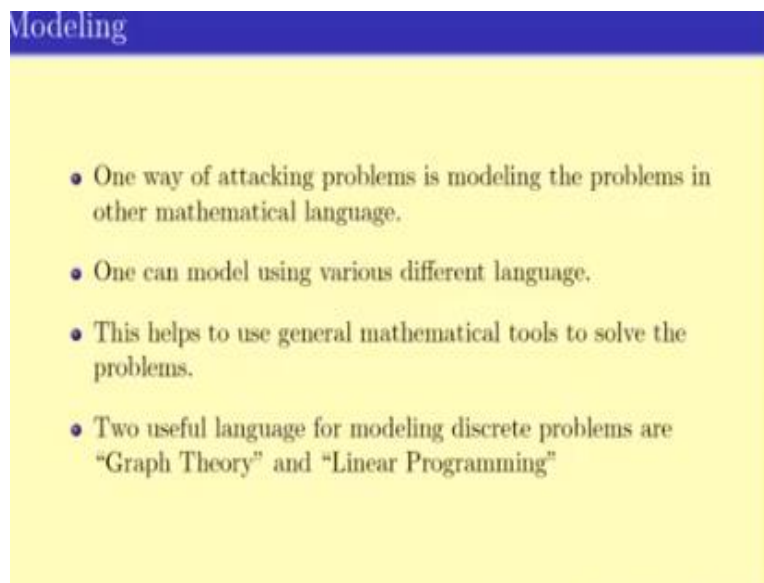


Discrete Mathematics
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Lecture - 49
Modelling: Graph Theory and Linear Programming

Welcome back. So we have been revising whatever we have been doing for the last few weeks in this discrete math course.

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So today, we will be focusing on graph theory and the linear programming methods. The most important thing here is that, is modelling one of the ways of attacking problems is modelling the problem in a very nice usable mathematical language. Now, one can use different mathematical languages to model or one can use different models and depending on the structure of the problem one has to decide which model to you.

We, in this course, we have looked at 2 particular models, number one is Graph Theory, number 2 is Linear Programming. We have spent quite a lot of time on graph theory but graph theory as a whole deserves a full course in itself and hence just a few weeks on graph theory is not going to be enough to do justice to this subject but (()) (01:22), we did an introduction to graph theory.

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Graphs

- Vertices - set of elements.
$$V = \{v_1, \dots, v_n\}$$
- Edges - set of pairs of vertices.
$$E = \{e_1, \dots, e_m\}$$

$$e_k = (v_i, v_j)$$
- Given the set of vertices and edges we have a graph
$$G = (V, E)$$

So, let us quickly recap what we have. So first of all, graphs are a set of vertices and the set of edges. The edges kind of denote the relationships, the binary relationship in vertices. The vertices are elements of some set and given the set of vertices and the set of edges, we have the graph.

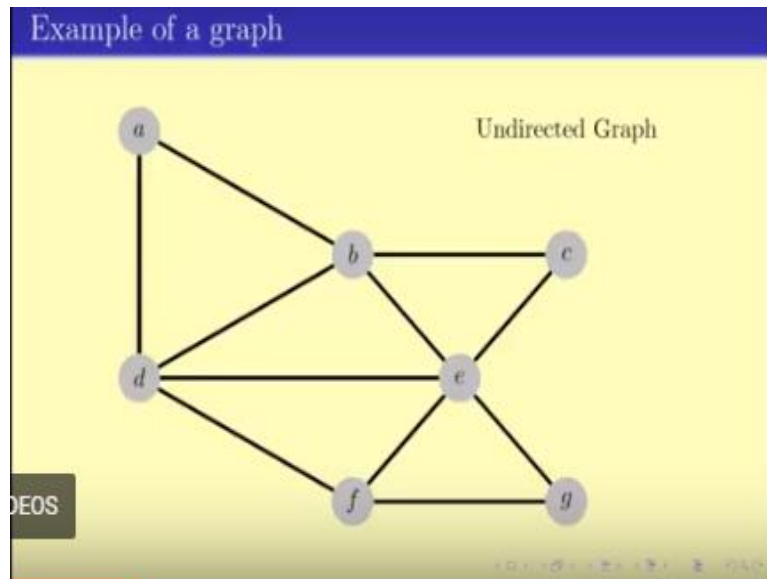
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Basic Definitions

- Let $G = (V, E)$ be a graph.
- If there is an edge from vertex u to v we say v is a neighbor of u .
- For an undirected graph the total number of u such that $(u, v) \in E$ is called degree of v .

