Discrete Mathematics
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Lecture - 46
Generating Functions (Part 3)

Welcome back. So we have been looking at how to solve recurrence relation using generating functions. So to quickly recap recurrence relations, we have gone through this definition a few times already.

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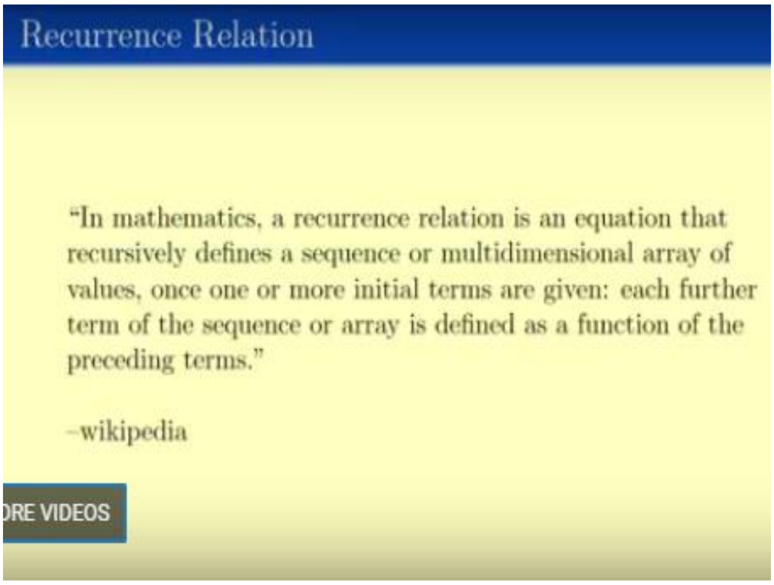

It is a sequence of numbers or we have been given the initial set of numbers and nf term-is written as a function of the previous ones. Recurrence relations is simply used in various subjects.

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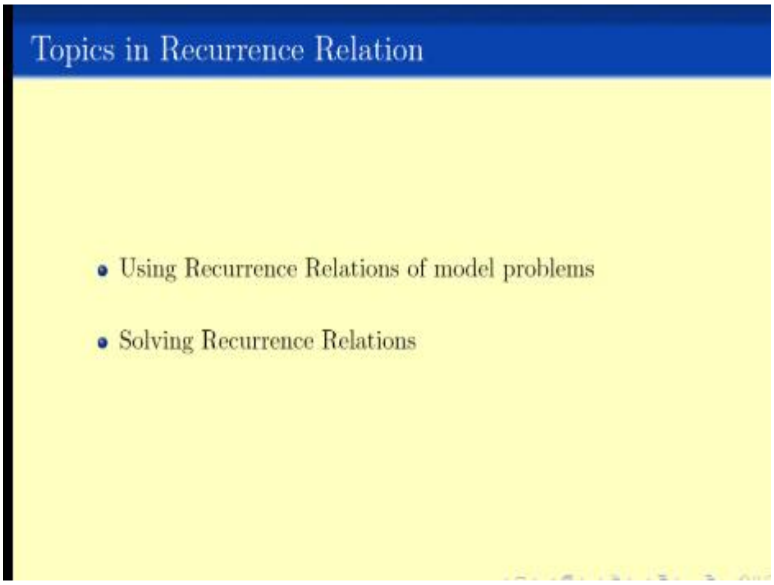

We have seen how recurrence relations can be used for modeling problems and we have been trying to see how to solve recurrence relations.

