Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky
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Lecture No 10
Morera's Theorem at Infinity, Infinity as a Pole and Behaviour at Infinity of Rational and Meromorphic Functions

Yeah so let us continue with our discussion which is about trying to see what kind of results you can expect or a function which is analytic at infinity, so I told you last class I mean the last lecture that you know if a function is analytic at infinity then you cannot expect Clausius theorem to work okay and the easiest demonstration of that is taking the function one by Z which is analytic at infinity but if you integrate it over a simple closed curve in a neighbourhood of infinity you are going to get there is a possibility that you do not get 0 okay, so Clausius theorem will fail could fail but what works is and before that we saw that you know if you have a function which is analytic at a point in the plane then you know that all its derivatives are also all the derivatives exist, derivatives of all order exist and they are also analytic at that point that anyway is true for a function which is analytic at infinity.

If a function is analytic at infinity then all its derivatives are also analytic at infinity and we in fact those derivatives at infinity will be 0 and the crucial point to note is that you do not define actually the derivative at infinity, it does not make sense but you deduce that derivatives are all 0 at infinity okay. Then the last thing that I said was about Morera's theorem okay and that Morera's theorem actually works okay even for a function which is analytic in a domain in the extended complex plane okay.

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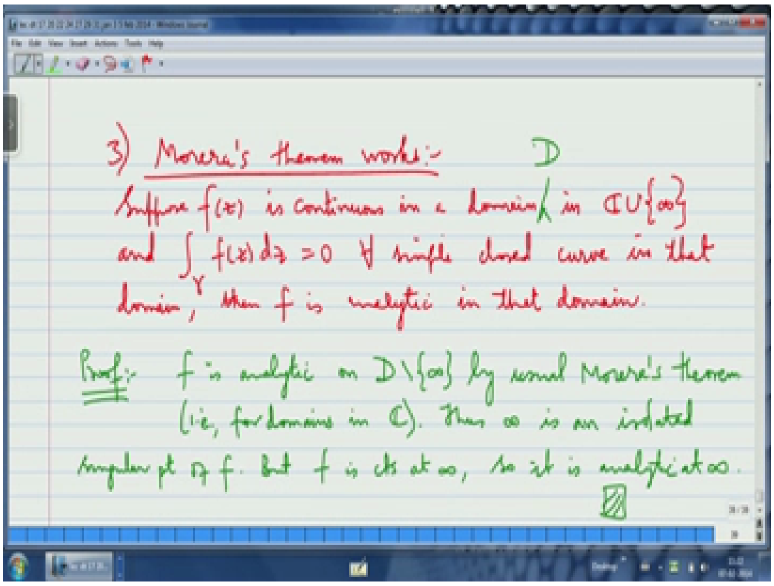

So let me explain that so suppose f of Z is analytic in a domain in C union infinity is which is extended complex plane okay and integral over gamma f of Z dz is 0 for every simple closed curve in the domain okay. So then I should not say analytic it should be the conclusion suppose f of Z is continues, so let me remind you that the whole point about Morera's theorem is that you are trying to get a converse of Clausius theorem okay and the Clausius theorem is that the integral over closed curve is 0 okay integral over simple closed curve is 0 and you want to say that if and that is true for an analytic function.

For a function which is analytic on and inside simple closed curve the integral over that curve is 0 but you want the corners that is you want to say that if a function, if the integral over every curve is 0 every simple closed curve is 0 in the function is analytic then you will have to add this condition of continuity because if you do not add the condition of continuity and then you can get into trouble okay, so that is also the same statement in case of a domain including the point at infinity suppose f is continuous in a domain including the point at infinity and the integral over any simple closed curve is 0 then f is analytic in the domain that is the statement but the point is I want to stress that when you say integral over a closed curve and you are saying for a curve in the domain which contains the point at infinity you must understand that this is not a curve that will pass through infinity okay.