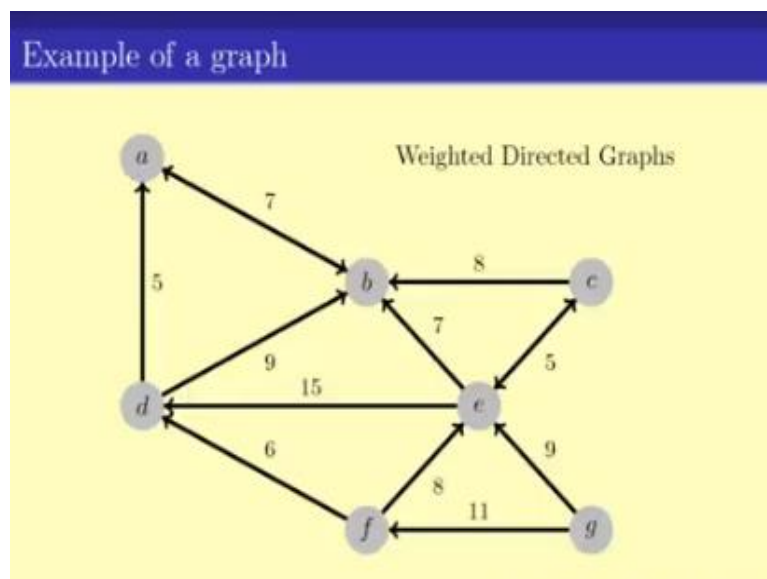
So, the graph is used to represent binary relationships if the graph is symmetric, we call it undirected as the relations symmetric. We can also have weights assigned to each of the edges and that is called a weighted graphs, we have the notion of what is the neighbour is and the degree of a vertices, vertex.

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So, in fact, we represent these graphs using by drawing on the plane where these blocks represent the vertices, the edges between they are drawn by lines that join the 2 vertices which they are supposed to represent.

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We can have weights from the edges; we can also have direction on the edges if the original binary inflation is not symmetric.

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Advantages of a graph

- Mathematical way of expressing relations among objects.
- Very simple and very general.
- Many other problems in real life can be designed as a problem in graph theory.
- So studying the structure of graphs and designing algorithms for graph problems is an important field.

The advantage of the graphs is that it is the nice mathematical way of expressing relationships between objects. They are very simple and very gentle. We have seen this example of many problems in real life, can be designed as a problem in graph theory and hence studying the structure of graphs and designing algorithms of graph is an important field in the modern world of algorithms and Complexity.

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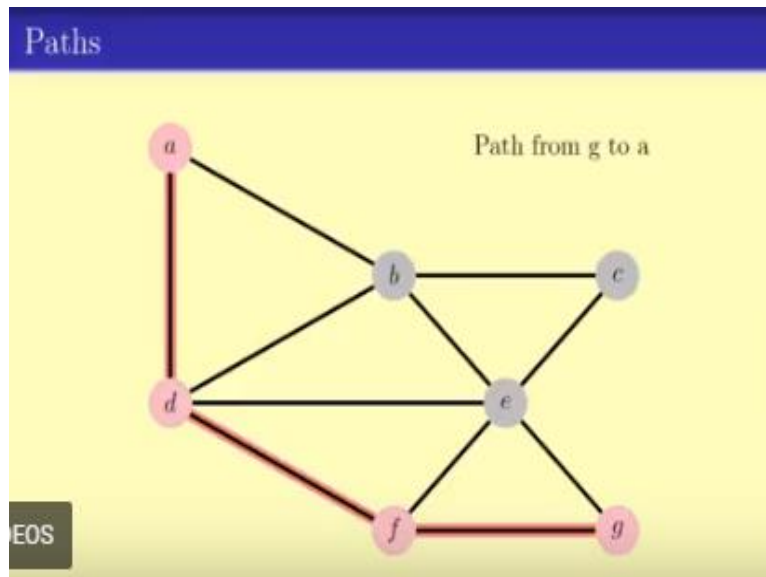
Introduction to Graph Theory

There are a number of properties/structures in graphs that keeps of arising again and again. We we have special names for theses.