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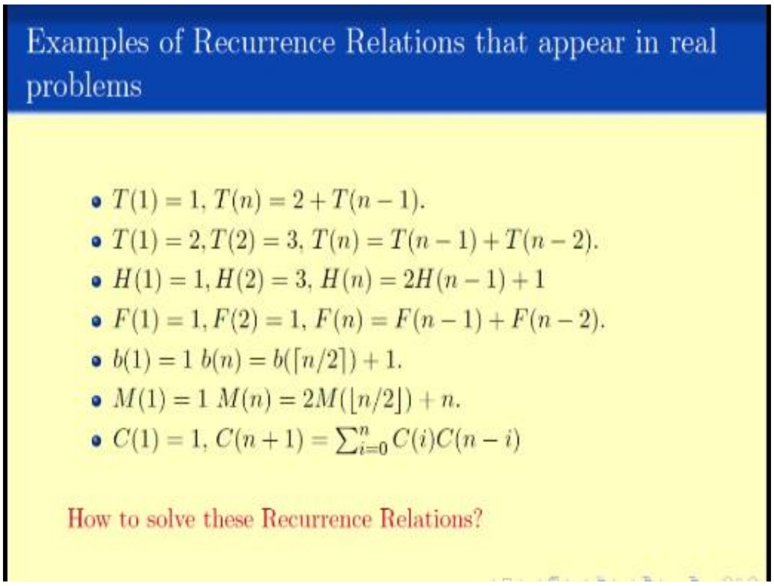

Now there are quite number of recurrence relations example that we have looked at. The main question is how do you solve that?

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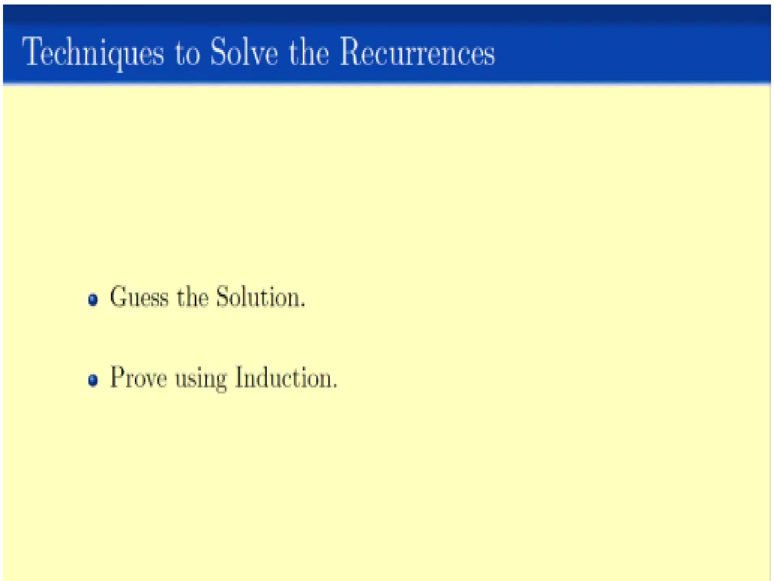

So the technique that we looked at it that if you can guess the solutions then we can prove it using in depths. Once you can guess it proving it using induction is quite a straight forward thing, the main issue being how do we guess the solution.

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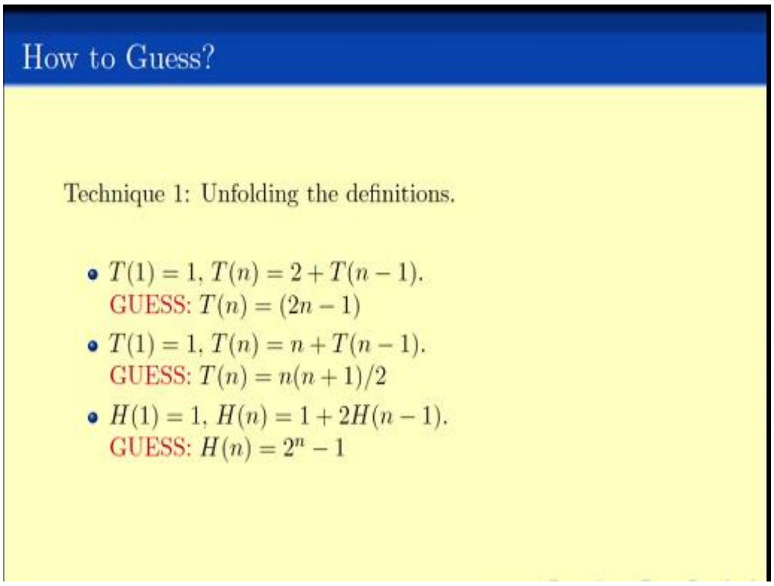

First technique is of course we can guess it by unfolding the definitions and we have seen how that can be used to guess it correctly.

