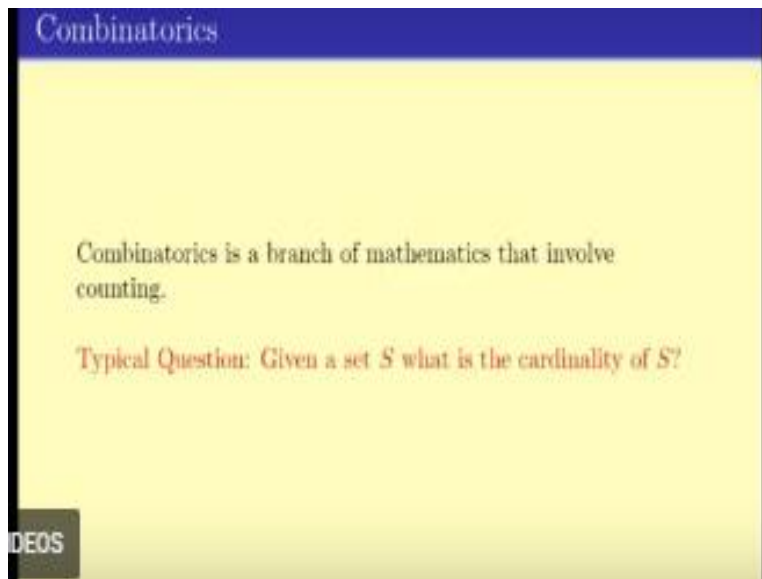


Discrete Mathematics
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Lecture - 50
Combinatorics

Welcome to the last video lecture in this course, so today we will be revising on the topic of counting.

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So, combinatorics is a branch of mathematic that involves counting the very big part of mathematics and we have seen that big names like Ramanujan and many other people have worked on this particular field quite a lot. Typical question is given a set S what is the cardinality of the set S ?

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Finding the cardinality

Typical Question: Given a set S what is the cardinality of S ?

How is the set given?

Usually the set is described in words.

Discrete Mathematics Lecture 50: Counting

Now, the main question is how is this set given? Usually the set is given in terms of words and you have to count it.

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For example

How many element of a set (universe) satisfy the certain set of conditions?

Equivalently: How many ways you can draw an element from the set (universe) such that the element satisfies the set of conditions?

For example, one can ask how many elements of a set satisfy the certain set of conditions or if you draw, how many ways can draw an element from the set such that the elements satisfied set of conditions.

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For example

- How many n digit numbers are there (in decimal representation) where no consecutive digits are same?
- How many functions are there from $\{1, \dots, n\}$ to $\{1, \dots, k\}$ that are non-decreasing? (That is, if $x, y \in \{1, \dots, n\}$ and $x \leq y$ then $f(x) \leq f(y)$).
- How many ways can you distribute n identical toffees among k kids?
- Number of 0,1 - strings of length n which does not have any consecutive zeros.

There are quite a number of examples that we have already seen in this course.
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How to count?

- Each problem is unique and each has to be solved by applying a technique that fits it.
- Counting is one of the most challenging subjects in mathematics.
- Some of the best works of Srinivasa Ramanujan was on counting.
- There are some handy tricks and tools to attack the problems that we will learn in this set of lectures.

Now, each problem is unique and each has to be solved using applying a technique that fits it. In fact, combinatorics is possibly one of the most challenging subject in the mathematics and hence some of the most creative ideas come out in the field of combinatorics. Some of the best work of Srinivasa Ramanujan was on count. There are quite a number of handy tricks, and tools that one can be used to attack these problems and that what we have learnt in this picture.

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Counting for selection

Selecting k objects from n objects

	Order Important	Order NOT important
Without Repetition	$\frac{n!}{(n-k)!}$	$\frac{n!}{k!(n-k)!}$
With Repetition	n^k	$\frac{(n+k-1)!}{(n-1)!k!}$

In particular, we looked at 2 special cases, case 1 was how many ways can you select k objects from n objects and the idea was that is all depends upon whether the k objects are identical, n objects are identical or not? Whether repetitions are allowed or not? Whether the ordering once we chose it matters or not? and so on.

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Problem

How many ways to distribute n balls in k bins?

- Are the bins distinguishable or are they indistinguishable
- Are the balls distinguishable or are they indistinguishable
- If the balls are distinguishable then does the ordering of balls in the bins matter?
- Can some of the bins be empty?

