Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

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Lecture No 1

Properties of the Image of an Analytic Function: Introduction to the Picard Theorems

Okay so welcome again to lectures on advance complex Analysis, so what we are going to do today is ask very basic question okay so let me switch to the writing board.

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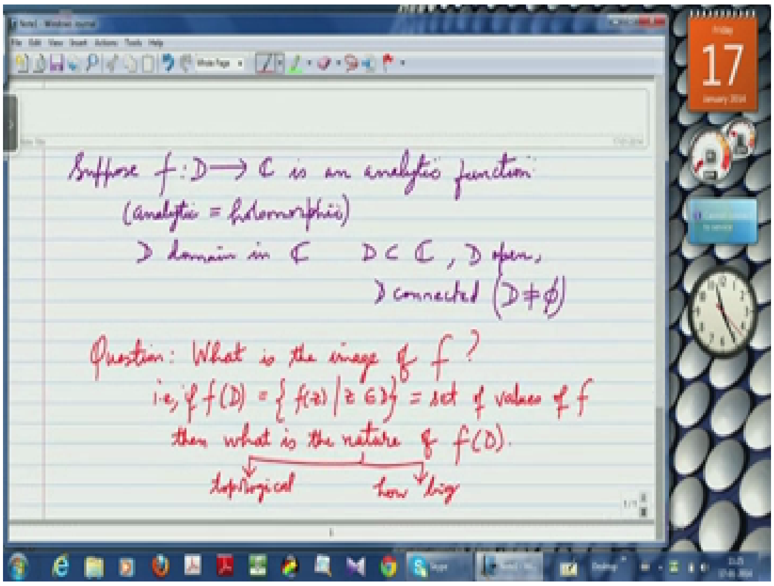

So suppose f from D to C is an analytic function, of course the analytic means the same as holomorphic, so analytic is the same as holomorphic and always as usual we assume D is a domain in the complex plane okay so D domain in the complex plane so that means that D is a subset of the complex plane D is open, D is connected and of course you know D is certainly non-empty I mean because by definition you know the empty set is also open okay, so of course we are not interested in looking at the empty set, so D is connected and of course you know in this context that for an open set connectedness is equivalent to path connectedness, so D is also part connected so you have a function f, f is analytic, f is a function which is complex value function, it is a complex value function of 1 complex variable and that one complex variable you might call it as Z.

So you can think of the function as f of the Z and Z varies over D okay and what is the fundamental question that we are asking? The fundamental question that we are asking is what is the image of f? So that is the question so here is the so let me change color to something else so here is the question, what is the image of f? So this is the question so in other words so that is you take you look at f of D okay, what is f of D? It is set of values of f okay so this is equal to the set of all f of Z where Z varies in D okay this is the set of values of f, this is all the values that f takes on D okay and obviously it takes complex values, so f of D is a subset of the complex parent, the question is what kind of a subset is this okay so here yes so what is the image of f that is that is if f of D is equal to this then what is the nature, what is the nature of f of D.

So when I say what is the nature of f of D you know you ask what do you mean by nature? Nature of course one can ask a lot of things one is of course topological nature okay, so f of D is subset of the complex parent you know complex plane is a topological space so you can ask whether f of D is open, whether f of D has any one of these properties that substance of a topological space satisfy okay. Properties that you know are open sets, sets being opened, sets being closed sets being connected, sets being part connected, sets being compact and so on okay.

So you can is what is the nature of this set of f of D is in the topological (())(5:09) okay then you can ask another question, how big is f of D okay what is the… How much of it or how big it is when compared to the whole complex plane okay, so when I say nature I can ask topological and the other thing is how big, how big is it? So and it so happens that complex Analysis gives you several nice theorems which answers go along way and answering these questions okay. So let me you know so let me go ahead and look at in just a minute let me resize the screen so that I get…

So and so let me look at the following let us ask them simple questions let us ask them simple questions so there are 1st few obvious things that you can say, see f is an analytical function so you know analytical functions are in fact infinitely differentiable that is what you learn in the 1st course in complex analysis once differentiability implies infinite differentiability on an open set and that is one of the characteristics properties of an analytic function and therefore in particular they are certainly continuous and you know continuity preserve certain properties or example the continues image of connected set is connected, continues image of a path connected set is path connected, the continues image of compact set is compact so you can say immediately that f of D is certainly a connected set okay so that is very basic topology it just uses the fact that the continues image of a connected set is connected.