So when I say when you say curve in a domain okay it could in principle passed through any point of the domain but when I say curve in a domain in the extended complex plane, the domain the extended complex plane could contain the point at infinity and in the case when it contains the point at infinity when I say a curve in the domain it is necessarily means that I am not thinking of a curve passing through the point at infinity, it does not make sense okay, so whenever we talk about simple closed curve whether it is in the complex plane or in the extended complex plane a simple closed curve is always a simple closed curve in the complex plane okay even if even if you are looking at a domain in the extended complex plane, it includes the point at infinity.

A simple closed curve always means just a simple closed curve in the usual complex plane, the point at infinity is not involved okay, so here is the conclusion, the conclusion is that if f of Z is continuous in a domain in the extended complex plane and integral over every simple closed curve in the domain of f is 0 then f is analytic in the domain, so this is Morera's theorem and the fact is that and the proof is pretty easy as I was saying by words last time, the proof is the proof uses the usual Morera's theorem and then and it uses the definition of being analytic at infinity okay.

So let me write this f is analytic on the domain so you know so let me call this domain as let me call this domain as the, so f is analytic on D minus the point at infinity okay throw away the point at infinity f is analytic on that by the usual Morera's theorem. Of course while usual Morera's theorem I mean the version of Morera's theorem that you study on the complex plane okay it is analytic because it is continues even if you throw if a function is continuous on a domain then it is also continuous on every subset of that domain, so if a function is continuous on the domain D then if you remove something from D it continues to be continuous okay.

The restriction of a continuous question is always a continuous function, so the function continues to be analytic in the domain with the point at infinity thrown out okay and that is the domain in the usual complex and integral over every closed curve is 0 that is that is one of the conditions of Morera's theorem, so usual Morera's theorem by that I mean the version of Morera's theorem for the usual complex plane that works okay and f becomes analytic and because f becomes analytic on D minus infinity what it tells you is that infinity is a singular point okay, it tells you that infinity is a singular point because D minus infinity is a neighbourhood of infinity okay and of course you know in all this am assuming that infinity is in the domain. If infinity is not in the domain is nothing to prove because it is a usual Morera's theorem which we assume okay.

So I am assuming at infinity is in the domain and if infinity is in the domain, then infinity as an interior point and if I say that outside infinity function is analytic that means the function is analytic in a neighbourhood of infinity. The domain itself any domain which contains infinity is a neighbourhood of infinity and f is analytic on that neighbourhood of infinity in the deleted neighbourhood of infinity but there is also this this extra assumption that f is continuous at infinity because f is continuous on the domain, it is continuous at every point in the domain so infinity is also there in the domain.

So f is continuous at infinity but then you know we cheated by saying that continuity at infinity is the same as analyticity at infinity and this is kind of inspiration we drew from the removable singularities theorem in the usual complex plane, so what you get is that 1st you get that if you throw the point at infinity out the function is analytic okay and that will tell you that the point at infinity is a singular point is an isolated singular point and then continuity at infinity will tell you that it is also analytic at infinity and therefore it is analytic at all points in the domain and your new version of Morera's theorem works okay. So let me write this down by usual by usual Morera's theorem I mean that is for domains in the complex plane because D minus infinity is the domain in the complex plane thus infinity is an isolated singular point of f but f is continuous f is continuous at infinity.

So it is analytic at infinity. So that is the end of the proof and you see what you see is that you know it is just being one is just being very clever when I say infinity is an isolated singular point and then when I say f is continuous at infinity you know that means I am actually saying that infinity is a removable singularity and infinity being a removable singularity was are clever way in which we define f to be analytic at infinity okay and the installation was taken in the removable singularity theorem, Riemann's removable singularity theorem because you know the straight the straight forward way of trying to say that the function is analytic at a point if it is differentiable at that point and also in a neighbourhood of the point will not work for the point at infinity because you cannot define the derivatives at infinity at the point for the point at infinity that is the problem, okay. Fine, so Morera's theorem works.