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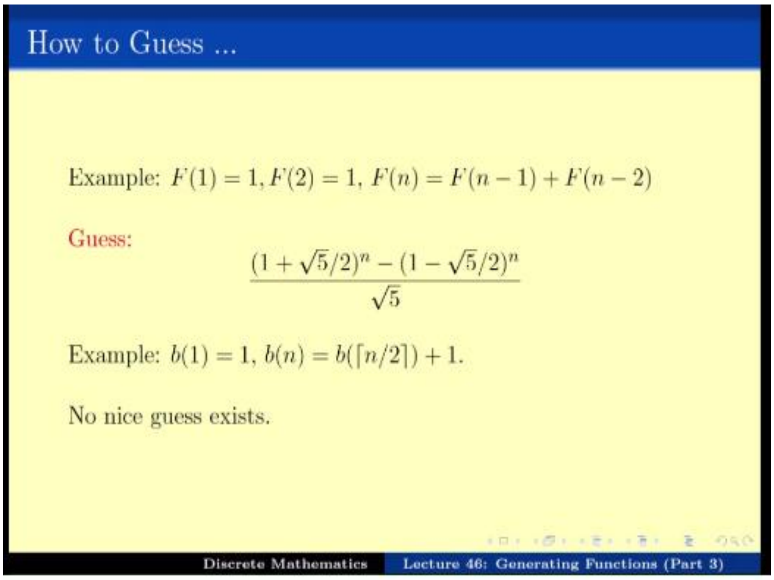

But then there were other things like the Fibonacci number which we did not know how to guess and then there are other things like the recurrence relation coming from the binary search and so on where again there was no nice guess.

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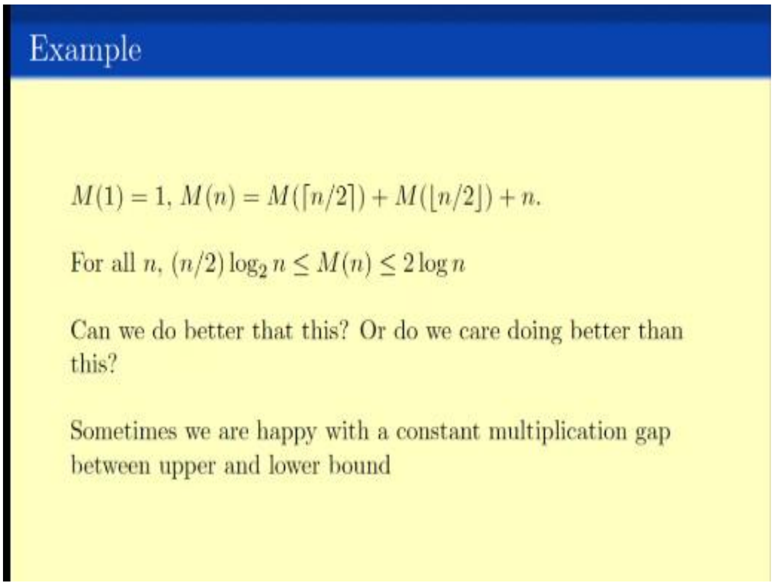

So in that case, the second case when there is no nice guess we told that okay maybe we can come up with a upper bound and lower bound and that should be good enough for us at least using the asymptotic notations of big O, big Omega, theta and so on. We can possibly get a asymptotic function for the recurrence.