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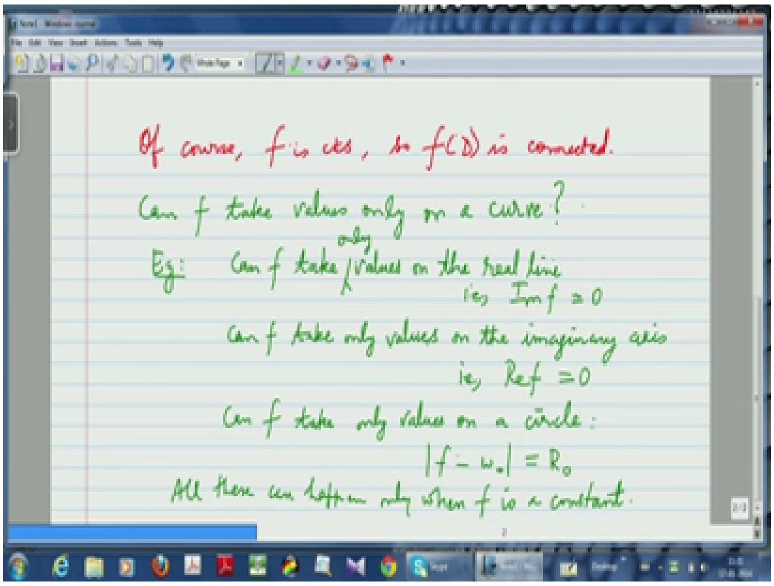

So let me say the following thing of course f is continues so you know I will use some abbreviations CTS assistance for continuous, so f is continuous so f of D is connected this is of course very basic well then you want to know more so you know let us let us try to see lets ask a few questions, so for example we ask a question like can f take values on a line or can f take all values only on the curve okay, so let us ask this question, so let me again change color, can f take values only on a curve? Of course with the curve I mean any simple curve that can think of like a parabola or a circle or something like that.

I mean particularly also it could very well be a straight line this is also curve okay so can f take values on the curve, so say for example you know suppose I take, so example so let me look at a few examples can f takes values on the real line? I mean take only values on the real line this means you are saying that the image of f is the subset of the real line okay and you know if f takes values only on the real line this is equivalent to saying that the imaginary part of f is 0 okay because if f is a complex valued function, normally we write f is equal to U plus IV where U is the real part of f and V is the imaginary part of f and you know very well from the 1st course in complex analysis that you and we have to you know be harmonic and in fact they will satisfy the Cauchy Riemann equations okay because f is analytic but the point is that if you say that f takes only values on the real line it means you are saying that V is always 0 that means the imaginary part of f is 0.

So this equivalent to saying that you know imaginary part of f is 0 okay and you know I can also ask can f take a values on the imaginary axis okay that is the y-axis consider it as there imaginary axis on the complex plane okay and that is equal in to saying that the real part of f is 0 okay, so there is another case of f taking values only on a line then of course they can ask can f takes values on a circle okay, so let me ask that also. Can f take only values on a circle so you know if you think that the circle is centred at the point w naught and has the radius R naught, this is equivalent to saying that the modulus of f of modulus of f minus w naught is equal to R naught this is what it means, this is the condition that f takes values on a circle okay.

Now surprisingly, not surprisingly in fact there is something you should have seen if you just recall that these are all the conditions that will ensure that the derivatives of f vanishes and therefore f has to be constant, so if the imaginary part of f is 0 at is the same as saying the imaginary part as a constant okay. It is a special case of the fact that the imaginary part of the constant and if the constant is 0 this is equivalent to saying that amounts to saying that the imaginary part of f is 0 which means it takes for the real valued and if the real part of f is 0 that is a special case of the real part being constant okay and the f taking values on a circle is the same as saying that the function f minus w naught which is f added to minus w naught which is a constant function okay.

