Discrete Mathematics
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Lecture - 45
Generating Functions (Part 2)

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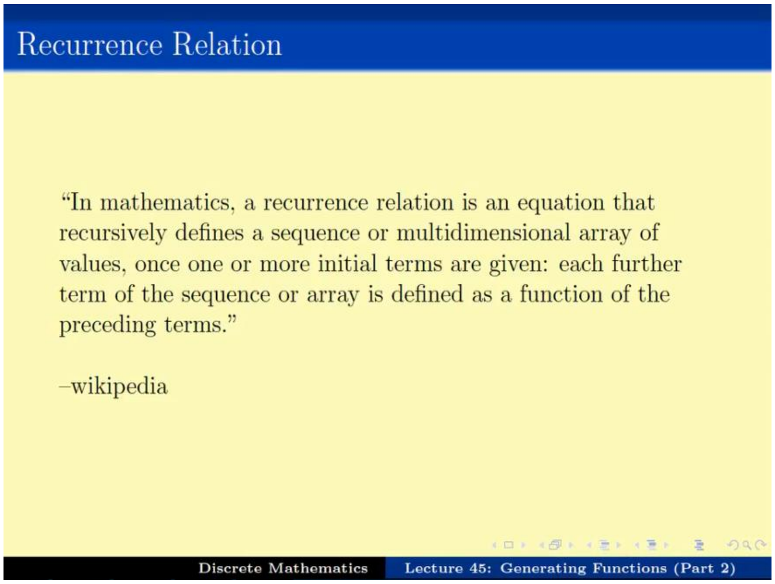

Welcome back. So, we have been looking at recurrence relations and how to solve recurrence relations using generating functions. So quickly recap, so recurrence relations is basically a sequence of numbers where the initial set of numbers are given and another term of the sequence is given as a function of the previous terms.

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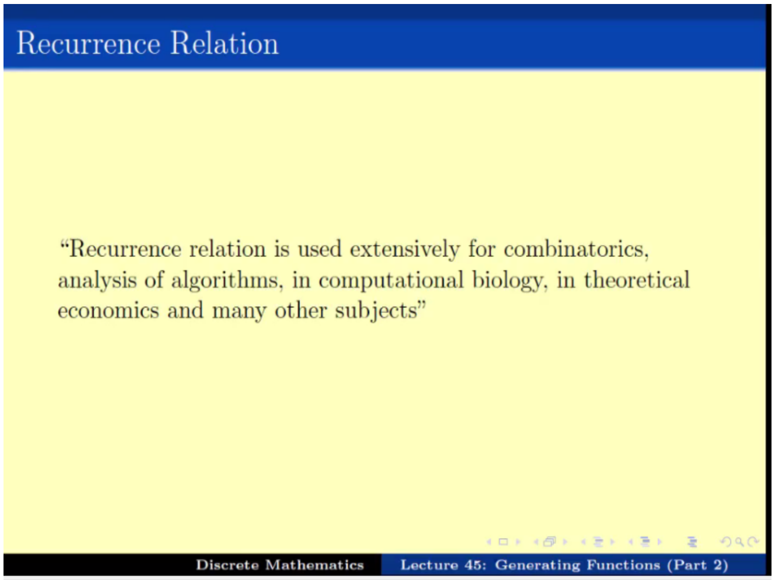

Now recurrence relation is used extensively in various subjects.

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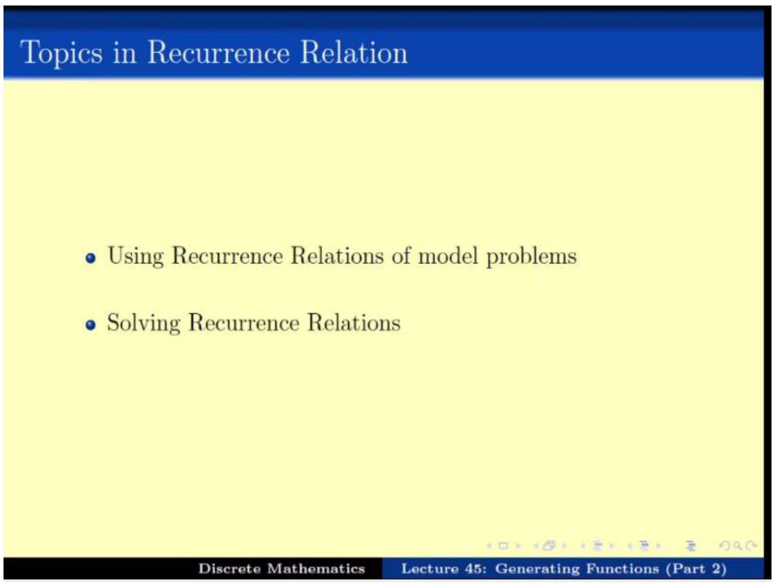

