point it does not of course mean that the function is analytic but the fact is that if you have an analytic function at a point in the residue at that point is 0, so long as that point is a point of the usual complex plane but if it is a point at infinity okay it may be analytic at infinity but yet the residue may not be 0, so we are going to talk about residue at infinity and we are going to talk about you version of the residue theorem at on the on the extended complex plane okay.

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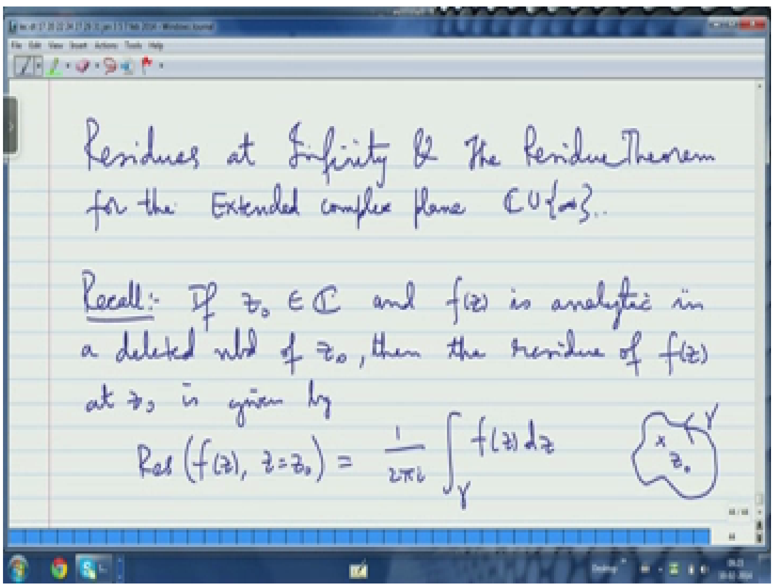

So let me write this down residue at infinity and the residue theorem or the extended complex plane, so this is what we are going to talk about, so you know let me begin by (())(3:17) what usual idea of residue is? Okay, so recall if Z naught is a point of the complex plane okay and f of Z is analytic in a deleted neighbourhood of Z naught okay, so when I say this I am saying that Z naught is actually a singular point okay, so it could be removable singularity which means that it is not really a singular point it could be analytic point but on the other hand it could be (())(4:01) singularity it may be a pole or an essential singularity and then the residue the residue of f at Z naught is given by well residue of f of Z at Z equal to Z naught is equal to well 1 by 2 pi i so, so you integrate you integrate the function fz dz around a simple closed contour you a simple closed contour which goes around the point the positive sense okay.

So basically so Z naught is this point and of course there is a deleted neighbourhood of this point sufficiently small deleted neighbourhood of this point so in particular there is a there is a disk of sufficiently small radius open disk surrounding that point where the function is analytic and then I am just taking a simple closed curve gamma okay of course this is simple closed contour, so simple means of course that it does not intersect itself. It has positive orientations which means that it is going anticlockwise around that point so in other words the interior of the contour contains the point okay.

So the point lies to the left of the contour as you walk along the contour and the region to the left of the contour as you walk along the contour in the direction specified by the contour this is called the interior of the contour okay and of course when I say contour you must remember that gamma has to be piece wise smooth, so gamma is whenever you parameterise gamma mind you gamma basically is continues image of an interval okay that is what the part this but it can always it can also be a piece wise continues image okay and well in fact we always take gamma to be a continuous path so but the thing is that it is also continuously differentiable at least piecewise okay that is the condition for a contour and this much is required or this integral to be defined as a Riemann integral alright.

So this is the residue and well I am wondering I am wondering if this that is right, so I was just wondering whether the fraction 1 by 2 pi i is correct, it is so for example if I plug-in 1 by Z minus Z naught if I take the function f of Z to be 1 by Z minus Z naught if I integrate 1 by Z minus Z naught over simple closed contour sufficiently small contour then I am going to get 2 pi i so the residue is one and well see of course it is very important that I choose this gamma in such a way that there are no other singular points of f inside other than Z naught and that is of course true because I am choosing gamma inside a deleted neighbourhood of Z naught where f is analytic so there are no other points where f is singular other than Z naught okay.

