Discrete Mathematics
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Lecture - 47
Generating Functions (Part 4)

Welcome back, so we have been looking at how to solve recurrence functions using generating functions.

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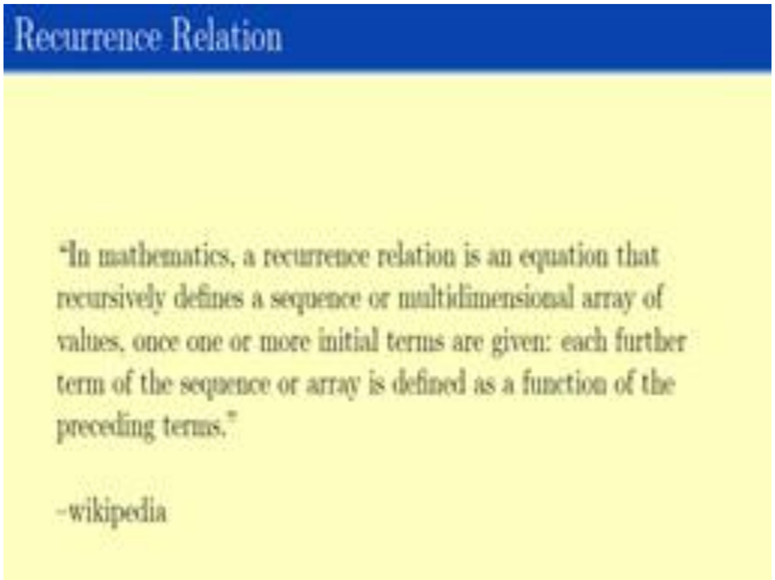

So, we have seen the definition of general recurrence functions quite number of times and we have also seen the generating functions recurrence functions can be used or is used in various subjects

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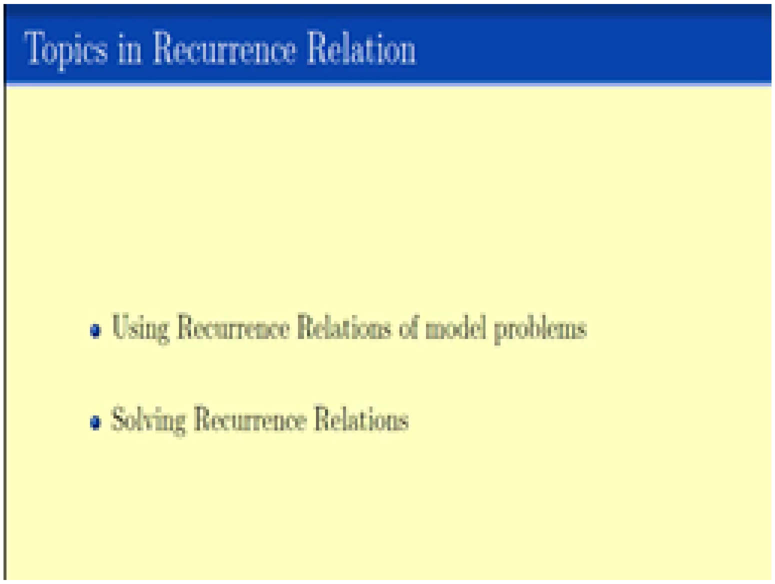

Particularly, we have seen how to use recurrence functions for modelling problem, the goal is to check or the goal is to find out how to solve these recurrence relations.

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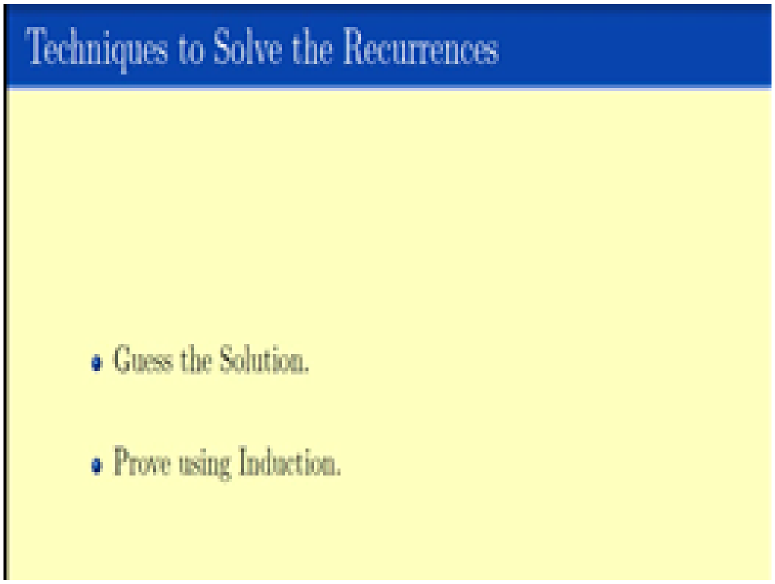