We have seen how recurrence relations can be used to model counting problems and how one can try to solve recurrence relations.

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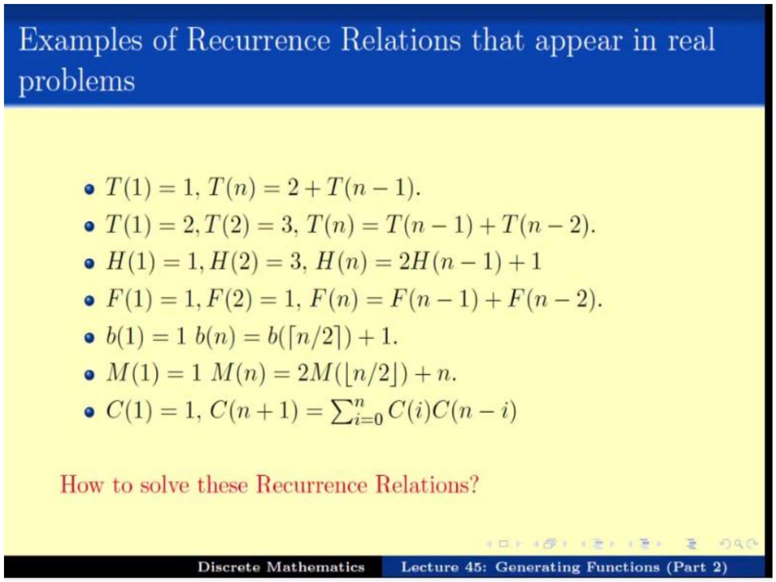

So, there are some of the examples that we looked at and for some of them we have managed to give you ideas how to solve them.

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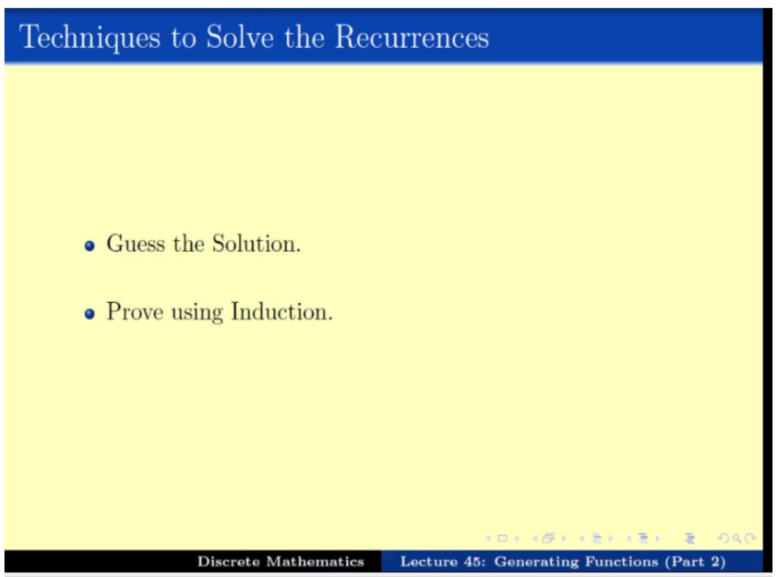

So the first idea was to guess the solution and then prove by induction.