And the other thing is that how many ways to distribute n balls into k bins and whether the bins are distinguishable or indistinguishable relative, balls are indistinguishable or distinguishable whether the matter ordered inside the matters can be bins be empty.

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Counting for Distributing

Distributing n items among k bins.

	Items Indistinguishable	Items Distinguishable	
		Ordering inside bin matters	Ordering inside bin don't matter
Bin Labeled (can be empty)	$\binom{n+k-1}{k-1}$	$\frac{(n+k-1)!}{(k-1)!}$	k^n
Bin Labeled (can't be empty)	$\binom{n-1}{k-1}$	$\sum_{i=0}^k (-1)^{i+1} \binom{k}{i} \frac{(n+k-i-1)!}{(k-i-1)!}$	$\sum_{i=0}^k (-1)^{i+1} \binom{k}{i} (k-i)^n$
Bin Unlabeled	$P(n, k)$	$\frac{(\sum_{i=0}^k (-1)^{i+1} \binom{k}{i} \frac{(n+k-i-1)!}{(k-i-1)!})}{k!}$	$\frac{(\sum_{i=0}^k (-1)^{i+1} \binom{k}{i} (k-i)^n)}{k!}$

These are various cases that we handled and by doing so, we have come up with the - we kind of went to the whole of it. So we have looked the various cases and we saw how to compute for each of these for it, each of the various cases. So these are the some small tricks that we used to solve these problems.

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Recurrence Relation

“In mathematics, a recurrence relation is an equation that recursively defines a sequence or multidimensional array of values, once one or more initial terms are given: each further term of the sequence or array is defined as a function of the preceding terms.”
-wikipedia

But one of the more general technique to solving the counting problem is using recurrence relations or in other words, recurrences is a very useful, is a very useful way in which, the inner term is written as a function of the previous terms and that extensively used for various other ways.

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Recurrence Relation

"Recurrence relation is used extensively for combinatorics, analysis of algorithms, in computational biology, in theoretical economics and many other subjects"

So, the main thing to study is that how to use recurrences model problems and how to solve the recurrence relations and we have in fact, seen quite a few examples of how to model problems using recurrence relations.

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Topics in Recurrence Relation

- Using Recurrence Relations of model problems
- Solving Recurrence Relations

And we also saw various techniques for solving recurrence relations.

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Techniques to Solve the Recurrences

- Guess the Solution.
- Prove using Induction.

The technique that we are seen, the most obvious one was first guess the solution and then proof by induction.

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But ...

If we can correctly guess the solution then we can prove using induction.

But how do we guess the solution?

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If you want - if one can guess it correctly then, that is it. But guessing the solution can be a tricky problem.

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How to Guess?

Technique 1: Unfolding the definitions.

- $T(1) = 1, T(n) = 2 + T(n - 1)$.
GUESS: $T(n) = (2n - 1)$
- $T(1) = 1, T(n) = n + T(n - 1)$.
GUESS: $T(n) = n(n + 1)/2$
- $H(1) = 1, H(n) = 1 + 2H(n - 1)$.
GUESS: $H(n) = 2^n - 1$

So we have looked at a few techniques, the technique one was the unfolding the definitions and we saw how this can be used to solve this or guess recurrence solutions.

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How to Guess ...

Example: $F(1) = 1, F(2) = 1, F(n) = F(n - 1) + F(n - 2)$

Guess:

$$\frac{(1 + \sqrt{5}/2)^n - (1 - \sqrt{5}/2)^n}{\sqrt{5}}$$

Example: $b(1) = 1, b(n) = b(\lceil n/2 \rceil) + 1$.

No nice guess exists.

But then there are techniques where unfolding technique does not work. Problem always unfolding techniques does not work. Particularly when this functions that we guessed is very complicated or where the recurrence relation is not that, not that nice.

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Example

$M(1) = 1, M(n) = M(\lceil n/2 \rceil) + M(\lfloor n/2 \rfloor) + n.$

For all n , $(n/2) \log_2 n \leq M(n) \leq 2 \log n$

Can we do better than this? Or do we care doing better than this?

Sometimes we are happy with a constant multiplication gap between upper and lower bound

So to handle the second case, when the recurrence relations are not that nice, we what we did was that, told was that, we can come up with upper and lower bounds.