Now one of the technique of solving it is first guess the solution and then prove by induction.

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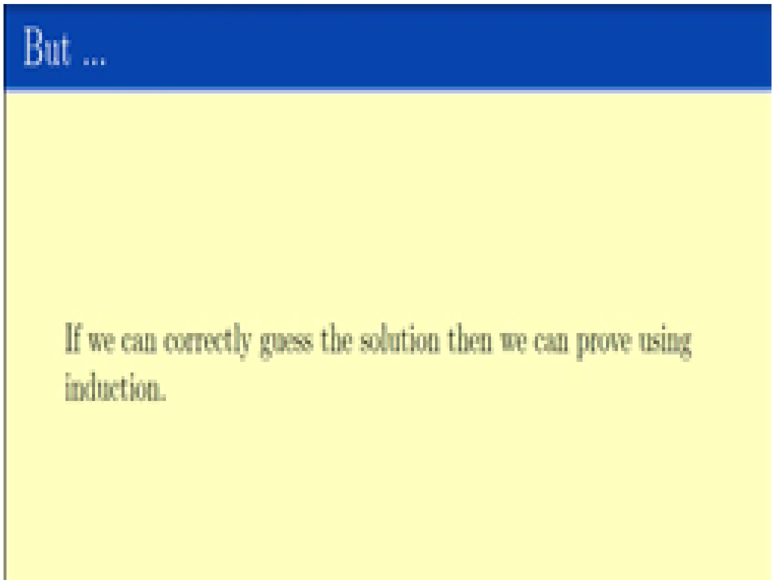

And if you can guess it correctly, then proving it by induction is quite a straight forward thing, but the question is how do we guess the solution?

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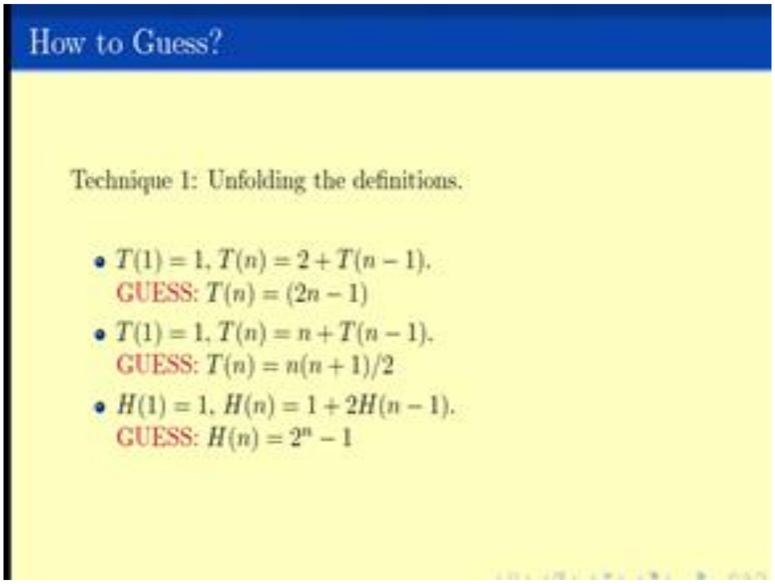

The technique one that we have looked at is the unfold the definitions, and using this technique, we have managed to test the solution for number of them.