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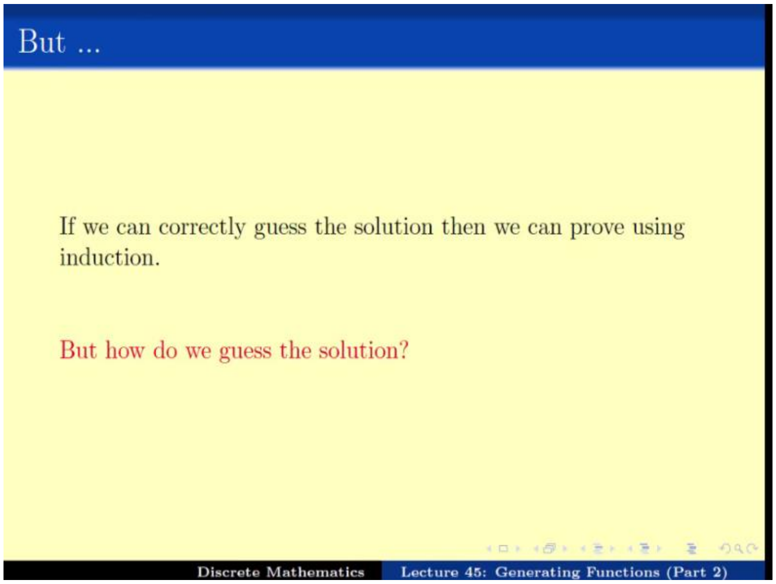

Now of course, if you can guess the solution then proving by induction is a very simple technique. But the main question is how do you guess the solution?

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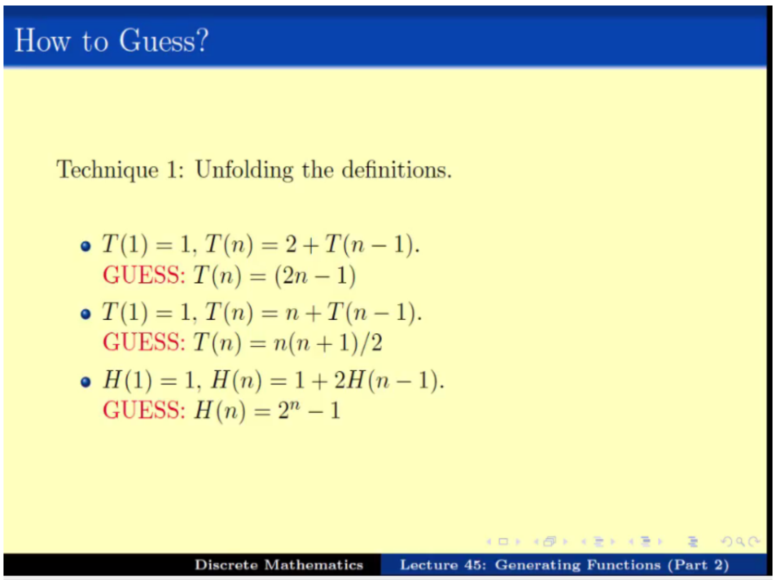

The technique one that was given to you was that you can unfold the definitions and that might help you to guess the solution. We saw some of the examples of that.

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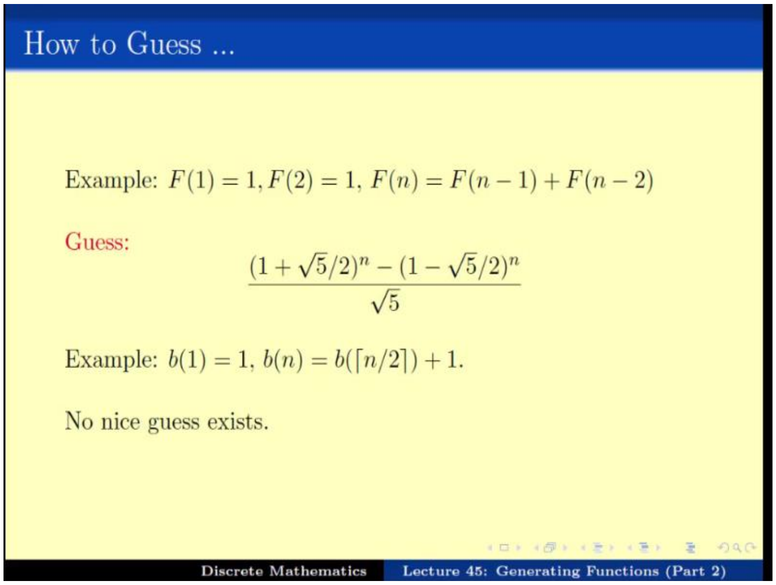

But then there are some problems, for example this one, which is the Fibonacci number where getting the solution is pretty complicated and one of the reasons to believe that it is complicated is that the actual formula of F n will come out to be this number, which is something that of course carry enough and hence one do not expect to guess such a, such a formula by itself. The other one is if the formula has something like this floor or ceiling or some kind of stuff like that in which there does not exist any nice guess.