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Asymptotic Notations

If $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$ then

- $f = O(g)$ or $g = \Omega(f)$ if for all for large enough x ,
 $f(x) \leq cg(x)$
- $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$
- $f \sim g$ is $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$
- $f = o(g)$ or $g = \omega(f)$ is $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$

And maybe that will be good enough for us and that introduced us to the field of asymptotic notations which are extremely important mathematical language or stating relationship between functions. It has been used quite extensively in algorithms and in various other fields also. We saw that how one can use the asymptotic notations to solve the recurrence relations, not exactly but asymptotically.

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Example

$$M(1) = 1, M(n) = M(\lfloor n/2 \rfloor) + M(\lfloor n/2 \rfloor) + n.$$

- Guess $M(n)$ for some n .
In this case for $n = 2^k$ we can guess $M(n) = n(1 + \log n)$
- Then prove by induction $M(n) = \Theta(n \log n)$
- Prove an upper bound, that is $M(n) \leq cn \log n$ for some c .
- Prove a lower bound, that is $M(n) \geq dn \log n$ for some c .

For example, problem like this, could be solved using asymptotical notations and proves that it might be as theta of $n \log n$ and again the way to prove it is by induction. So these all the things are basically somehow guess the solution and then prove by induction, either the exact value or the asymptotic value.

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How about ...

$$F(0) = F(1) = 1, F(n) = F(n-1) + F(n-2) \text{ for all } n \geq 2$$

How to you guess $F(n)$?

Even an upper bound and lower bound?

Actually

$$\frac{(1 + \sqrt{5}/2)^n - (1 - \sqrt{5}/2)^n}{\sqrt{5}}$$

But then, there are other recurrence relations, for which situations are much more complicated and we getting even an upper bound or lower bound is not good enough.

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Generating Functions

Let a_0, a_1, a_2, \dots be a sequence of numbers.

Then consider the polynomial $p(x) = a_0 + a_1x + a_2x^2 + \dots$

$$\sum_n a_n x^n$$

This is called the generating function for the sequence a_0, a_1, a_2, \dots

Idea: If I can somehow compute the coefficient of the x^n in $p(x)$ I will get a formula for a_n .

And for those, we came up with a completely different technique, it gave an extremely powerful technique and possibly the hardest thing we are talking about course till now namely which is called generating functions. So the idea is that if you have a sequence of numbers a_0, a_1 to a infinity, we considered this polynomial p of x as $a_0 + a_1x + a_2x^2$ and this is called the generating functions of the equation and somehow by using the $(\)$ (07:20) generate the recurrence relation, we can write with P of x as a function of x irrespective of the all the a_i 's.

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Generalized Binomial Theorem

Theorem (Generalized Binomial Theorem)

For all $n \in \mathbb{R}$ we have

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

Where $\binom{n}{k}$ is

$$\frac{n!}{(n-k)!k!} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k(k-1)(k-2)\dots 1}$$

We saw a various example of this one and we use the concept of generalized binomial coefficient and the Taylor series that we have seen to solve them.

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Taylor Expansion to be used

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots = \sum_n (-1)^n x^n$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots = \sum_n x^n$
- $(1-ax)^{-1} = 1 + ax + a^2x^2 + a^2x^3 + \dots = \sum_n a^n x^n$
- $(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots$
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_n \frac{x^n}{n!}$
- $e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \frac{a^4x^4}{4!} + \dots = \sum_n \frac{a^n x^n}{n!}$

So these were the techniques that we have seen for counting problems of combinatorics problems.

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Thus ...

We have covered

- Proof Techniques
- Introduction to Graph Theory and Linear Programming
- Combinatorics - Recurrence Relations, Asymptotic Notations, Generating functions.

To wrap up the whole course, we have seen we have covered in this course proof techniques, introduction to graph theory, linear programming, combinatorics, recurrence relations, asymptotic notations and generating functions. Now, let me be honest with you that whatever we did in this course is just a small part of what the actual course of discrete math is. The discrete math is the way more, the vast subject actually.

I have only been able to introduce you to some of the subjects and show you the nice exciting parts of them. There are lots and lots and lots of things to do outside whatever we have done. I pointed out to things like graph theory or generating functions and so on. And the world does not end there. Discrete math is the basic foundation for mathematics, it has been mathematical foundation for whole of computer science, whole of various kind of, part of mathematics and so on.

So if you are interested in discrete math, please continue with your work, finding out other subjects that will be that fall in this category as keep on reading then. Thank you.