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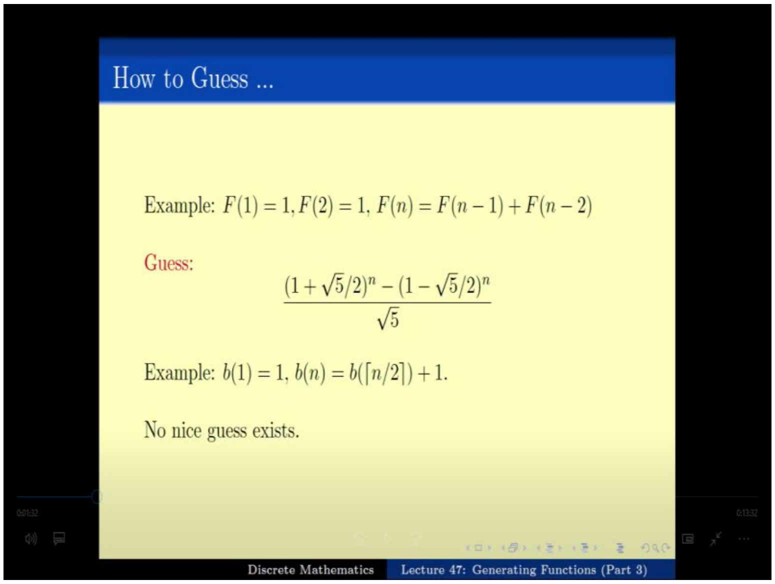

But then there are functions like below references like this, Fibonacci sequence, when the function is quite hard to guess, and this is what it is, and this is what we will be proving in this video and sometimes the recurrence functions does not have a nice being formed.

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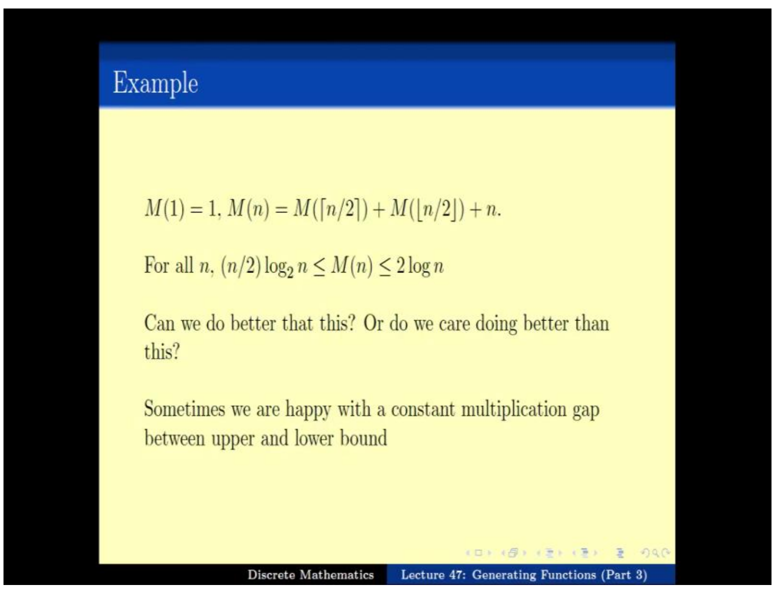

So, we have seen the indirect, upper bound and lower bound functions.

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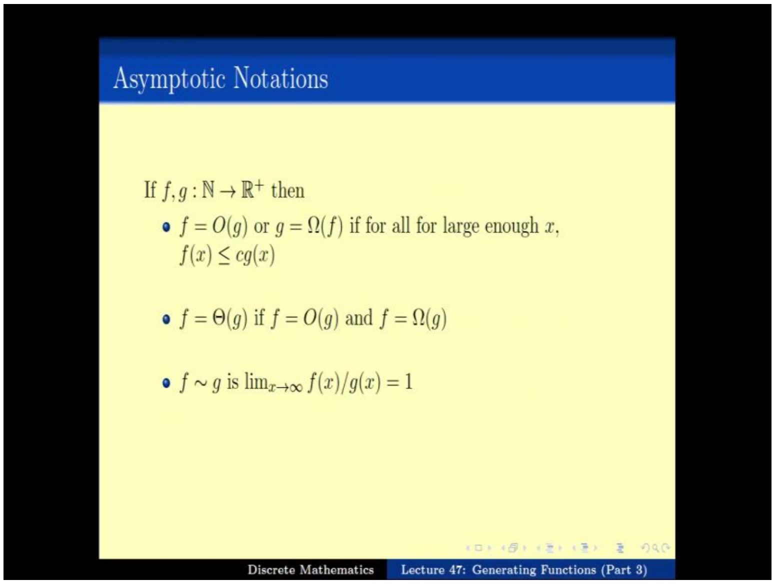

And that could be sometimes good enough for us at least asymptotically.