That has constant modulus that modulus is R naught okay now you have done this in the 1st course in complex analysis probably by using Cauchy Riemann equations that you know if the function has imaginary part constant or the real part constant or the modulus constant in the function has to be constant okay so all these things can happen only if f is constant okay so all these all these can happen only when f is a constant, f is a constant function okay takes the same value, so of course therefore the question is we are not certainly we are not interested in studying constant functions because there is nothing special about these constant functions they constant function maps the whole plane onto a single point which is the value of the function of that constant, we are not interested in…so we are not interested in such constant functions, we are worried about non-constant functions okay.

So what this will tell you immediately is that if you have a non-constant analytic function okay it cannot take values at least on a line or something like a circle okay but then so what is it so what this tells you is that the either you have the case that you are looking at a constant function in which case the images a single point okay it is that single constant value that it takes or the image cannot be a set of just the real line or it cannot be a subset of only the imaginary axis, it can be a subset of the circle. Such things cannot happen that is what this says okay but what is the that you have more generally, so more generally we have you know we have very nice theorem, so here is the theorem.

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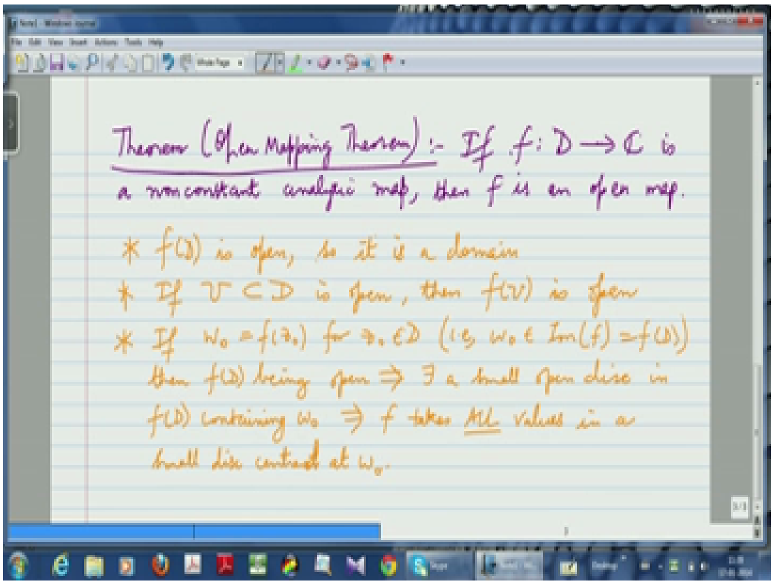

So here is the theorem and it is called the open mapping theorem and is a very fundamental theorem, what it says is that non-constant analytic map is always an open map okay, so if f from D to C is a non-constant analytic map then f is an open map okay it is an open map, so let us try to understand what this means? It means that see what is an open map? An open map is a map which for which the image of any open set is again an open set okay, so in particular what this will tell you is that f of D is an open set because D is already a domain, the D is the domain so D is already an open connected set, so f of D will become open okay and it is already connected so it is the same as it being path connected so f of D is again a domain okay so what this what, so let us try to understand what this means?

If f of D is open so it is a domain so that is something that comes immediately okay and mind you f is not a constant function, so it takes more than one value so f of D is non-empty of course okay and the other important thing is the following thing. What is the condition of open mapping? If you take U a subset of D which is an open subset then f of U would also continue to be open okay so if U is subset of D is open then f of U is open, this is what an open map means it maps open sets to open sets. The image of an open set under an open map is again an open set that is the definition of what an open map is? Okay and…so let us go a little bit more into this and you know try to see what it really means, what is the meaning of an open set?

An open set is a set which every point is an integer point okay that means you take any point in the open set then there is a small disk open disk surrounding that point which is also in that set that is what an integer point means okay, so what is f being open mean suppose f takes a certain value w let us say takes a value w naught okay then there is a it means you are saying w naught belongs to the image of f okay if f takes the value w naught okay that means w naught is in the image of f because image of f just consist of the values of f okay and then but since the image of f is open w naught is a point of an open set therefore there is a small open disk centered at w naught which is also in the image.