Now the point is…so this is the this is the definition of residue this is one definition of residue and as you can see the importance of this definition is that if you multiply out the 2 pi i on the right side if you multiply both sides by 2 pi i what you will get is 2 pi i times the residue is the integral of the function over gamma, so what it tells you is that it tells you how to integrate a function around a singularity okay that is the important thing, so the fact is that why is this definition interesting? It is interesting because you get to know what the integral of a function is around singularity at the important thing is that you should be able to compute left side without having to do the integral okay and that is the method of residue that you have learned in the 1st course and so there is.

So you also can recall he you know the usual definition of the other usual definition of residue in terms of Laurent expansion, so the point is that since f of Z is assumed to be analytic in a deleted neighbourhood of Z naught there is a Laurent expansion or f centred at Z naught then you write out this Laurent expansion it will contain both positive and 0 and negative powers of Z minus Z naught and the residue is precisely the coefficient of the of 1 by Z minus Z naught namely we Z minus Z naught to be minus 1 okay.

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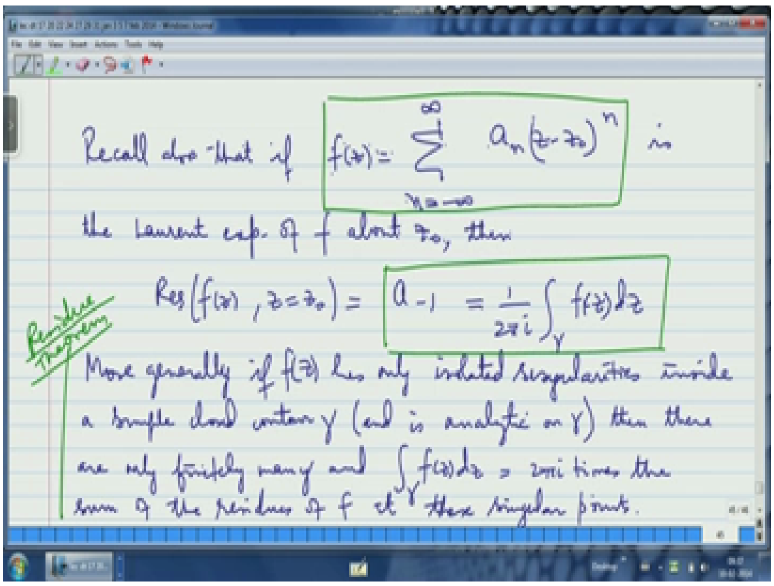

So that is another so let me write that down recall also that if f of Z is equal to Sigma n equal to minus infinity to infinity a n times a n times Z minus Z naught to the power of n is the

Laurent expansion of f around Z naught or f centred at Z naught or f about Z naught mind you that that exist because of Laurent's theorem. Laurent expansion exists because of Laurent's theorem and you know what is the big deal about Laurent's theorem, it is the it is the analog of Taylor's theorem, so Taylor's theorem tells you that a function is analytic at a point Z naught then you can expand it as a power series in Z minus Z naught namely you can expand it as a series which involves only 0 and positive powers of Z minus Z naught.

Laurent expansion tells you that you can do something and similarly something as good if even if Z naught was a bad point if Z naught was singular point and the only singular point in that neighbourhood in its neighbourhood then you can expand f and both positive and negative powers of Z minus Z naught okay, so the fact is that if you if you are dealing with a singular point you must allow negative powers of Z minus Z naught also in the series expansion and of course if Z naught is a removable singularity you know roughly I mean this is exactly what Riemann's removable singularity theorem is that if you write out the Laurent expansion you will only get a Taylor expansion in Z naught is actually removable singularity okay which is equivalent to saying that f is analytic at that point.

So anyway so the point is that if f has this Laurent expansion then then the residue of f of Z at Z equal to Z naught is none other than a minus 1 and so you know so if I write this down this is also equal to as I wrote earlier it is 1 by 2 pi i integral over gamma fz dz okay and so the advantage of this formula is that you know in many cases if I can write out the Laurent expansion at Z naught then I can explicitly and out what this a minus 1 is and that a minus 1 will give you that multiplied by 2 pile i is going to give me give you the integral of the function around that point okay so that is the advantage of this you know and you have used this is a 1st course in complex analysis to a (())(11:52) integral and so on and so forth, so it is a very useful thing.