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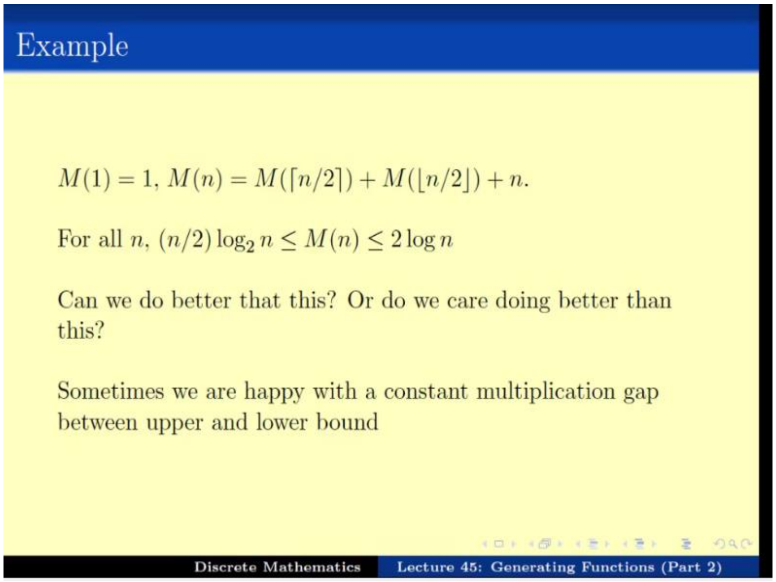

But in the second case, one can use or one can prove some upper bound and lower bound and that might be good enough for us.

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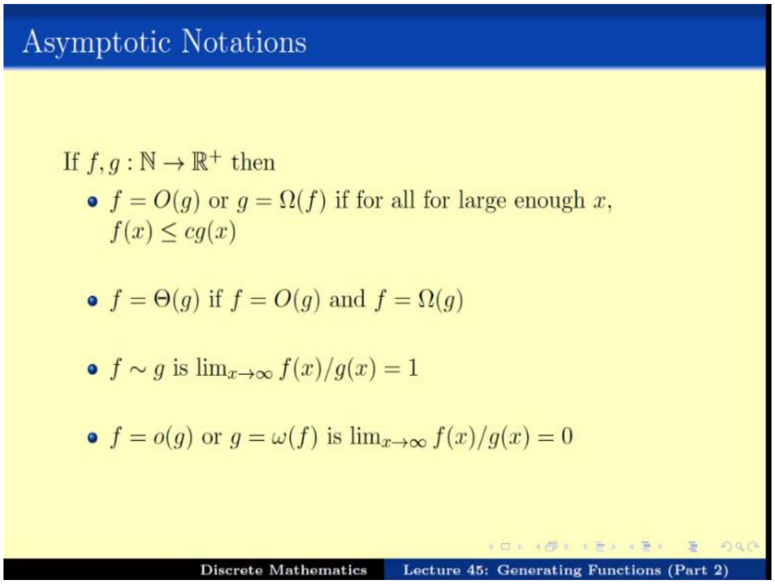

And in fact we can use the asymptotic notations of Big-O, Big-Omega, theta, Small-Omega, Small-O and sim to come up with an asymptotic expressions for the sequences, which is possibly good enough.