Paths: Given a graph $G = (V, E)$ a path from u to v ($u, v \in V$) is a sequence of vertices v_0, v_1, \dots, v_k such that $v_0 = u$, $v_k = v$ and for all $0 \leq i \leq (k - 1)$ the edges (v_i, v_{i+1}) is in E .

Now, there are quite a number of properties that keeps on arising again and again and we have special names for this.

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So, we have (()) (03:46) of paths which basically - if you have to look at a path from g to a, it is a set of edges is a path from g to a. They can be various a path from g to a. Paths can be directed or undirected.

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Connectivity

We say " u is connected to v " if there is a path from u to v .

An undirected graph is called connected if for every vertices u and v there is a path from u to v .

In an undirected graph if there is path from u to v there is a path from v to u .

And we have a notion of connectivity that we looked at.

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Problems on Paths

Given any graph G prove that the relation " u is connected to v " is an equivalent relation.

And the notion of whether u is connected to v that if there is a path from u to v is an equivalent relation.

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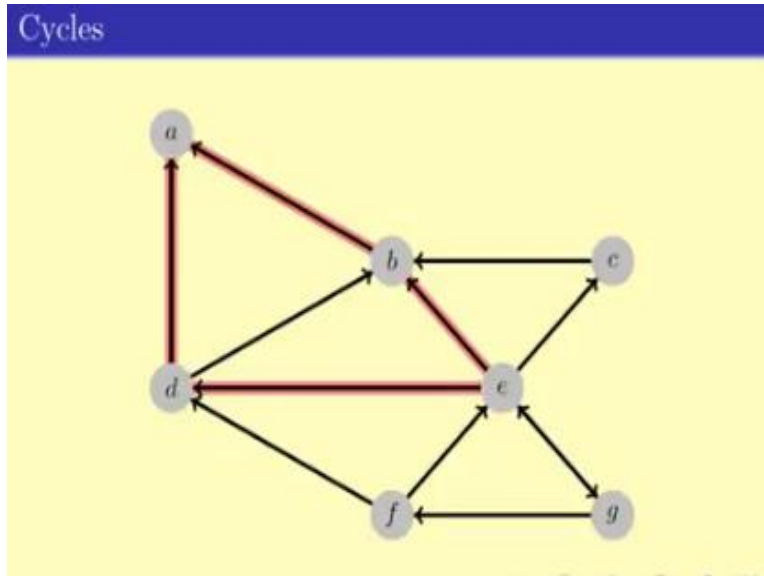
Connected Components

In an undirected graph the set of vertices connected to a vertex u is called the connected component of u in the graph.

A graph can be written as a disjoint union of connected components.

And hence can be written as a - hence the graph can be written as a disjoint union of connected components.

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We have also seen cycles like this.

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Problem on cycles.

If G is an undirected graph such that every vertex has degree ≥ 2 then G has a cycle.

And we also have looked at some cases when cycle exist.

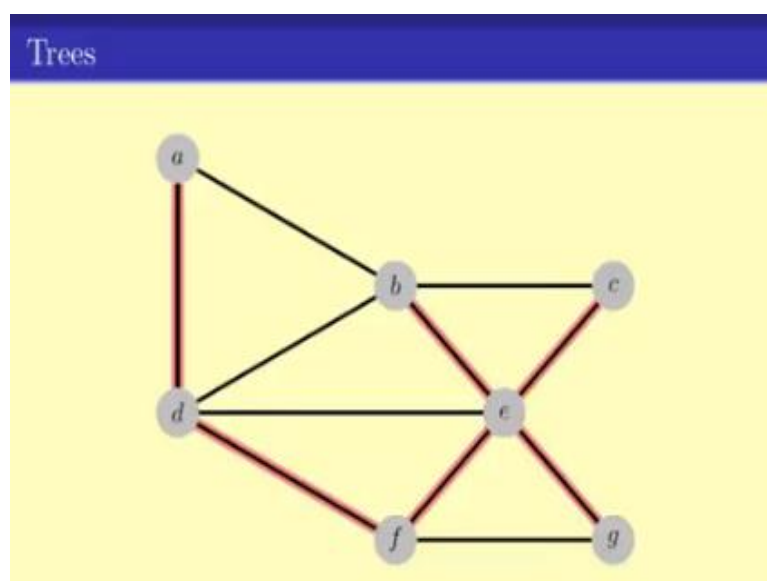
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Trees

- A directed graph that has no cycle is called an **acyclic graph**
- A connected undirected graph that does not have a cycle is called a **tree**.