Now the point is that you know and then of course what is the generalisation of this, the generalisation of this is that suppose you are going to integrate a function around a simple closed contour and assume that inside that region the function has only isolated singularities okay and you know they are going to be finitely many isolated singularities and then the formula reads the formula that you get is essentially the residue theorem says that if you integrate the function around a bunch of isolated singularities what you are going to get is 2 pi i times the sum of the residue of the function at each of those similar points okay and that is the residue theorem and what we have written down here is that the integral of the function is 2 pi i times the residue at Z naught because Z naught is the only singularity and the point is that this extends to 2 pi i times some of the residues at various points which are the isolated singularities of f inside the contour okay.

So that is the residue theorem essentially so the point is that you know in a way it is very easy the residue theorem is very easy if you look at it like this. It is rather not a theorem it is more part of the definition except that if you want to really prove it you will actually be applying Clausius theorem okay by surrounding each of the singular points by a sufficiently small disk and noting that outside this disk and inside your contour the function is actually analytic okay and you will have to apply the version of Clausius theorem or multiply connected regions and you have to use this definition of residue to get the residue theorem, so let me write that down more generally more generally if f of Z has only isolated singularities inside simple closed contour gamma and of course I am assuming that on the contour there are no singularities okay and it is analytic on gamma.

So this means that it is analytic on a small neighbourhood an open set which contains the contour gamma, so in particular I am avoiding the situation that there are singular points of the function on the contour okay and of course you know if there are similar points of the function on the contour then you know at those similar points the function will fail to continuous if they are on a single points and once the function fails to be continuous it is very difficult to define the Riemann integral of the function over the contour, so you know you must understand that the moment you define the Riemann integral more or less you are assuming that the function has nothing wrong going on the contour is only what happens inside that matters okay.

So well then there are there are only finitely many and integral over gamma f of Z dz is equal to 2 pi i times the sum of the sum of the residues of a residues of f at the single points. So this is essentially the residue theorem okay and of course that there are only finitely many follows from the fact that if you take the if you take the contour along with the…of course whenever I say contour mind you it is always an oriented contour and without if nothing is mentioned always the contour is oriented and it is given the positive orientation which means that you go around the point in the anti-clock wise sense okay and that is always there is always taken for granted unless something else is mentioned, so whenever integral over contour is mentioned you must understand that the contours is already oriented and the orientation is positive okay.

So well so this is essentially the residue theorem. So let me write that down so this is this is just the residue theorem and well as I told you it is very useful to evaluate integrals which integrals of function around contour which have only isolated singularities inside the contour okay so this is it. Now the point is that…what we want to do is that? We want to do this for the point at infinity and we want the residue theorem for a domain the extended complex plane so how we are going to do it, so the idea is very simple we start off by using this by adapting a little using the same philosophy namely the residue should be you know you integrate the function okay around the point okay and then divide by 2 pi i okay and then that should give you the residue, so if you want to get the residue of a function at the point at infinity 1st of all it should be analytic in a deleted neighbourhood of the point at infinity okay and then I must take a contour that goes around infinity in the positive sense okay and I have to integrate the function around that contour okay and divide by 2 pi i and I must get the residue at infinity, okay.

So this is so the definition for residue remains the same namely you just integrate the function around by going over a contour that goes around the point in the positive sense as far as the point is concern and then you divide by 2 pi i okay that is the definition alright and let us see what that that brings up, so I incidentally I wanted to point out something here which I just remembered, so let me tell you so for a moment in a way once you once you believe Laurent's theorem.

Once you believe Laurent's theorem that this is exactly I mean that this formula is correct or something that you can more or less see because you know so you know I have this expression I have this expression of the Laurent series okay and what I need to compute is that I need to compute integral of f over gamma okay and that is the integral that is the same as integral over gamma of the series on the right side and now the point is that I can push the integral inside some because the fact is that the Laurent series like a Taylor series you know wherever it converges, it converges normally that means it converges uniformly on compact subset and since I am going to integrate over gamma, gamma is of course mind you any simple closed contour is a compact set because it is both closed end bounded.