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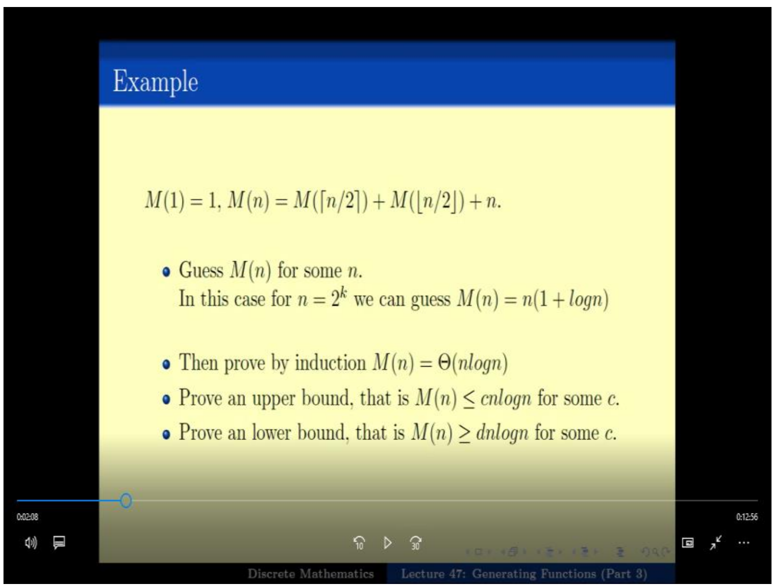

So, the idea is that in this case, you can solve it for some n, and then prove it using induction, that recurrence relation, the function is theta of some number by two kind of induction, induction for upper bound and induction for a lower bound. This is where if at all you can do simple techniques.

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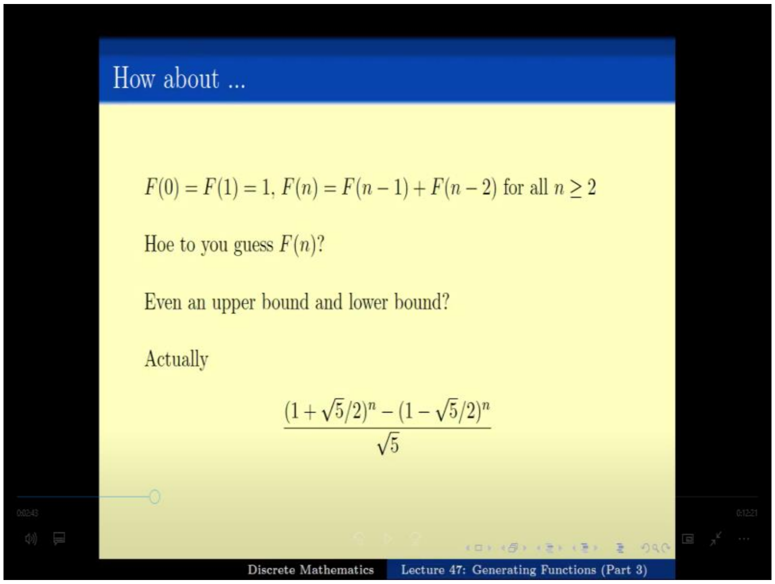

But still there are some differences for which you cannot do it. For example, this Fibonacci sequence, how do we guess this Fibonacci sequence? how do we even come with upper and lower bound, when actually the value is like this and I told you this is what we will be proving that actually this value to take, not by induction, by guessing, but actually come up with this formula.