And if graph is not cyclic or does not have a cycle, it is called a tree and a connected graph without a cycle gives the tree.

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The tree is a very useful notion again. So, these are small of small notions of graph that we have seen for various properties of graphs that we have seen in this course. We have gone through various of these examples, right.

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Properties of Graphs

- A tree has a degree 1 vertex.
Such a vertex is called a leaf.
- If you remove a leaf from a tree it is still connected.

We have seen nice properties of trees, for example, a tree has some - has a degree one vertex, if you remember leaf from a trees in still connected.

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Problems on Trees

How many edges are there in a tree on n vertices?

Answer: $(n - 1)$

Proof by Induction on the number of vertices.

And we have also prove that every graph has a spanning tree that the tree that touches all the vertices.

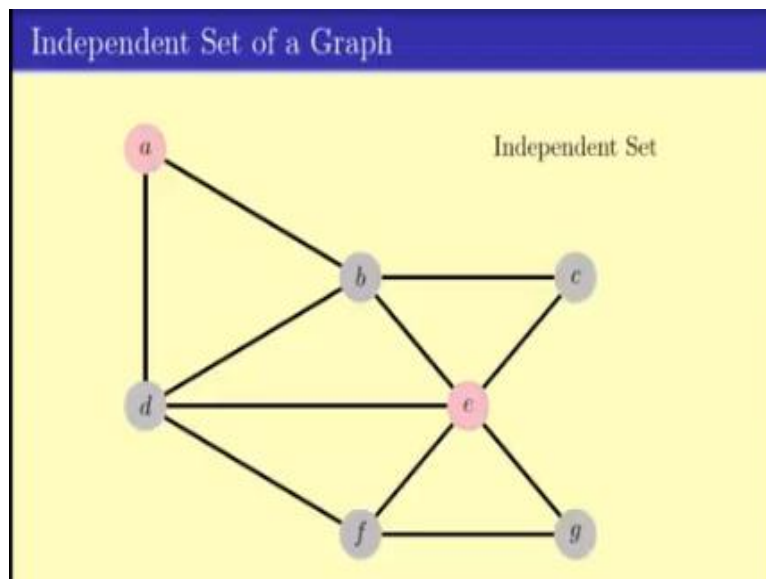
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Independent Set and Cliques

- Let $G = (V, E)$ be an undirected graph.
- An independent set is a set of vertices such that no two vertex in the set has a edge between them.
- A clique is a set of vertices such that there is an edge between any pair of vertices in the set.

So, we can also define other structures in graphs namely set and cliques that we have defined. So independent set is a set of vertices, such that there is no edge between any pair of them and a clique is the opposite of that.

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So, in other words, here is a set of delayed vertices or sets of independent set and we can have clique and now there are a lot applications, that one can apply or one can model using graph theory and I am not going to go through this application one by one.

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Application: Washing Machine Usage

Everyone's preferred time:

Rahul: 6PM - 8PM
 Joy: 7PM - 9PM
 John: 8PM - 11PM
 Anita: 10PM - 11PM
 Papu: 5PM - 6PM
 Jack: 8PM - 10PM

But please go back and refer to the original video in the original places.

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Vertex coloring of a graph

Given $G = (V, E)$:

- Color the vertices with k colors

$$C : V \rightarrow \{1, 2, \dots, k\}$$
- Such that for all edge $(v_i, v_j) \in E$