So therefore the integral of the sum is same as some of the integrals so I can push the integrals across the some and when I push the integral across the sum what is going to happen is that you know you will see that if I take positive powers of Z minus Z naught the integrals are going to vanish because the positive power of Z minus Z naught are analytic functions then Clausius theorem is going to tell you that whenever you whenever you integrate analytic function around a simple closed contour you are going to get 0 in fact the positive parts are all entire functions okay they are just polynomials and the same thing is going to happen the constant term okay and then if you take the if you take the negative powers of Z minus Z naught greater than 2 okay.

So I mean what I mean by that is if you take terms involving 1 by Z minus Z naught the whole square, 1 by Z minus Z naught the whole cube and so on the integrals of those sums will also vanish that is because they all have anti-derivatives, so you know it is something very basic that I want you to understand, the fact that an integral vanishes is you know basically it is equivalent to saying that the integral is independent of the path and especially when the integral when the integral has an anti-derivatives then you know that the integral is actually the final value minus the initial value okay of the anti-derivative and if the final value is equal to the initial value is what will happen if you go around a closed path you are going to get the integral be 0 okay and so you know all the integrals which involves Z minus Z naught is the power of n where n is minus 2, minus 3, minus 4 and so on they are all going to vanish.

So the only thing that is going to survive is going to be integral over gamma a minus 1 Z minus Z naught power minus one dz okay and you know the integral over gamma Z minus Z naught to the minus 1 dz is just going to be 2 pi i and that is just because of Clausius theorem because the integral over gamma is not going to really depend on the shape of gamma you can replace gamma by a small circle okay going around Z naught and then you actually parameterise a circle as usual a write it as Z equal to (())(21:47) to the i Theta, Theta varying from 0 to 2 pi and compute the integral you will get 2 pi i okay so therefore what I want to tell you is that once you know Laurent's theorem okay once you believe Laurent's theorem is formula for the residue that a minus 1 gives you the residue is more or less direct okay, so that is something that I want you to recall.

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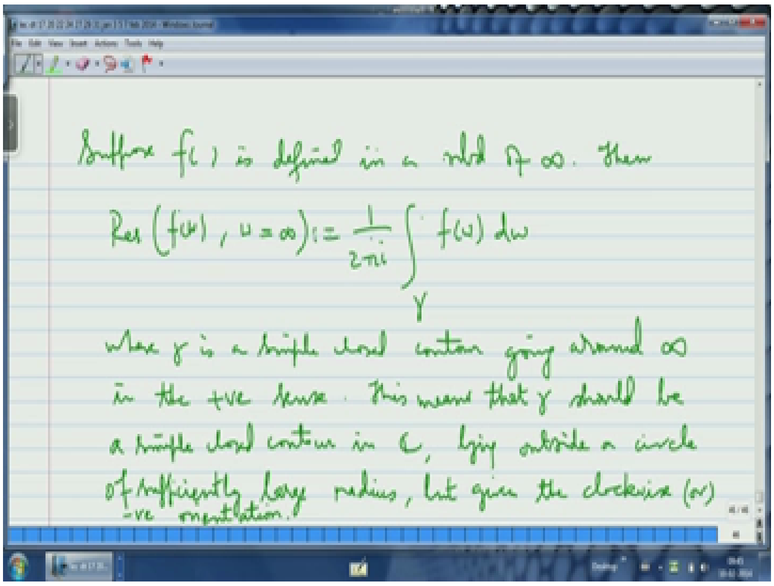

Now let me go back and define the residue at infinity, so suppose f of Z is defined in a neighbourhood of infinity okay so mind you that what this means is that f of Z is defined on a circle on the exterior of a circle of sufficiently large radius that is what it means. Neighbourhood of infinity is by definition you know by the stereographic projection the same as the exterior of a sufficiently large circle okay and so at this point what we do is that, we define the residue of f at infinity will be just the integral over gamma 1 by 2 pi i integral over gamma fz dz where gamma is simple closed contour that goes around infinity in the positive sense okay this is exactly the way we define it as we defined it for the point in the usual complex plane, so let me write that down.