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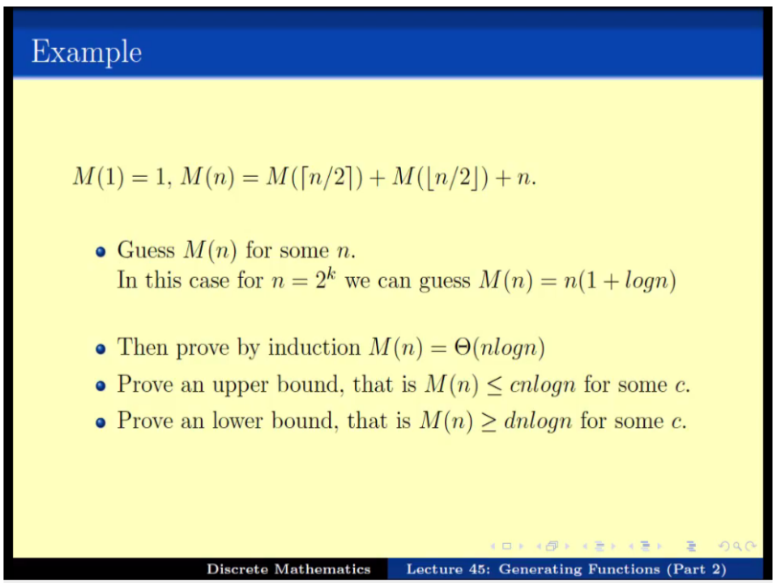

For example, one way of going about this example, is you first guess the M n for sum M, which is the sum of good M. For example, here we saw that we can guess it for n equals to power of 2 and then you fear M n does become something like nlogn and you proved that M n is equal to theta of nlogn and you do it by proving M by using induction to prove an upper bound and the lower bound separately.

Now this is good, if the formula again has a nice expression for some particular M.

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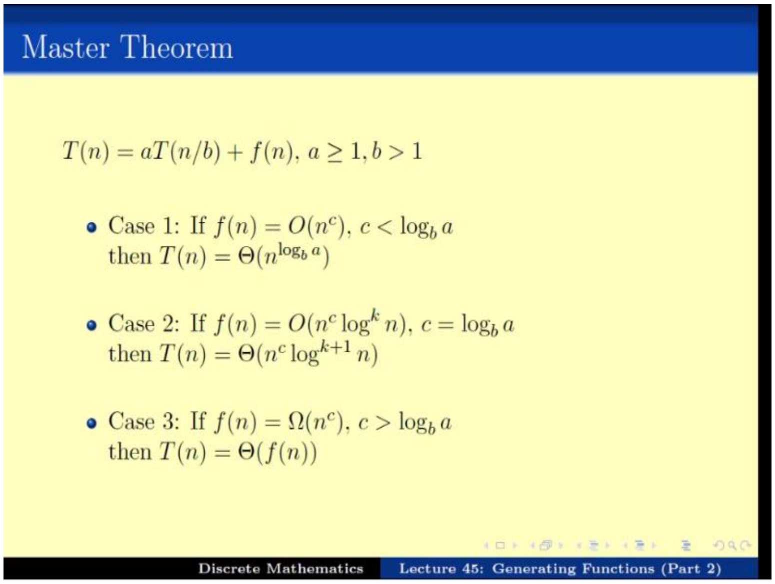

We saw some Master Theorem, which kind of helps us to identify the formula or guess the formula when the expression is of a particular kind.

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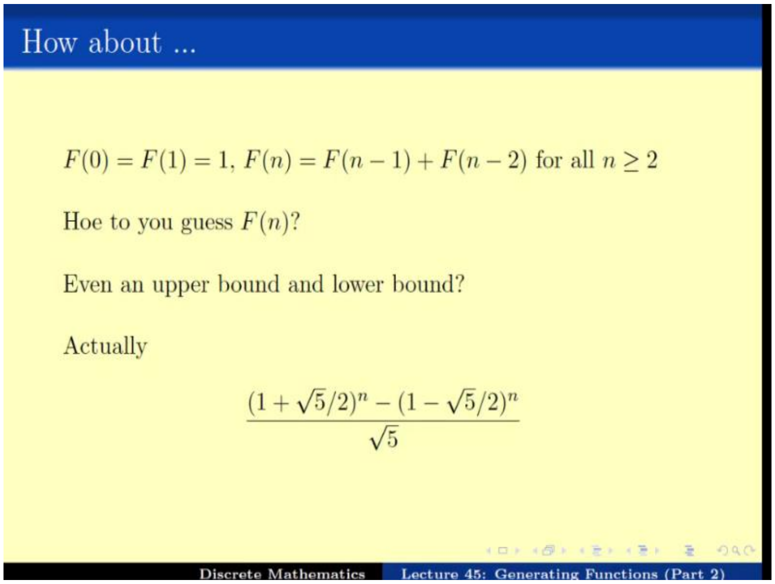

But what do we do when we have a formula like this, F 0 equals to 0 F 1 first 1, so F 0 equals to 1, F 1 equals to 1, and F n equals F n minus 1 plus F n minus 2. Now when it is of this kind, problem is that there is no easy way to guess it. How do you guess F n? Even getting an upper bound and lower bound, which is close tight enough is not an easy job. So this is the particular expression of this F n and it clearly shows that even getting a proper or theta notation for this F n is not a easy job.