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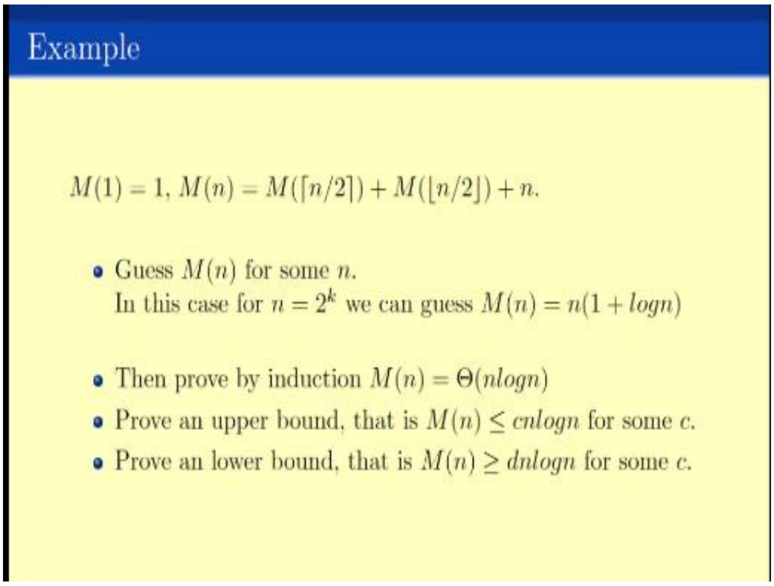

For example, we looked at this recurrence and the idea is to first guess a M (n) for some particular N. And then you use induction to prove that M(n) is theta of some function with whatever you have guessed till now that is done by proving a upper bound and lower bound. So this is what we were there using the simple techniques of doing recurrences or solving recurrences.

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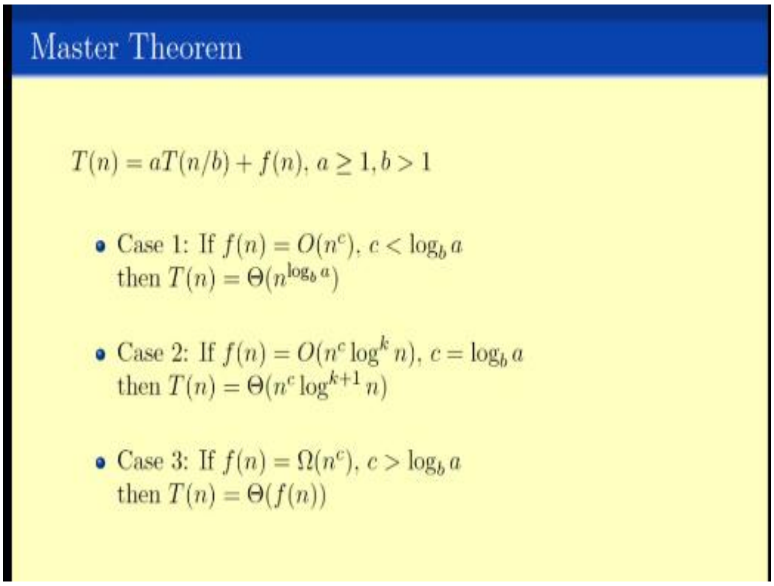

There was also the Master Theorem that was kind of a nice theorem to deal with a quite number of different style of recurrences in one go.

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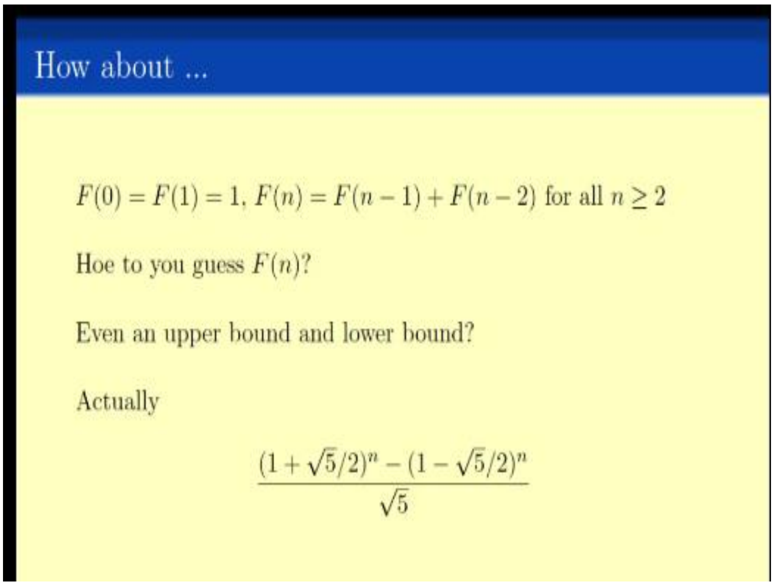