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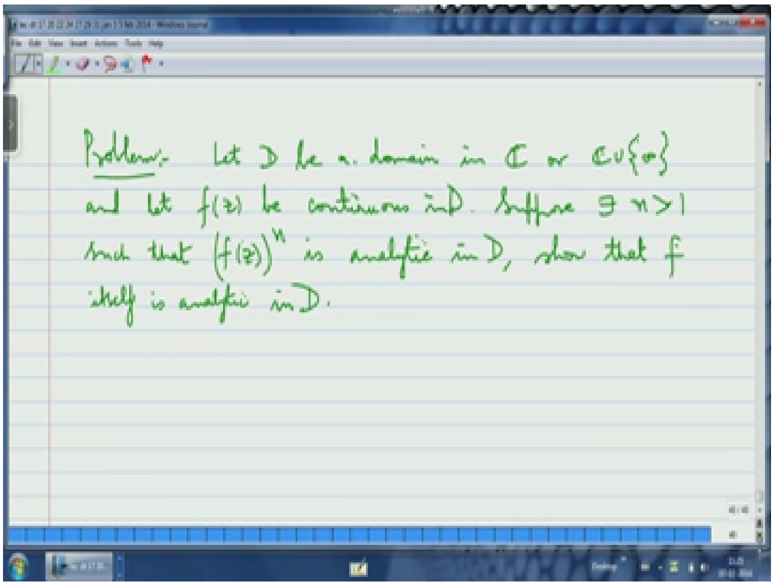

Now what I want to tell you about is about a problem that I want you to try so here is the problem for you to try let D be a domain in C or C union infinity and let f of Z be analytic in the let f of Z be continuous, let f of Z be continuous in D. Suppose there exist an n greater than 1 such that goes when I say n greater than 1 it is an integer n greater than 1, n is not a real number okay greater than 1 suppose there is an integer n greater than 1 is that f of Z to the power of n is analytic in D show that f itself is analytic in D okay.

So this is a problem which I want you to try so the idea is very simple you have a domain of course nonempty connected open set the domain you can take the domain is in the complex plane or you can include the point at infinity if you want. First may be you first try it for a domain in the complex plane and then we will see that the same arguments will work for a point at infinity. Suppose f is continuous in D and there exist an integer power of f greater than 1 which is analytic then f itself has to be analytic okay, so you can try this now what I am going to do as well we have now more or less seen something about the infinity being a point of analyticity or which is same as saying infinity is a removable singularity then next kind of singularities that of a pole okay, so let us go to the study of infinity as a pole okay.

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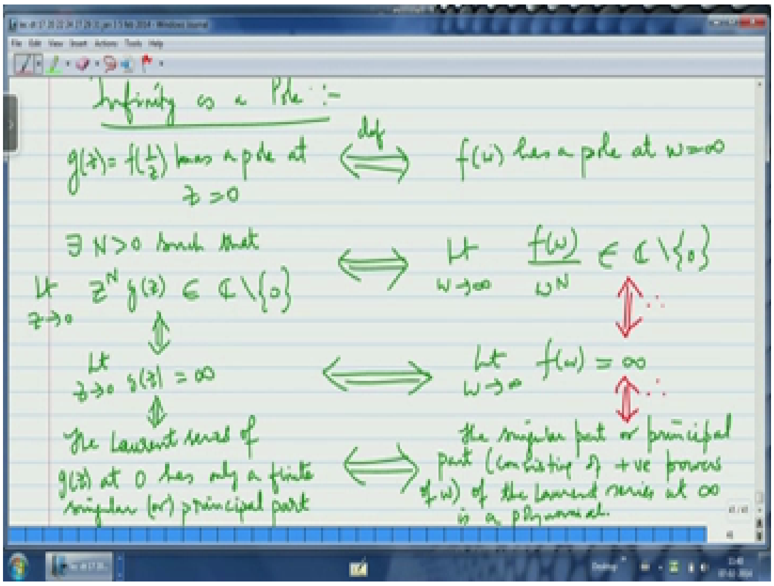

So let me go to the next page, infinity as a pole so here is so here is a situation you have a function f which has infinity as an isolated singular point okay that means it is defined in a deleted neighbourhood of infinity which you should think of as a the exterior of a sufficiently large circle on the complex plane okay and the question is that, when is infinity a pole? Okay so how do you define this? So the of course there are the way we do it is that we call the independent variable as W, we write the function as f of W and then we say the behaviour of f of W at W equal to infinity is the same as behaviour of f of 1 by Z at Z equal to 0 because you are plugging in W equal to 1 by Z okay.