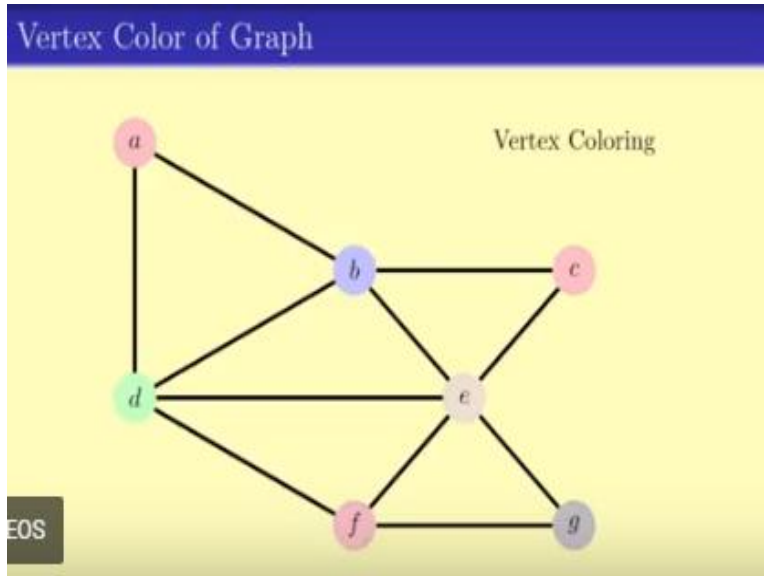
$$C(v_i) \neq C(v_j)$$

Can one colour a graph with k colors?

The minimum number of colors required to color a graph is called the **chromatic number** of a graph.

So, we can use the concept of colouring whether when we are allowed to colour a vertex with some K colours such that move to neighbours at the same colour. This is one more concept in colouring that is very useful and very handy and it used to and you study a lot. We called the chromatic number of graphs.

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So here for example, one can colour this one with 4 colours.

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Colouring is having applications in drawing on colouring of maps and we also looked at some problems and properties of colouring.

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Thus ..

- Graphs are very useful for modeling of various problems
- So studying graphs is an important subject - called graph theory.
- We have studied a small number of properties of graphs namely: Connectivity, Trees, Independent Set, Clique, Coloring
- Most of the properties of graphs can be deduced using proof techniques like induction, contradiction, case studies etc.

Thus, graphs are very useful for modelling various problems, studying graph is an important subject called graph theory. People who have (()) (07:25) attended this course, I strongly recommend you to go and learn a little bit more on graph theory, possibly attend one more course on graph theory. It is a subject that deserves a whole course and it is a beautiful subject that is very useful for solving their problems, mathematical problems as well as real life problems.

We have studied a small number of properties of graphs and we have of course, use the proof techniques of induction, contradiction, case study, etc. to solve (()) (08:07).

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Modelling using Linear Programming

We can model many optimization problems in the form of

$$\max(3x + 4y - 10z)$$

under the condition,

$$\begin{aligned} 5x + 8y &\leq 15 \\ x + 5y + 2z &\leq 10 \\ 7x + y + 8z &\geq 4 \\ 0 &\leq x, y \leq 1 \\ z &\geq 0 \\ x, y, z &\in \mathbb{R} \end{aligned}$$

This is called a **Linear Programming (LP)** .

There are packages to solve LP in R. *lpsolve*

Now one more model that is used extensively is linear programming model. The idea is that you can maximize or minimize a linear equation under a set of conditions and this is called the linear programming model.

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Linear Programming: input for *lpsolve*

$$\max(3x + 4y - 10z), \text{ such that}$$

$$5x + 8y \leq 15$$

$$x + 5y + 2z \leq 10$$

$$7x + y + 8z \geq 4$$

$$0 \leq x, y \leq 1 \text{ and } z \geq 0$$

$$x, y, z \in \mathbb{R}$$

	x	y	z		
Maximize	3	4	-10		
Condition 1	5	8	0	≤	15
Condition 2	1	5	2	≤	10
Condition 3	7	1	8	≥	4
Upper bound	1	1	-		
Lower bound	0	0	0		
Values	Reals	Reals	Reals		

Now, it is linear programming model can be, linear program can be solved if the variables can take real numbers. This is called linear programming and it can be solved very quickly using various softwares possibly in (R) (08:49), there is a very simple way of solving it.

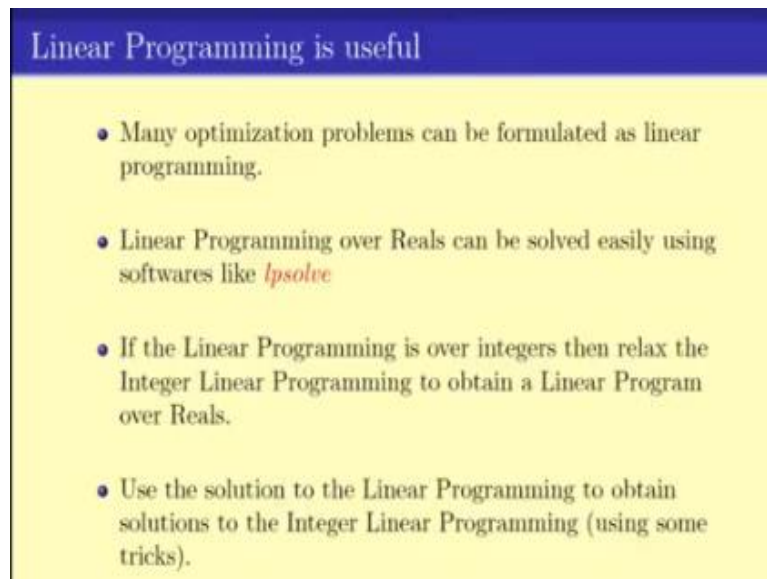
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How to Solve the LP