But still there are recurrences, for example, this one Fibonacci number for which we do not know how to solve it neither we know how to solve get a approximate solution and we have been telling it that it is hard to guess and it hard to get an upper bound and lower bound and the reason being that actually a fn is of this form. Now we are going closer and closer to actually proving to you that fn is of this particular form.

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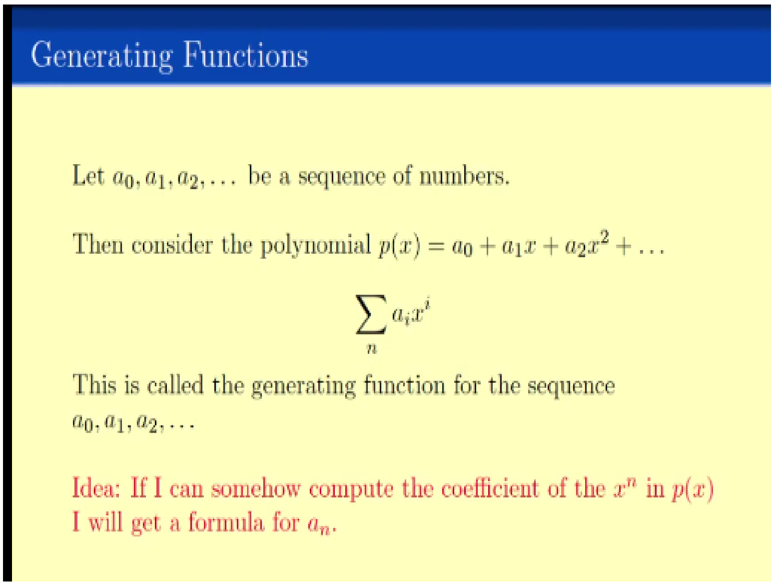

To start with we looked at this new technique called generating functions. The main idea is that if you have a sequence of number A0, A1 till A infinity you first represent this sequence of numbers as a polynomial infinite polynomial which we call some type of power series where P(x) A0+A1 X+A2 X square it goes on or in other words, summation of A(N) X power N.

Now once we have this polynomial, this is called the generating function for this sequence A0, A1 and so. The idea being if we can somehow compute the NF coefficient of X power N in P(X) then I get a formula for A (N). Now in the last video, we saw how this technique can be applied to solve when the recurrence was of the form A = 3 times AN-1. In this video, we will see one more application of this generating function.

And that will be again something that is not new we will be solving the Tower of Hanoi using generating function, but we will see how one can apply generating function there. In the next video, we will be seeing how to apply generating functions to get the formula for Fibonacci numbers.