So this is a philosophy which we have justified earlier because W equal to 1 by Z is a is a homeomorphism of the extended complex plane onto the extended complex plane which is an analytic isomorphism of the punctured plane onto itself okay. So and under an analytic isomorphism the type of singularity must coincide okay, so let me do that, so how do you define f has f of W as pole at W equal to infinity, this is the definition, this is I put def definition if and only if g of Z which is f of 1 by Z has a pole at Z equal to 0, so this is the this is the way we do it okay but what are these conditions, so you know so what is the condition that a function g as a pole at a finite point of the complex in there are 3 equivalent conditions we know one is based on the limit of the function as your (())(16:24) the limit should tent to infinity that is one way of looking certifying that something is a pole.

The other way is to look at the Laurent series at that point and then see that there are only finitely many terms in the singular part and the 3rd one is to say that you know a pole has to have a certain order okay, a pole has to have a certain positive integer order and that positive integer is actually the power of the variable minus the Centre which you have to multiply with the function to neutralise the pole because it is like if a function has pole of order n at Z naught okay then multiplying the function by Z minus Z naught power n should neutralise the pole okay, so you neutralise the pole by multiplying by the by killing the 0 of the denominator okay.

So I will write down all those 3 conditions, so the 1st thing is there exist an N greater than 0 such that you know Z power N g of Z limit Z tends to 0 is equal to this nonzero okay is nonzero and of course in fact it is a nonzero complex number is not…when I say it is nonzero it cannot be I do not want this to be infinity so this should belong to the punctured complex plane okay, so this is 1 definition of a pole of order N. g this one definition of g having a pole of order N at Z equal to 0, had it been a pole of order N at Z equal to Z naught you know you will have to modify it limit as Z tends to Z naught, Z minus Z naught to the power of N times g of Z is a nonzero complex number okay, so this is one definition okay.

Then the other definition is of course it is a definition that has got nothing to do with order of the pole is the fact that the function approaches infinity as you approach the pole which means that the function in modulus approach is infinity as you approach the poll, so that 2nd condition is limit Z tends to 0 g of Z is infinity okay and mind you should think about this in terms of know what in terms of a limit being in finite that we have defined earlier okay, so the way they should be made sense of his that as Z comes closer and closer to 0 that is if Z is restricted to a small deleted neighbourhood of 0 in the values of g as Z will lie in a small deleted neighbourhood of the point at infinity which is supposed to be the exterior of a sufficiently large circle in the complex plane, okay.

So that is what you should think of this and well the third condition is that the principal part or the singular part of the Laurent series of g is at the origin has only finitely many terms of okay and the highest negative power is of course going to be you know the subscript is going to be this n which is the order of the pole okay or rather minus of that subscript okay so or rather let me say this you take minus of the highest negative power of Z that should be N, so the Laurent series of g of Z and 0 has only a finite singular or principal part and what you must remember is that this is the same as saying that it has only finitely many terms involving negative powers of Z okay.

Now a fact is that each of these definitions have their analogues for f at infinity okay having a pole at infinity. Well you know I just want to point out few subtleties, so you know for example the one in the middle goes through very easily limit Z tends to 0 g of Z is infinity is the same as saying limit W tends to infinity f of W equal to infinity. This is this is direct because basically Z is 1 by W and Z tends to 0 is equivalent to W going to infinity and the limit going to so I just make a change of variable from Z to W by putting W equal to 1 by Z for Z equal to 1 by W okay so these 2 are one and the same is obvious okay and especially again you are using the fact that W is equal to 1 by Z which is Z going to 1 by Z is a homeomorphism okay that is there that is implicitly used here okay essentially because basically under continuous map image of a convergence equal to convergence (())(22:04) okay.