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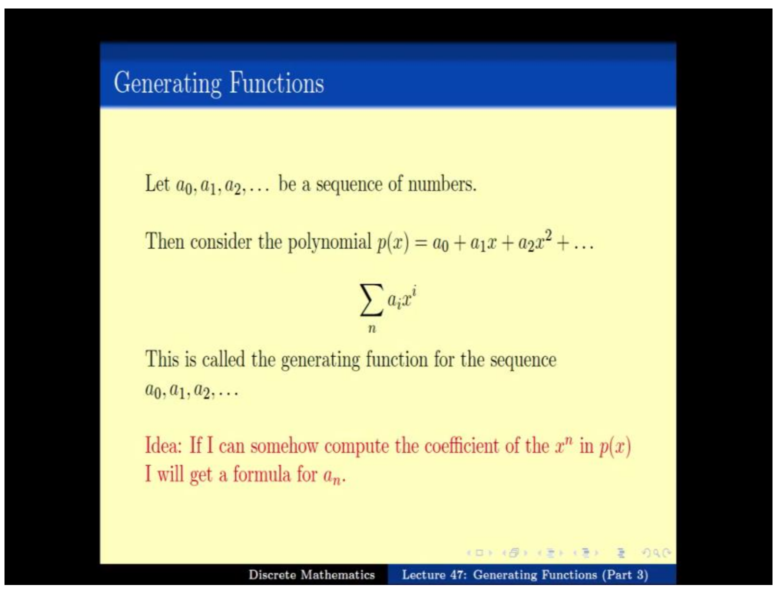

The way we have to do is what we are looking at for last three videos is the generating functions. Now we can quickly recap if I have a sequence of a0, a1, a2…infinity. The generating function is defined as the polynomial or power series which is a0 plus a1 x plus and a 2-x square and so on for namely, summation of a1 X Power F. This is called the generating functions of the sequence of a0, a1 till infinity.

The question is that or the idea is that if somehow, we can compute the coefficient of x power n in this px then, we can get the formula for an. We have seen couple of examples in the last two videos where we did it. Namely we wrote power series or generating the functions and then we used recurrence relation to get right p of x as a function of x and then we used or time to understand the coefficient of x power n and the function by looking at its Taylor series.

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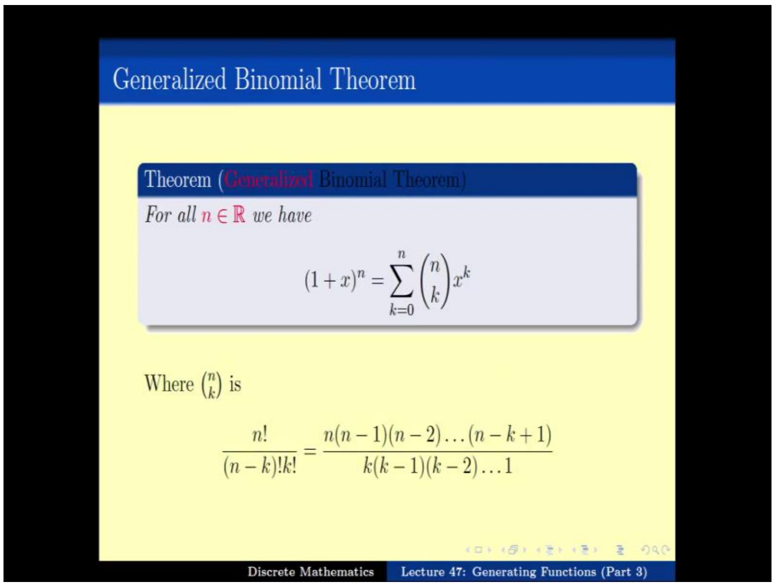

So, for Taylor series basically a way of getting, expressing the function as infinite polynomial.

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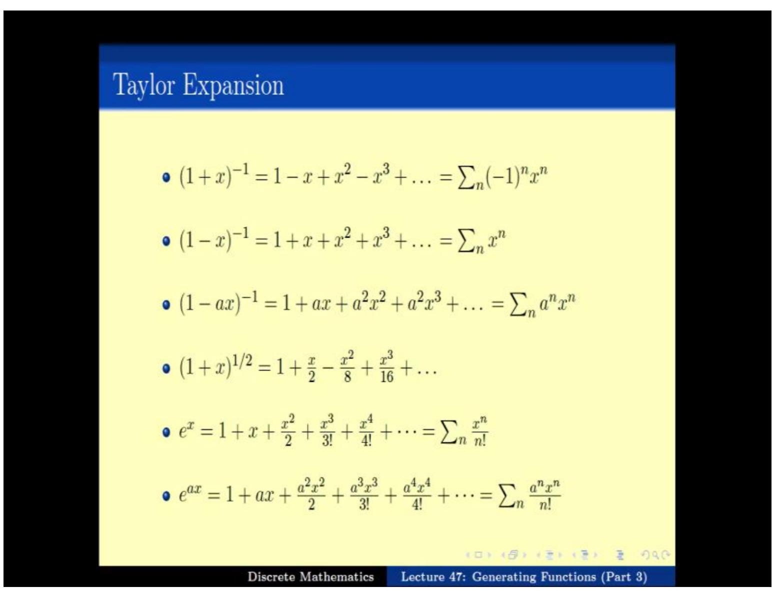

And we have heard some of the examples of Taylor series expansion and that one might use. So, in this particular video let us go to the hardest problem of all namely the Fibonacci numbers.