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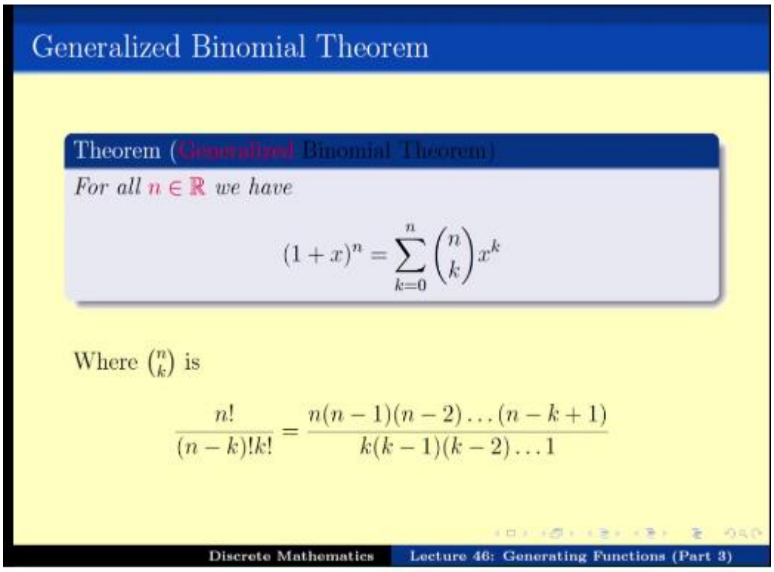

Before we go up on to generating functions one of the crucial components of this technique is to use the generalized binomial theorem to get the Taylor expansion.

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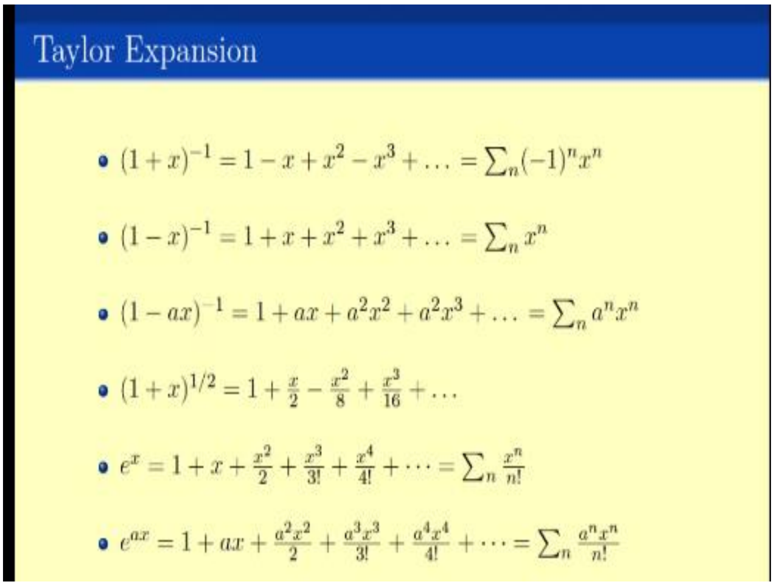

Now once you have the Taylor expansion you do not really need to prove it again and again you can use this Taylor expansion if and when it required. Here is some of the useful Taylor expansion that is there.

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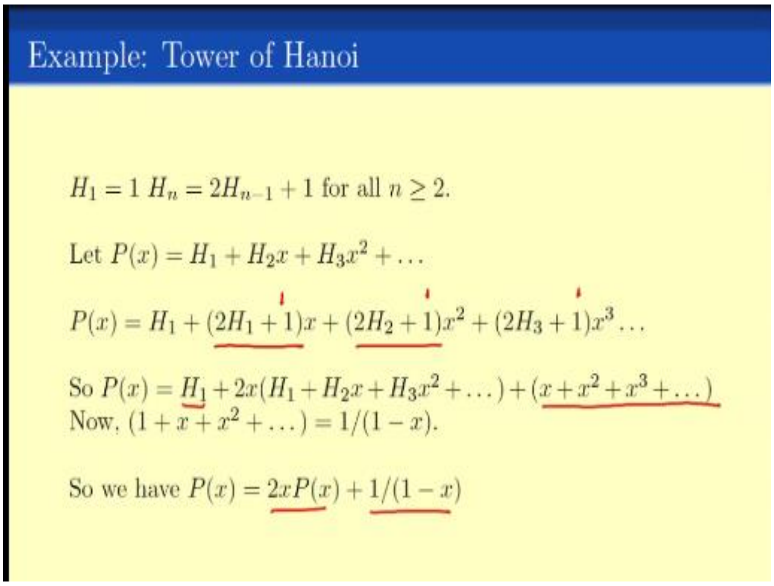

Now let look at the example up to date which is the Tower of Hanoi. This is our old friend he says that H1=1 and HN = 2 times HN-1+1. You remember that we did solve this particular generating recurrence relation using first technique only which is unfolding the definition. But today we are going to solve it in a different style using the generating function. It would be useful for us to see how the generating functions can be used to solve this kind of stuff.