Then residue of f of Z at Z equal to infinity is defined to be so I put a colon and equal to 1 by 2 pi i integral over gamma so you know let me as let me change notification to f w because I will need to appeal to something I need to appeal to change in the variable to 1 by Z, so let me do that, so f w, w equal to infinity 1 by fw dw okay 1 by 2 pi i integral over gamma, so let me write that is very important where gamma is a simple closed contour going around a point at infinity in the positive sense, so you know this is the this is definition and well you know so this is exactly the definition that you would have made for a point in the complex plane and this is the same definition I am using for a point for the point at infinity but then there are 2 or 3 things that one has to be careful about the 1st thing is that what do you mean by contour that is going around infinity the positive sense.

So the fact is that you know you must think of a contour in so here is where you again appealed to the stereographic projection okay. You know that sufficiently small neighbourhood of infinity is given by the exterior of a sufficiently large circle centred at the origin okay, so in some sense a sufficiently large circle centred at the origin should be a contour which goes around the point at infinity okay and you can you can see this you can see this so more generally if you take a sufficiently large contour which goes around the origin okay simple close contour which goes around the origin sufficiently large means it encloses a sufficiently large area okay.

Then for that matter I mean it lies in the exterior of I should rather say it lies in the exterior of a circle of sufficiently large radius okay then that itself is an example of a simple closed contour that goes around the point at infinity and that is because of the that is because of the stereographic projection and the only thing is that what is this, what is this business of the positive orientation of that contour with respect to infinity mind you the positive orientation means at infinity should lie in the interior of that contour okay so you should orient the contour in such a way such that the interior of the contour contains a point at infinity. Now if I take a circle sufficiently large circle centred at the origin and oriented in the usual way added which we do namely give it the anti-clock wise orientation in the origin becomes comes into the interior, the interior is just the interior of that circle okay and the exterior will be the exterior of the circle and that will contain the point at infinity, so you can see from this argument that will have 2 orient it clockwise okay.

So the gamma must be a simple closed contour lying in a sufficiently small neighbourhood of infinity, so it should be line in the exterior of a circle of a sufficiently large areas and it should be given the clockwise orientation that is what it means okay. So let me write that down this means that gamma should be a simple closed contour the complex plane lying outside a circle of sufficiently large radius but given the clockwise or negative orientation. So here you see you have to be careful I am saying that gamma should have negative orientation here but in the statement before that I am saying it should have positive sense, so mind you in the statement preceding it was positive sense with respect to the point at infinity and positive sense with respect to point at infinity is negative sense with respect to the origin okay. So these are not contradictory statements you have to understand the subtlety and well of course you can also see this pictorially more or less.

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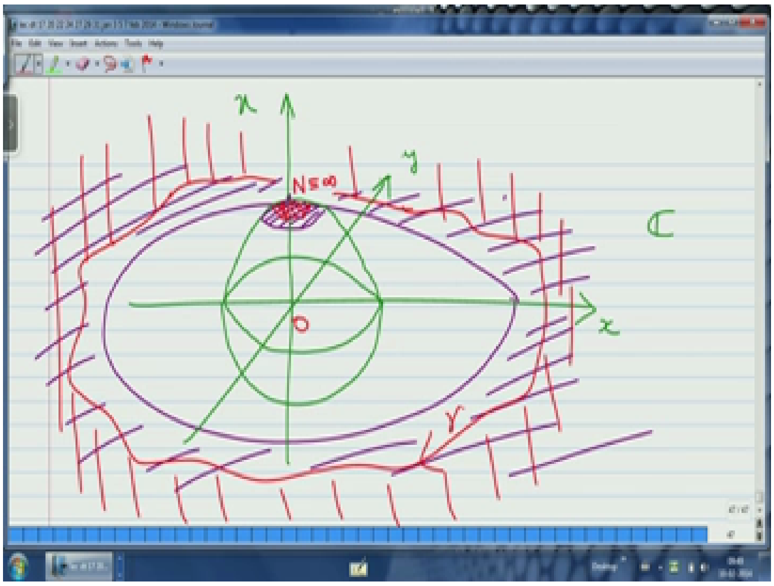

See you know if you draw the stereographic projection, so here is the x y plane which is a complex plane Z plane and then and then you know we draw this 3rd axis which we call it as u and then you take the unit sphere, surface of the unit sphere in 3 space I am going to get something like this this is the stereographic projection so you know if I take now you know if I take sufficiently if I take a circle in the complex plane of sufficiently large radius okay and then you know well you know the point at infinity corresponds to do this point here on the on the Riemann sphere which corresponds to the North pole so this is the this corresponds to point at infinity okay.