So this is very clear so if you want test of whether a function is having a pole at infinity you just have to see whether it goes to infinity as you approach the point at infinity okay so and what does it mean in terms of the topology of the point at infinity it means that if mod Z is sufficiently large then mod I mean if mod W is sufficiently large then mod f W is also sufficiently large, you can make mod f W as large as you want provided you choose mod W as large as you want that is just a verbal (())(22:41) statement of saying that f tends to infinity as W tends to infinity okay.

So the one and the Centre is very pretty easy the one in the top needs a little bit of thought you see you see if you want to neutralise this pole of order n of g at 0 mind you pole of order n should be thought of as a 0 of order n of the denominator or of the reciprocal of the function, so if you want to neutralise a pole you should multiply a power by a power of the variable okay or the power of variable minus the Centre, the point which is a pole okay but to do at infinity you can imagine you have to divide.

If you want to neutralise a pole at infinity okay you will have to divide, so that you can see by simply substituting Z equal to 1 by W in this so this is equivalent to limit W tends to infinity f of W by W to the power of n is a nonzero complex number okay so this is so this is the point that you have to notice to neutralise a pole at a finite point in the complex plane, a finite point Z naught in the complex plane you have to multiply by a sufficiently high-power of Z minus Z naught okay and in fact it is a particular power of Z minus Z naught okay and that is the order of the pole okay at Z naught but if you want to neutralise the pole at infinity you have to divide by the variable, by the correct power of the variable okay, so for example the simplest case is take the identity function f of W equal to W okay that has a pole at infinity because f of W equal to W if I take limit W tends to infinity I am going to get infinity, limit W tends to infinity of W is just infinity okay.

So infinity is a pole and it is the pole of order 1 because I can neutralise it by dividing by W you take the function f of W equal to W divided by W I will get one, now that has constant function which even at infinity remains one okay, so you can see that in this way you can see that if you take a polynomial of degree n, if you take a polynomial of degree n in W polynomial of positive degree in W that has a pole at infinity and the order of the pole will be simply the order of the decay of the polynomial okay, so if you have polynomial of degree n in W if you divided by W power and then take limit as W tends to infinity you will see that you will and with you will end up with a nonzero complex number okay.

So polynomial of degree n has a pole of order n at infinity okay that is something that you can see and coming to this last condition on the left which is that the Laurent series of g at 0 has only a finite singular or principal part, what will that translate to? It will translate to the fact that the singular part of the Laurent series of f at infinity okay will have only finitely many terms okay but mind you the singular part of a function at infinity is supposed to be the part at consist of positive powers of variable okay see you must remember that the singular part and the analytic part a will interchange if you move from 0 to infinity okay, so at 0 it is the positive power of the variable that behave well and the negative powers of the variable do not behave well.

At infinity it is the other way round, at infinity it is the negative powers of the variable that behave well and they form the analytic part along with the constant term okay and the positive powers of the variable misbehaves and they form the singular part, so saying that the right way to say the function has a pole at infinity is to say that if you take the Laurent expansion at infinity then there are only finitely many positive powers of the variable okay and that is equivalent to saying that the singular part at infinity or the principal part at infinity is a polynomial okay, so let me write that down the singular part or principal part consisting of positive powers of W of the Laurent series at infinity is a polynomial. Of course of course you know the constant term is not included, the constant term is taken to be part of the it is part of the analytic part.

So the constant term which is the 0 power of the variable okay equation of the 0 power of the variable and then and the other terms which involves a negative powers of the variable okay. Powers of 1 by W this form the analytic part of infinity okay and the positive power of the variables okay they form the singular part at infinity and that has to have only finitely many terms which means that it has to be a polynomial okay. A Polynomial in the constant term 0 and of course the degree of that polynomial will be the degree of the pole okay.