And help us to understand generating function machineries better. Now for generating function we always start with defining the generating function namely the polynomial. So here it is so P x = one more that is here I am defining it slightly differently. I am defining as H1+H2X+H3x square and so on. So that means coefficient of X power N is HN+1 remember that okay that is most important.

I have not defined H0 because H0 is not known. Now again what is the idea is that I can write from H2 onwards I can write it as H2 again 2 times H1+1, H3 is 2 times H2+1 and so on. So this one note that here everything breaks down to 2 times H1X+X, 2 times H2 square +X square. So in other words I can club them together to get H1+2X of course we can take out and then I get H1+H2x +H3X square and so on + the remain like this 1s which is X+X square + X cube + so on.

Now of course like last time this one is nothing, but P (X) so I get before I go on to do that note that this number is what we have done in the Taylor series is 1/1-X. So this +H1 is 1/1-1-X. So I get P of x = twice X times P (x) + 1/1-x and this is the kind of place where we want to be because this is as we saw in last time also. This helps us to say that if I define P(X) like this I can get P(X) = 2X times P (X) + 1/1-x.

Or in other words P (X) = 1/1-x * 1-2X. Now this is a very useful way of doing it because now again we have written down the polynomial or this function P(X) as a function which has nothing to do with returns So all I need to do is to now understand what are the coefficient of this formula or this function and of course we have our Taylor series in handy. This is let me pause here a little bit.

The idea is we started with this polynomial and then we use the recurrence relation to write down the polynomial as right equation on the polynomials and we got something like this P(X) = Twice X P (X) + 1/1-X. And hence P(X) = 1/ 1-X and 1-2X. So it is a very simple thing that I did or this generating functions are extremely powerful tools and this is how we basically goes always.

You started the generating function and then use the recurrence and whatever things that you know to get a function or write the polynomial P (X) as a function of X and once you do it one goal is now reduced just to understanding the nf coefficient of this function that we have. Sometimes that might be easier said and done as you see now, but as we saw in the last time that we could we have something like 1/1-3X and we applied the Taylor series immediately.

So this is where we are now. We shall have written P(X) as 1/1-X times 1-2X and or we want to do is understand what X power N coefficient of X power N in the function. Now in the generalized binomial theorem or in Taylor series expansion one I could write the Taylor series of 1/1-X or I can also write the Taylor series of 1/1-2X. By writing down the Taylor series of product of them is not easy because there is no direct way of writing it.

We do not know the Taylor series of this.

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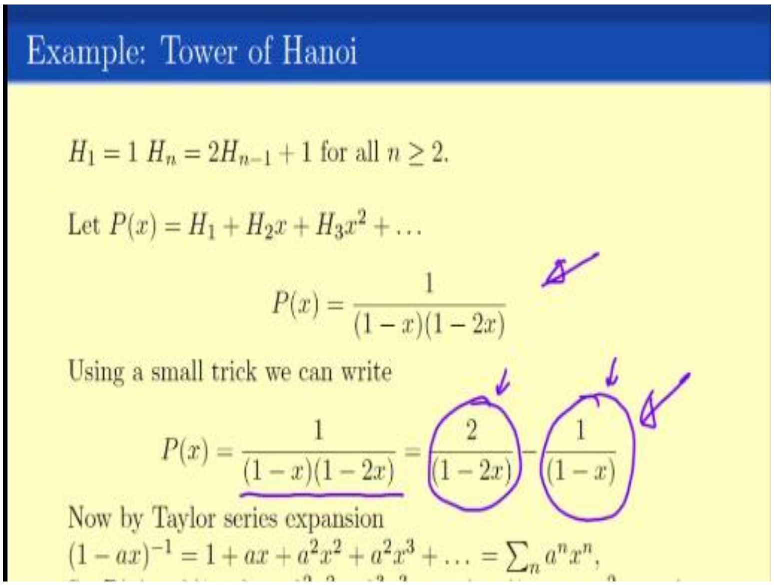