So it means that if f takes a certain value will take all values in a small disk surrounding that value okay so this should immediately tell you that f cannot simply take values on the curve because the moment f takes values at the point will take all values in a small disk surrounding that point and you know no curve can accommodate a small disk however small okay therefore you immediately get this idea that you know the image of an analytic mapping, non-constant analytic mapping cannot go into a curve, we saw special cases, we saw that it cannot go on into a real axis, it cannot go into the imaginary axis, it cannot be a circle okay and go into the circle but is more generally the reason is the image is open okay and of course curves are closed sets okay.

So let me write that now if w naught is equal to f of Z naught for Z naught in D that is the same as saying that w naught is an image of f which is f of D then f of D being open implies that there exist small open disk in f of D containing w naught and that implies that f takes all values in a small disk centred at w naught, so this is what is very important if f takes certain value it will take all values in a disk about that value okay this is a very important property and this is true for of course for a non-constant analytic function okay. So this is about this is about at the moment this is about the topological property of f of D this theorem tells you that f of D the image of f is certainly a domain it is an open connected set.

It is very important that it is an open set and in fact going into a higher geometric point of view okay what actually happens is this so let me tell it to you in words, what actually happens is that the mapping f from D to f of D becomes what is called ramified cover of Riemann surfaces okay so it means that it there are set of points these are the points where the derivatives of f vanishes okay these are called the points of the ramification and outside those points in the complement of those points this is actually a covering map okay it is covering map in the topological and is and also in the analytical or holomorphic sense okay so this open mapping theorem this so important that tells you that essentially every analytic non-constant analytic mapping is ramified cover okay fine so now what I am going to do is I am going to go to ask a more specific question.

So we are trying to look at the image of a domain under analytic function, so let us look at the cases where 1st at the case where you know the function is analytic on the whole plane so these are the entire functions, so what is an entire function? An entire function is a complex valued function which is analytic on the whole plane okay then the question is what is the what is the image of sets of functions? So that is a very deep theorem namely it is the so-called little Picard theorem which says that the image is either the whole complex plane or it is the punctured plane, it is the punctured plane namely it is the complex plane minus single point that means an entire function okay we will take all values except for omitting at most one value okay and this is called the Little Picard theorem okay.

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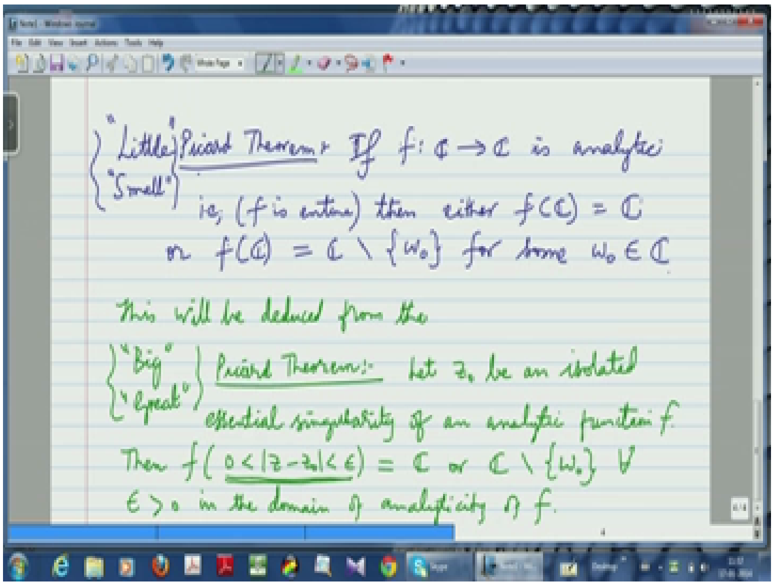

So let me state that so here is the Little Picard theorem, sometimes people also use the adjective Small Picard theorem, so what is this? If f from C to C is analytic that is so let me write it here f is entire then either f of C is equal to C or f of C is equal to minus w naught for some w naught C so this is a little Picard theorem, so you know it is the it is a very tremendous theorem it says that you take an entire function, you take the image, the image is huge I mean the image is literally everything at the worst if the image omits it can omit only one value okay and the case where the image omits a single value is of course the simplest example is that of the exponential function, you know if you take the function Z going to E power Z that is an entire function okay and the image will not it will be the whole punctured.