- Linear Programming is a very well studied subject with many different algorithms for solving Linear Programming.
- When the variables are allowed to take real values then LP can be solved quickly in polynomial time.
- In most languages there are software libraries for solving them (like *lpsolve* in R).
- The trick is in modeling the problems in LP form.

Now linear programming is again another subject that is very well study and many different algorithms are solved using linear programming. If the variables are reals can take the real value, then it can be solved in polynomial time but the trick is of course, how do you model it in the LP form.

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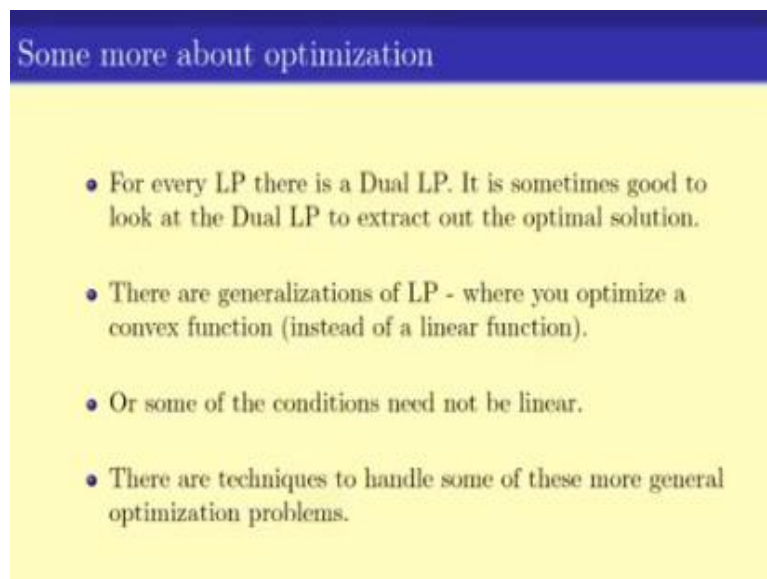


Linear Programming is useful

- Many optimization problems can be formulated as linear programming.
- Linear Programming over Reals can be solved easily using softwares like *lpsolve*
- If the Linear Programming is over integers then relax the Integer Linear Programming to obtain a Linear Program over Reals.
- Use the solution to the Linear Programming to obtain solutions to the Integer Linear Programming (using some tricks).

So many optimization problems can be formulated in this linear programming format. If the linear programming, these variables are over integers and unfortunately it is not necessary we can solve it using polynomial time. In that case, there are various tricks to solve them.

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Some more about optimization

- For every LP there is a Dual LP. It is sometimes good to look at the Dual LP to extract out the optimal solution.
- There are generalizations of LP - where you optimize a convex function (instead of a linear function).
- Or some of the conditions need not be linear.
- There are techniques to handle some of these more general optimization problems.

The subject of linear programming is a very vast subject and I am not going to - we did not do too much work in this, course on this. The main idea was that can we or how can we modulate using linear program. So in this course, we spent a quite a bit of time on how to model a problem using either graph theory or linear program. Now how to solve such problems is something that I have not done exactly in the class and unfortunately it will require independent set of courses on linear programming and/or graph theory to solve them.

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Conclusion

- Many problems can be modelled as Graph Problems.
- Some can be modelled using Linear Programming or other optimization problem.
- There is lot of techniques in the literature to solve these problems.
- The trick is to model the problem in a mathematical way for which techniques are available.

So I again, encourage you guys to attend these courses. So to conclude this particular revision, many problems can be modelled as graph problems or linear programming or optimization problems. There is lot of things known in the literature. The trick is to model the problem in our mathematical way in for which techniques are available. Thank you.