So because of all this what you can say is that therefore these 3 are you know these 3 are equivalent so let me put that here this is equivalent to this and this is equivalent to this and I will put therefore the point is at somehow you may have to go to the behaviour at 0 and mind you even at that these 3 conditions for a pole in the finite complex plane that they are equivalent if you have tried to write down a proof of the equivalence you will see that somewhere you might use removable singularities theorem okay so that is something that you should try to do or if you have not done it, fine.

So the moral of the story is you think of a function having a pole at infinity essentially to be polynomial plus something which is analytic at infinity okay, so the model for function having a pole at infinity is a polynomial, so polynomials are functions which have a pole at infinity okay and the degree and the order of the pole at infinity is just the degree of the polynomial okay and therefore you must understand that you know and mind you these are entire functions polynomials which are entire functions.

So the moral of the story is that if you take an entire function okay and if it has a pole at infinity you should expect it to be only a polynomial okay and more generally this is not exactly correct what you could get is that you could get quotient of polynomials also with the numerator polynomial bigger than the denominator polynomial okay that is also possible because that is also something that will as an improper rational fraction will work out to something that is a proper fraction plus polynomial and the proper part will be analytic at infinity okay.

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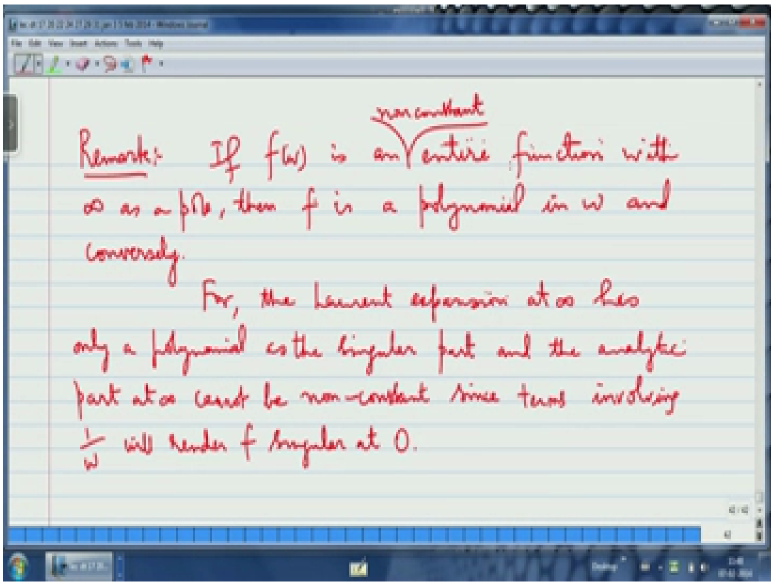

So let me so let me say that in more detail so here is a remark if f of W is an entire function with infinity as a pole then f is a polynomial in W and conversely, so the converse is something that we have just seen if f is a polynomial then a polynomial is of course entire and a polynomial does have a pole at infinity of course you know in all these things you have to assume that you are not working with a constant function but as a constant function it should always be excluded, so because even constants are thought of as a polynomials so let me just put non-constant entire function, so let me write non-constant here okay let me write non-constant here mind you if you have an entire function which is analytic at infinity it has to be constant okay.

This is non-constant entire function will have a problem at infinity, it will have a singularity at infinity and that singularity cannot be removable okay that is just another avatar of Liouville's theorem okay so, so how do you prove this proof is very simple. Well let me say it in words f has a pole at infinity so if you write the Laurent expansion at infinity a singular part is a polynomial okay and then there will be an analytic part which consist of a constant and it will consist of negative powers of W okay but then this is also supposed to be analytic at 0, so you cannot really have any negative powers of 0 because they will negative powers of the variable because they will they will simply the negative powers of the variable will simply misbehaves at W equal is to 0, so all the coefficients of the negative powers of the variable have to vanish which means that f is essentially a polynomial.