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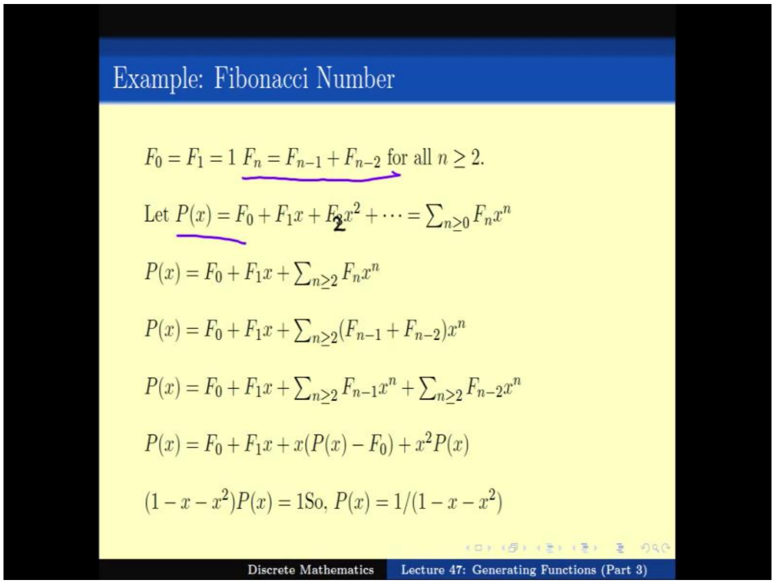

So here is the recurrence function. F0 is equal to F1 is equal to 1 and Fn is equal to fn minus 1 and f1 minus 2, now of course we start with writing the power series which is P of x is f0 plus f1 x plus f2 plus f3x but it is not f3 it is f2x and so on So that is which is the summation of n. f n x power n. And so, P(x) equals to let us write down here, F0+ f1x + fnx power n if greater than or equal to 2.

I have not done anything till now, I have just written everything after this second thing. But the fact is that, the fact is that for n greater than or equal to 2, I already have something like this. So, I can write it down as P x equals toF0 + F1x + n greater than or equal to 2. Fn will be written as fn-1+fn-2. Good? Let me expand it, this is of course Fn greater than or equal to 2 is Fn-1 times x power n plus Fn-2 times x power n.

Now like you did for many things, let us look at this first start, let us go back one step. Le us look at this one, what is this one, If I take away x this is x into summation n equal greater or equal to 2 Fn-1 x powered n-1. So, what is this stuff? Here n is greater or equal to 2 and here F n-1, that means, I guess this is F1x+F2x square+F3x cube and so on which means this quantity is nothing but the p(x) that we have minus f0 right and f0 here is one for us of course.

So, it is P(x)-f0. So here if I take x square as common I get summation N greater than or equal to 2 Fn-2 x powered n-2 and this is nothing but you can see F0x F0+F1x+F2x square and so on this is p(x). Right. So, but again the same argument we did in the last two videos we first note down the power series and use the recurrence relations to split up the later terms which helps us to get the formula like this.

This is x and Px-F0 and you do not plus x squared p(x). Now we can take the p(x) to the one side and we get of course P(x)-x P(x)- x squared P(x) equals to F0+F1x-F0x. now since F0 is equals to 1 so this is f1 equals to 1, 1+x-x which is 1 and hence I get P(x) equals to 1 by 1-x-x squared, so without much detain we have managed to write down P(x) as the function of x. All that is left to be done is now to understand or write down the Taylors series expansion of this function and if we can do it as we can understand the nx coefficient of that.