So we do a small trick let see what we do. We started with this P (X) and we wanted to do this one. The trick that we do is I write P (X) so this is P (X). I write this expression 1/1-X is 1-2X as 2/1-2X -1/1-X. Just check out that this is actually true. So in other words this is a 1 by a polynomial of degree 2 I have written it as sum of 2 inverses of polynomial of degree 1. You can check that this equation is right.

And once we see that this equation is right this one I know how to use what is a nf coefficient of this one using the Taylor series expansion and also on the coefficient of the X power N in this function again using the Taylor series expansion and hence in this substraction I will be able to understand the coefficient of X power N by subtracting the coefficient here – coefficient here.

And this is what we will have. For example, P (X) here will become 2 times now just apply the Taylor series here which is 1 by (()) (13:15) 2X the Taylor series expansion 1+2X+ 2 square X +2 cube X cube and so on and 1-X is 1+X –X square or so. So P (x) is this expression and hence you can already see the coefficient of X power N is 2 times 2 power N - 1 and this is what we wanted.

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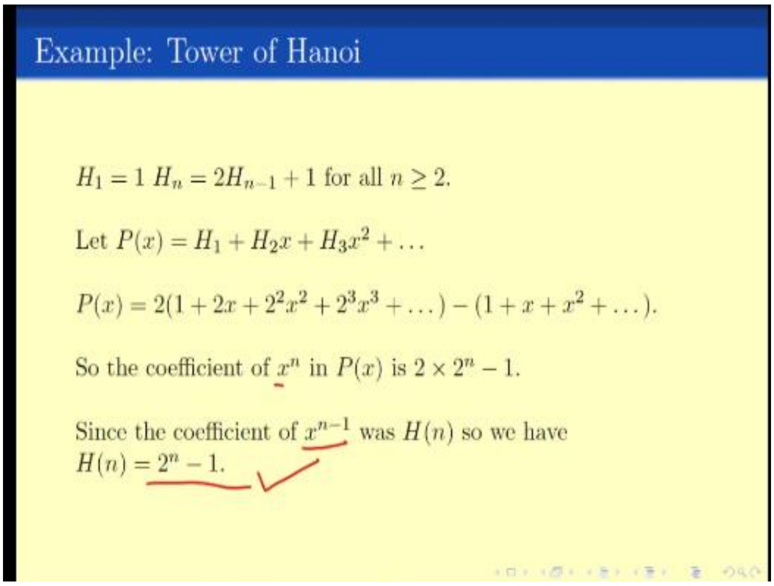

So we have this one and we have got P (X) as this expression. So the coefficient of X power N is 2 power 2 times 2 power N-1. Now since the coefficient of X power N is HN and we are just like it is N-1HN so to understand what is the function of HN is 2 times is the coefficient of X power N-1 which is 2 times 2 power N-1 it is 2 power N-1 and we know that this is right because we have done it earlier also.

The idea is not whether the generating function can solve this one or not, but this is a generic approach to solve the recurrence relations. It is a very powerful approach and we will see in the next video how this approach can be used to solve the Fibonacci number once that will be possibly the high point of this course where we solve the Fibonacci numbers using generating functions.

So we have seen how one can use generating functions to solve Tower of Hanoi. We have seen that we have first represented it as a polynomial then use the P (X) and write P (X) as a function of N and X and once we have a function of X you use amount of tricks and some Taylor series expansion to write the Taylor series expansion the whole thing which helps us to understand the coefficient of X power N and hence get a compact form for the recurrence relations.

In the next video, we will be doing the Fibonacci numbers. Thank you.