So it is pretty easy to see that this is true okay, so let me write that down. For the Laurent expansion at infinity has only a polynomial as a singular part as the singular part and the analytic part at infinity cannot be non-constant since terms involving 1 by W will render f singular at 0 okay, so that is the proof okay. So let me again repeat it you write f of W you write it as a positive negative powers of W including the constant then the positive power of W there can be only finitely many because that is what will happen if f has a pole at infinity, so the positive powers of W they will be polynomial okay and the constant and the negative power of W that will be the analytic part at infinity okay but then as expansion should be also valid at 0, so at 0 negative powers of W cannot be allowed okay.

So there are no negative powers of Z as a result you see that the analytic part of f is just a constant okay, so the analytic part is a constant the singular part is a polynomial when you add it together you get a polynomial, so f has to be a polynomial so the only functions entire functions, non-constant entire functions which are analytic which has a pole at infinity are polynomials okay and of course if you if you relax the condition that f is entire but you add the condition that f is Meromorphic okay which means that it has only isolated singularities which are poles in the plane then you can prove that f as to be a rational function okay it has to be a quotient of polynomials the degree of the numerator polynomial greater than the degree of the denominator okay.

So this is something that you can show but we will come to it was eventually we have to come to Meromorphic functions that is what I want to do and you know just to connect things back in our discussion we are trying to prove the big Picard theorem and the big Picard theorem involves looking at chance which has essential singularity is at infinity and or probably functions which have poles at infinity and basically you want to study Meromorphic functions and families of Meromorphic functions. You want to do topology and in fact study compact families of Meromorphic functions, compact space of Meromorphic functions but in all these things you want to be able to work with infinity very easily you want to work with the point at infinity in a very easy way and that is the reason why we are analysing the point at infinity insomuch details okay, fine.

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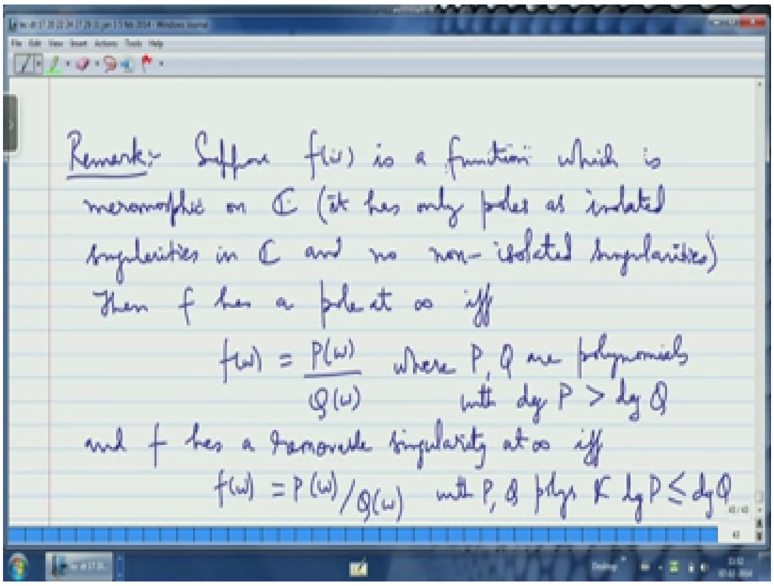

So now that we have this so let me write that as well so here is another remark suppose f of W is a function which is Meromorphic on the complex plane which means that it has only poles as isolated singularities in C and no non-isolated singularities okay, so this is what are Meromorphic functions is, it has only the only singularities it has they are all isolated, so it has no non-isolated singularity and every isolated singularities a pole okay this is what a Meromorphic functions is and then you also assume that infinity is an isolated singular point and I assume that infinity is a pole okay then the then the claim is that f is quotient of polynomials with the numerator polynomial adding degree greater than the denominator polynomial okay.

So let me write this then f has a pole at infinity if and only if f of W is equal to P of W by Q of W where P Q are polynomials with degree P greater than degree Q okay and f has a removable singularity at infinity if and only if f of W is of the same form P of W by Q of W but now the degree of P is less than or equal to degree of Q with P Q polynomials and degree of P is less than or equal to degree of Q okay so basically this remark tells you the obvious thing namely we know if you take to polynomials and you take the ratio we will take P of W and Q of W they are 2 polynomials and you take P of W by Q of W.