So I put this triple line to tell you that it corresponds to the point at infinity under this geographic projection and well what happens to the well you know the exterior of the circle the exterior of the circle in the complex plane and that is going to correspond to well on the Riemann sphere is going to correspond to the small disk like region though it is a curved surface it is a small disk like region, when I say disk like I mean topologically you can flatten it to look like a disk topologically and it is a disk like neighbourhood of the North pole on the Riemann sphere okay and that is how the shaded region on the plane namely the exterior of the circle and the small cap on surrounding the arrest like (())(30:41) cap okay.

If you imagine the Earth it is North at the North pole so now the point is that you see what you must understand is that if I now give this so in particular you know if I now take a sufficiently if I take a contour which lies in outside the circle okay if I take a contour like this. Now that contour is going to correspond to a contour here on the Riemann sphere that goes around the point at infinity okay and as I make this circle bigger and bigger that is going to give a smaller and smaller contour simple close contour that goes around the point at infinity the only thing that you have to worry about is the orientation, the orientation the way I have drawn it the orientation should be like this mind you it is the it is actually it is actually clockwise about the origin and the reason for that is that if you define it like that which is what you should do if you are dealing with the point at infinity in the interior of that contour is the exterior.

So this region this thing outside this is the interior of that contour okay and well and it is actually the region that lies to the left of the contour as you walk along the contour okay and you can see that this region corresponds to well this region on the on the Riemann sphere and that of course contains the point at infinity, so the point at infinity is an interior point for the contour oriented in the clockwise direction okay, so you can see this diagrammatically, fine. So very well now that we have defined what the residue at infinity is so of course this is gamma right so let us compute it.

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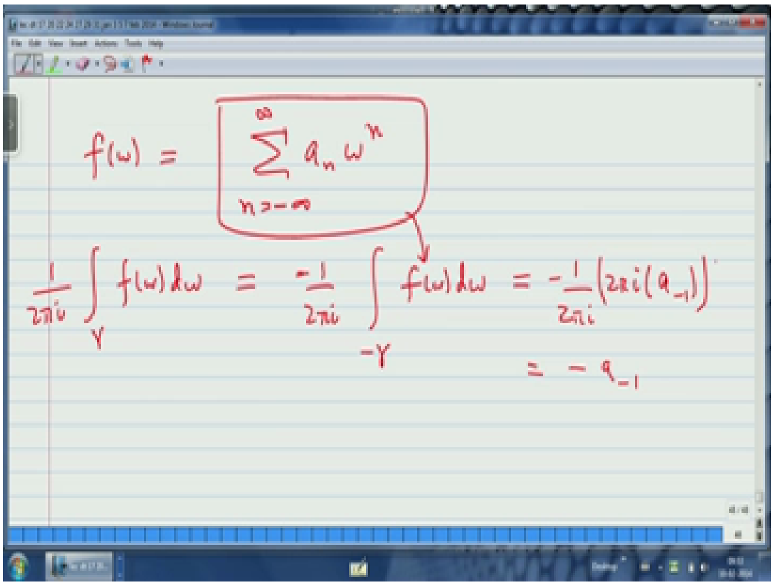

Let us compute what the residue at infinity is, so f of Z is well f of w let me write f of W, f of w is going to be Sigma n equal to minus infinity to infinity a n w to the n which is the Laurent series of f this is the Laurent series of f at in a neighbourhood of infinity okay and mind you in principle it is actually also the Laurent series of f at the origin in some sense centred at the origin but in a domain that is a neighbourhood of infinity and the only thing that you distinguish is that when you say it is the Laurent series at infinity you see the singular part is the one that involves the positive powers of w and the analytic part is the one that involves 0 and negative powers of w because it is the negative powers of w that behave well at infinity, okay.

So that is the only difference but what you are actually looking at is actually the Laurent series of f at the origin okay. Now but anyway the fact for the same reason that I told you earlier if you compute if you if you now compute integral over gamma fw dw if I do this what I am going to get is and you know of course I think it is 1 by 2 pie i this is what it is, well you know mind you if I if I compute this integral the way I normally would compute the integral on the plane, what I first do is I always compute integrals with my contour being positively oriented with respect to the usual plane that is with respect to the origin.