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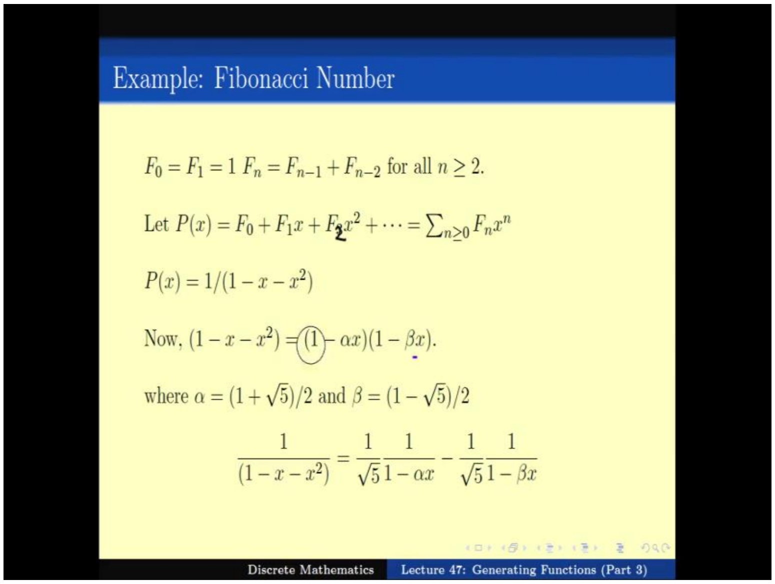

So let me be quickly see this one it is F2 there is mistake here, and we have got P(x) equals 1-x-x square. Question is that how do you solve it. If you remember the last time what it was 1 by 1-x is 1-2x and we wrote it as a sum of 2 and in reciprocals of two linear functions. But here, we cannot do that we have to do something similar but we do it carefully. The main way of doing is to first factorise 1-x-x squared.

Note that 1-x-x square is nothing but 1minus alpha x time for 1 minus beta x and again factorise it. Like you do where alpha is equal to 1 plus square root 5 by 2 and beta is equal to 1 minus square root 5 by 2 and in that case, you can see that. You can check that this is can be done. This is something easy thing to check.

This is the usual factorization of quadratic formulas. The formula we use a minus b minus 4ac and b square minus 4ac by x that formula. You can factorise 1 minus x square in this way. You will release alpha and beta. And in that case, we can write 1 minus x and in this expression as this. This is something I am not going to say how to do it.

I will give it to you guys to check that it is true and find one how will one apply to do it. It is very simple way to put it through. Right. So, we first factorise the denominator and then write it down as this expression. Which is again here note it down this one is linear expression and not this is also linear expression. It is 1 by r linear polynomial or polynomial of 51.

Note that this is what we exactly wanted. Because we know the Taylor's expansion of inverse of linear functions.

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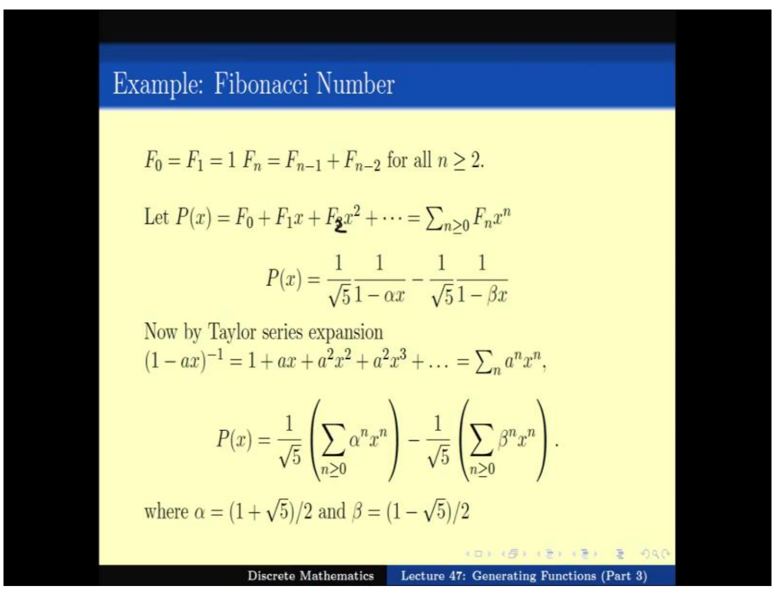

So again, we have this sum and based on Taylor series expansion we can write down 1 minus alpha is and 1 minus beta is, so 1 minus alpha x is equal to 1 plus so it is the summation of alpha power x of n and beta power x of n and alpha and beta are these two numbers. Right and thus we know and therefore we have written p of x as a polynomial where the coefficient of x power n. From this thing is 1 by square root of 5 alpha power n minus beta power n and that is exactly equals to F of n.