You know the P by Q will be analytic function at all points where Q is not 0 okay and the points where Q is 0 there will be poles okay and so P by Q will be a Meromorphic function okay and this Meromorphic function will behave well at infinity if the degree of P is less than or equal to degree of Q okay and that is because the degree of the numerator is less than or equal to a degree of denominator as the variable goes to infinity you will get a limit okay whereas if the degree of P is greater than the degree of Q this quotient will go to infinity as you go to infinity, so infinity will become a pole okay and that is exactly what this remark says okay, so you can try this it is pretty easy to do what we will come back to it later right. Now what I want to shift my attention is to do something connected with residues okay, so you know one of the ways of trying to study a function at singular point is by the so-called residue of the function of that point.

Why is residue important? The residue is important because the residue actually gives you the integral of the function along a simple closed curve that goes ones around that point okay, so the whole importance of the residue is because it will allow you to integrate the function around a singularity that is why residues are important and that is why you use residue to compute so many integrals in the 1st course in complex analysis, you even compute many real integrals by converting them as real parts or directly into complex integrals and then trying to apply the residue theorem or the Clausius theorem okay, so the moral of the story is that residues are important just because you can integrate a function around a singularity okay and you know so now if you think of the point at infinity as a singularity then it is natural that you will you will ask about the residue of the function at infinity okay so you can talk about the residue or the function at the point at infinity and what you get?

So the so the answer to that is something very nice actually it tells you in a way as why Clausius theorem fails or the point at infinity. The beautiful thing is that since Clausius theorem has failed for the point at infinity you think you have loss something at the point is you getting it back different way there is a so-called extended version of the residue theorem which is called the residue theorem for the extended complex plane which tells you that you should expect the Clausius theorem will fail at infinity, so if I tell you in nutshell if I want to tell you in nutshell what the residue theorem or the extended complex plane is?

There is very simple it says take a function is on the extended complex plane which has only isolated singularities okay then I knew if it has only isolated singularities there are only finitely many okay because you are looking at an isolated subset of a compact set, the extended complex plane is a compact set mind you the extended complex plane is a 1 point compactification of the complex plane it is the compact set it is homeomorphic over the Riemann sphere okay which is compact okay and if you take if you give if you take an isolated subset of a compact set is should get only a finitely many points okay, so the moral of the story is that when I am looking at a function on the extended complex plane which has only isolated singular at these there are only finitely many singularities okay and then one of the singularities good be the point at infinity.

So that function may have infinity as a singular point or it may not have infinity as a singular point and the beautiful thing is the residue theorem says now if you integrate along any simple closed curves okay and assume that the simple closed curve goes around all the finite singularities, all the singularities in the finite complex plane okay the residue theorem says the sum of all the residue okay inside the curve plus the residue at infinity will add up to 0. The total sum of residue is 0 that is the residue theorem for…which includes the point at infinity.

The residue at infinity as to neutralise the sum of the residues at the finitely many points see that is the reason see when I try to integrate 1 by W or a sufficiently large circle okay I get if I put positive orientation for the if I would negative orientation for the circle so that it the infinity lies in the interior of that circle okay which actually becomes the exterior of because of negative orientation then what happens is that I do not get 0 okay even though 1 by W is analytic at infinity I get minus 1 and what is that minus 1 that minus 1 is to it is to neutralise the plus 1 which is the residue at 0 of 1 by W and you see sum of residues of 1 by W at 0 which is plus 1 and the residue at infinity which is minus 1 they add up to give 0 and this is exactly demonstration of the residue theorem for the extended complex plane.

So the factors that if you take a function which is another dig at infinity the integral infinity does not vanish because it actually tries to you know it neutralise a sum of residues at finite at the finite points at the points of the complex plane which are singular points okay that is the reason why the integral at infinity is not giving you 0 that is the reason why Clausius theorem fails, so Clausius theorem fails but it is not in vain you get the residue theorem or the extended plane okay so I will elaborate on this in the next class, okay.