It will be the punctured plane it will be the complex plane minus the origin because exponential function will never take the value 0 because 0 does not have a logarithm okay so if you take any non-zero complex number you can find the logarithm and exponential of that logarithm give you back that complex number of course you will get many logarithms okay but you can find at least one for a nonzero complex number, so it means that the exponential function will take all values except 0 okay and that is the…so in that case it is an example that illustrates Picard theorem if you take f of Z equal to E power Z then the image of f is actually C minus 0 which is a punctured plane.

Normally if you take the whole complex plane and remove single point that is called the punctured plane okay with puncher at that point because that point is being removed and of course there is also the case when function an entire function can take all values, the simplest case is that of a polynomial, so if you take a polynomial if you take f of Z equal to P of Z where P of Z is a polynomial then it will take all values because I can always solve for P of Z equal to w naught for any w naught and that is because of the fundamental theorem of algebra namely that the complex numbers of algebra (())(25:27) closed so I can always solve a polynomial equation in one variable okay so a polynomial is also an entire function and it is it gives the case the 1st case namely the image of the whole complex plane is the whole complex plane okay fine so this is the little Picard theorem.

Now somehow what I want to do is I want to really to prove this okay it is a deep theorem normally this theorem is only stated in the 1st course in complex analysis but since this is advanced was in complex analysis I think it is fitting to look at a proof of this. Now well you know interestingly it is very interesting that the proof of this that I am going to present is actually gotten by deriving this as a corollary were much more deeper theorem, it is called the big Picard theorem and the funny thing is that the big Picard theorem is a theorem which deals which again asked the same kind of questions, it answers the same kind of questions namely what is the image of a domain under an analytic map okay but the point is that the domain you are looking at is a disk around an essential singularity of an analytic function.

So you know so let me state that so here is so let me use something else this will be deduced from the from the big or great Picard theorem and that is let so here is the statement of the theorem, let Z naught be an isolated essential singularity of an analytic function f okay then f of…so let me write this 0 less than mod Z minus Z naught less than Epsilon is equal to C or C minus single value for every Epsilon greater than 0 in the domain of analyticity of f okay so I have just stated a part of the theorem there is still more to it, so I want you to look at this. What I want you to appreciate is I wanted to appreciate the following things to reduce the little Picard theorem which is theorem about a function is analytic on the whole plane if function is mind you if a function is analytic on the whole plane it has no singularities okay.

It has no singular points okay so the little Picard theorem is a theorem about a function which has no singular points okay and it says that the image of the whole plane under such a function is either the whole plane or a punctured plane okay but we are deducing it from a theorem about the image of function with a singularity, so that is the funny thing so it is like even to answer question about an entire function you are forced to study singularities this is the point I want you to understand okay.

See normally we would not like to dirty our hands with singularities, why study singularities when there are functions without singularities but the point is you know sometimes mathematics and theory teaches us that even to study good things we have to study bad things okay so if you want to prove the little Picard theorem which is a theorem about good things I mean the function is analytic entire you have to still study functions which are having singularities and so here is the big Picard theorem and obviously you know the adjective great or big should tell you that this big Picard theorem has to be a Big Brother of the little Picard theorem and therefore you know the little Picard theorem can be reduce from the support of the Big Brother and what is this big Picard theorem, what does it say?

See you are looking at an analytic function okay and you were looking at a singularity of an analytic function okay now so I come later to what a singularity is? Okay because that is motivation for me to recall these things okay so you look at a function f which at a point has isolated singularity, isolated means there is a whole his surrounding that point where there are no other singularities okay and a deleted disk surrounding that point is given in this form as I have written here in the on my board zero strictly less than mod Z minus Z naught strictly less than Epsilon is actually the disk centred at Z naught the radius Epsilon is an open disk but I have thrown out Z naught that is the reason for putting 0 strictly less than I am not allowing Z equal to Z naught that means it is a punctured disk.