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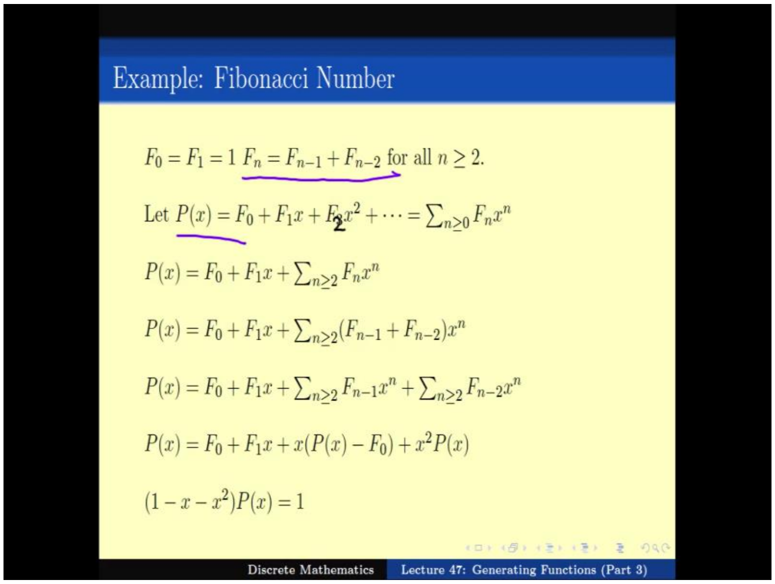

So, in other words, we get that coefficient of x power n if f n which is 1 minus this is alpha right. We need to substitute alpha here with 1 square root 5 by 2 power n minus 1 minus 5 square root by 2 n by square root of 5. So, as you can see almost like magic. One can come up with a compact form of the Fibonacci number with this quiet and impressive job.

Again, the idea is write the generating functions and try to use the recurrence function to write the generating functions as the function of x and then write down Taylor's series expansion of that using some tricks and then by looking at the coefficient of x power n in fn we have this number.

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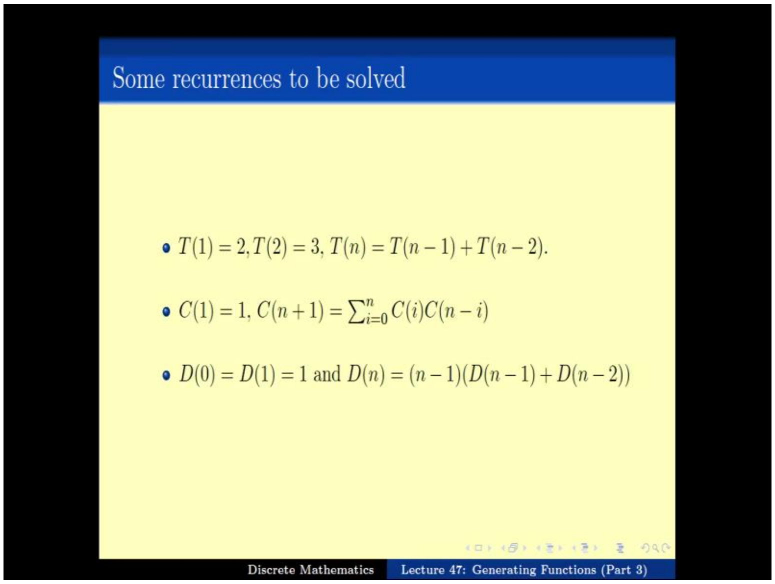

Some of the recurrence functions unsolved here are this one. That is T1 is equal to 2 t 2 is equal to 3 and t n is equal to t n minus 1 plus t n minus 2. I ask you guys to go and solve it by yourself. You do it as a kind of exercise. One more that is there which is this number if c of n plus 1 is equal to summation of ni is equal to 0, ci and c of n minus This is known as Catalan number. It is quite complicated one.

But again, solution is true generating numbers. So, we will be doing this solution for this thing in the next video. You also have this one formula d of n is equal to n minus 1d n minus 1plus d n minus 2. Again, we will be talking about this formula also in the next set of videos. Thank you.