So what I will do is I will 1^{st} write this as minus 1 by 2 pie i integral over minus gamma where minus gamma is the gamma oriented in the anticlockwise sense and that is the positive sense for the plane okay with respect to the origin if you want okay and then I will get this and you know now if I plug in the series here okay and member that I can do integration term by term because of the same reasons I told you earlier because the Laurent series always converges normally it converges uniformly and compact sets and gamma or minus gamma for that matter they are as sets their compact sets so if I do that again what is going to happen is that, what is going to survive is only the coefficient of 1 by w okay and that is going to give me that the coefficient when n minus 1 so I am going to get a minus 1 the only thing is that I am going to get I am going to get a minus a minus 1 okay.

So what you must see is that I mean what you will see is that I will get 1 by 2 pie i times 2 pie i times a minus 1 this is as before and I get minus a minus 1 okay so the moral of the story is that when you are looking at the residue at infinity okay what you do is you literally get minus of a minus 1 okay which is with an extra minus sign added to it okay whereas if you look at the usual Laurent series of a point around a point in the complex plane then the residue is actually a minus 1 which is just the coefficient of the 1^{st} negative power of the variable okay whereas in this case it is the minus of that and so you know you can now believe that you can see right from here you can see something happening suppose this suppose this function f had only singularities at the origin and at infinity okay which means that this Laurent series has in finite radius of convergence okay.

So by that I mean the Laurent series is valid for all w naught 0 and not infinity so it is valid on C star C minus the origin okay. If that is the case then you see this function if you calculate the residue at the origin you are going to get plus a minus 1 okay which is usual definition of residue if I if I take this function suppose it is also analytic in the neighbourhood of the origin and suppose the only singularities are at the origin and at infinity okay then you see you notice that the residue of the function and the origin is plus a minus 1 and the residue of the function at infinity is minus a minus 1, what is the sum of residues? It is 0 and that is exactly what the residue theorem is says for the for any function which has only isolated singularities in the extended planes.

So you know so the statement is that you take a function which has only isolated singularities in the extended planes okay which means that know there are only mind you it means whenever you say isolated singularities in the extended plane there are to be only finitely many that is because the extended plane is compact and any subset of a compact isolated set subset of a compact set is finite in this case okay. So therefore you know what the residue theorem will say is that if you take the residues at all the points at the points at the finite complex plane and you take the residue at infinity and you add them up you will get 0 and here that is exactly what happens if f where having a residue if f was having a singularity at the origin only and the singularity at infinity okay then you see that from this computation the residue at 0 is plus a minus 1 the residue at infinity is minus a minus 1 the sum is 0 okay.

So anyway so the moral of the story is that you know it gives you a very easy way of computing residue at infinity it is very simple, what you do is that you simply write the Laurent expansion of the function of the origin but) so that it is valid in the exterior of a sufficiently large circle okay I knew you must always remember this that so this is probably maybe I should spend a few minutes on this. See in the 1st course in complex analysis when you study about Laurent expansion at a point you must remember that there are several Laurent expansion there could be several Laurent expansion at a point that is because the Laurent expansion, the domains of the Laurent expansions are actually annuli which are whose a boundary contains the singular points okay.

So if I say if I talk about the Laurent expansion of function at a point okay it could be even analytic point does not matter what the problem is that because there are singularities there are other singularities the Laurent expansions will be different expansions course you will get different annuli okay and therefore when you write the Laurent expansion at the origin you should not look at the Laurent expansion at the origin at may be valid in a deleted neighbourhood of the origin which at host boundary there is a finite singularity we should not look at that, you should rather look at a Laurent expansion that is valid outside circle of a sufficiently large radius okay write that Laurent expansion that is the Laurent expansion at you need to work with infinity okay and in that Laurent expansion okay at again look at the coefficient of 1 by the variable and take it to the minus sign that is residue at infinity okay and address how you can very easily write out you can compute what the residue at infinity is an that combined with the residue theorem is another powerful tool or computing lot of integrals as we will see in the next lecture.