It is a punctured disk centred at Z naught and the punctured is exactly at Z naught I have thrown out Z naught okay and on this disk the I am assuming that this disk is full of points where function is analytic okay and that will be through at least for small values of Epsilon greater than 0 because the point Z naught is an isolated singularity okay and look at what the theorem says, it says you take the image of this when I write f of something okay it means f of this set which is the punctured disk that is the whole complex plane or it is a complex plane minus a single point and this is true for Epsilon sufficiently small and therefore it will be true for even larger Epsilon so long as this deleted disk is in the domain of analyticity of f because larger disk, larger deleted disk will contain smaller deleted disk and therefore their images will contain images of smaller deleted disk okay so.

So you see this is again you see the result of the conclusions of the theorem both the big Picard theorem and the little Picard theorem they are the same I mean the conclusion always says that the image under analytic function of a certain domain okay is either the whole complex plane or it is the complex plane minus a point okay and in case of little Picard theorem you are looking at the domain is the whole complex plane but in the case of great Picard theorem the domain is a very small neighbourhood deleted neighbourhood of an essential singularity of an analytic function and what is really amazing is in fact there is more to this Picard theorem what it says is you see so I want you all to observe is the following thing it is a very deep results.

It says take a very small neighbourhood of the essential singularity okay deleted neighbourhood that means of course you do not take the neighbourhood that you take should be a domain weather function is analytic so it cannot include the singularity, so when I say take a neighbourhood of essential singularity of course I mean delete that essential singularity so you are taking a deleted neighbourhood of the essential singularity and mind you a neighbourhood as small as I want, you see this Epsilon can be extremely small okay and the theorem is amazing it says you take no matter how smaller neighbourhood you take, the image of that neighbourhood is still the whole plane no matter how small your neighbourhood is.

The image of the very small neighbourhood no matter how small is still the whole plane it still takes all those values so what this tells you is you know it tells you that it tells you how the values of analytic functions change in a neighbourhood of an essential singularity, in our neighbourhood of an essential singularity this analytic function is taking all values at the worst it can omit one value okay and of course you know the example for this is just as in the case of the little Picard theorem where you where the example of an entire function omitting a value is exponentially E power Z a which omits the value 0, here you can take E power 1 by Z okay you can take the function E power 1 by Z and this E power 1 by Z and E power 1 by

Z at Z equal to 0 has an essential singularity and if you take any small deleted neighbourhood of 0 however small and you take the image under E power 1 by Z you will get the whole plane except the origin because exponential function will never take the value 0.

So you know it is an amazing, it is an amazing result, it is an amazing result and in fact there is a stronger version of the Picard theorem which says that not only does the image of any small neighbourhood however small of an essential singularity under under analytic function is a whole plane or plane minus (())(36:05) it says it takes the every complex value that it takes it takes infinitely many times, so there is in fact so that we write that down just to tell you how powerful the theorem is so let me write that.

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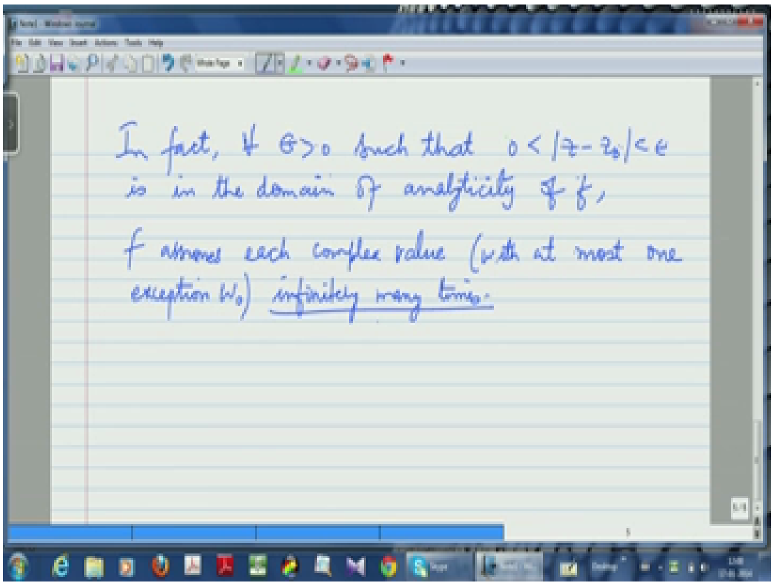

For every Epsilon greater than 0 such that 0 less than mod Z minus Z naught less than Epsilon is in the domain of analyticity of f. f assumes each complex value with at most one exception w naught infinitely many times, so in fact this, this infinitely many a times part of it which tells you the more it tells you with lot of force what is happening so the 1st part of the great Picard theorem says that you take an essential singularity and take a very smaller deleted neighbourhood about that, take a very small disk surrounding the essential singularity and take its image under the analytic function of course you do not take…the analytic function is not defined at the singularity okay so you do not take the value at the singularity there is no such thing.

So you are actually taking the image of a deleted neighbourhood but the point is no matter how small a deleted neighbourhood is, your image will be the whole complex plane or it may be complex plane minus a single point that is the 1st part of the theorem and in fact what this part of the theorem says that you know you take any value, any of the values in the complex plane except possibly for that one value w naught which it will not take okay, take any other of the values that it takes that value itself if you take the pre image of that value in that neighbourhood, the pre-majors and infinite set okay that means there are infinitely many points even in that small neighbourhood that are infinitely many points at which the function takes place that prescribed that value that you are pointing at and this happens for every value that it takes.

So what it does it is very funny it looks as if you take the small neighbourhood around the essential singularity, the function not only match that very small neighbourhood onto the whole plane or whole plane minus a point but it maps it infinitely many times okay it is like it maps it thousands and thousands of time okay and that is an amazing thing okay it is not that for every complex value there is one value here which goes to that, the fact is you take any complex value other than the exceptional value w naught then there is infinitely many points in this very small disk however small where that value is taken by f okay so that small neighbourhood it is really amazing to think of it, think of a very small infinitesimally small neighbourhood which is being again and again you know it maps thousands of times I mean probably unaccountably many number of times on to the whole plane or the whole place minus a point that is how the function behaves in a neighbourhood of an essential singularity and this is the key to…

This theorem on singularity is the key to proving or reducing as a corollary the little Picard theorem, so we will try to in the forthcoming lectures will try to give a proof of this theorem and so I will tell you roughly I will give you an idea of where we are going to go, so you know 1st of all I want to recall something about singularities, you would have studied singularities but I would like to recall them and some basic theorems about singularities especially the Riemann's theorem on removable singularities and then I want to reduce from that what is called the weak version of the big Picard theorem which is called the Casorati–Weierstrass theorem and Casorati–Weierstrass theorem is slightly weaker what it says is that while the big Picard theorem says that a function assumes analytical function assumes all values except with possibly one exception in every neighbourhood of an essential singularity.

What the Casorati–Weierstrass theorem says is that it will come arbitrarily close to every value okay so Casorati–Weierstrass theorem is a slightly weaker version of the great Picard theorem and that can be more or less reduced using the Riemann theorem on removable Singularities which I will prove okay so I have to recall something about singularities but then as we move towards the proof of big Picard theorem what you have to do is that we will have to study not one function but we will have to study a space of functions and we have to study functions with singularities and the kind of functions you going to study are functions with singularities as (())(42:05) and these are called Meromorphic functions.

So what I am going to do I am going to study topology of a space of Meromorphic functions and prove some fundamental ills like Montel's theorems and these are the keys to unlocking the proof of the big Picard theorem okay so what I am going to do in the next few lectures his 1st recall singularities then tell you something about Riemann removable Singularities theorem through the Casorati–Weierstrass theorem and then go onto Meromorphic functions studying Meromorphic functions and then trying to study families of Meromorphic functions topologically whether that spaces compact and things like that okay, so that is the that is the direction in which we will be proceeding so at least the 1st part of these lectures… our aim is to prove the great Picard theorem and you will see on the we will prove several other important theorems okay.