Instead what we can do is that we talked about a new technique, which is called generating functions.

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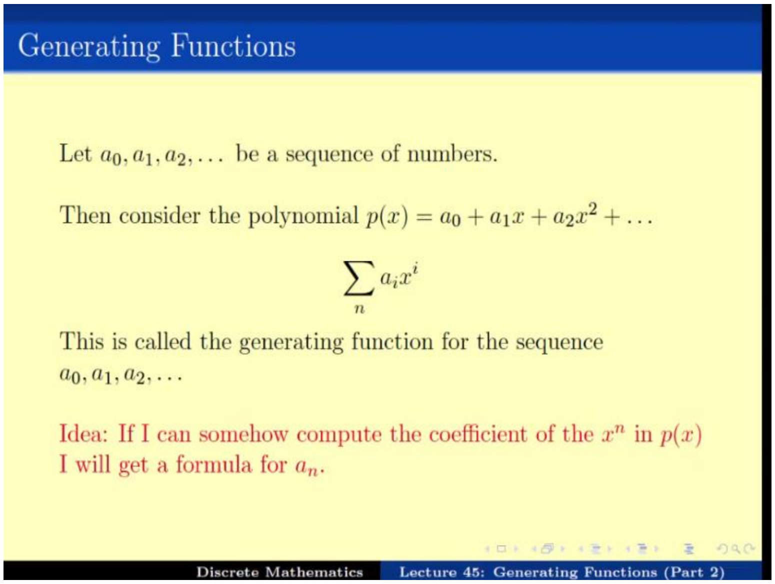

The main idea is that consider a sequence, so in this case you can think of the F 0, F 1, F 2 till F infinity. Now once you have been given the sequence of numbers, you can consider the polynomial, p of x equals to a 0 plus a 1 x plus a 2 x square and so on. So, in other words p of x is summation of a x power n, right. Now this is a polynomial and here this is called the generating function for the sequence a 0, a 1, to a infinity.

Now, so, I have done nothing and I am converting the generating function into a polynomial. Now if somehow, I can compute the coefficient of x power n in this polynomial, I will get a formula for the a n because a n is the coefficient of x power n in the polynomial p x. Now let us see what is going on here or how do we get to get the formula for p, for the coefficient of x power m.

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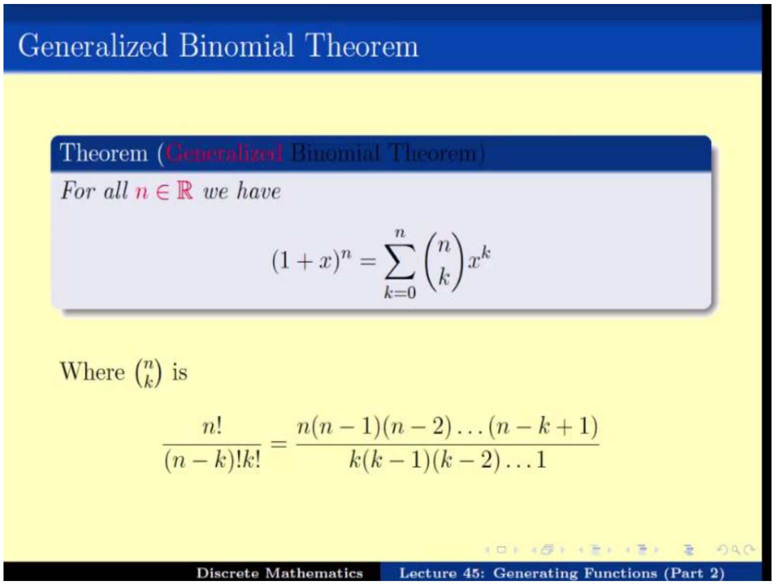

Before that we might have to use some of the Taylor expansion, we in the last class we saw that we can use the generalized binomial theorem

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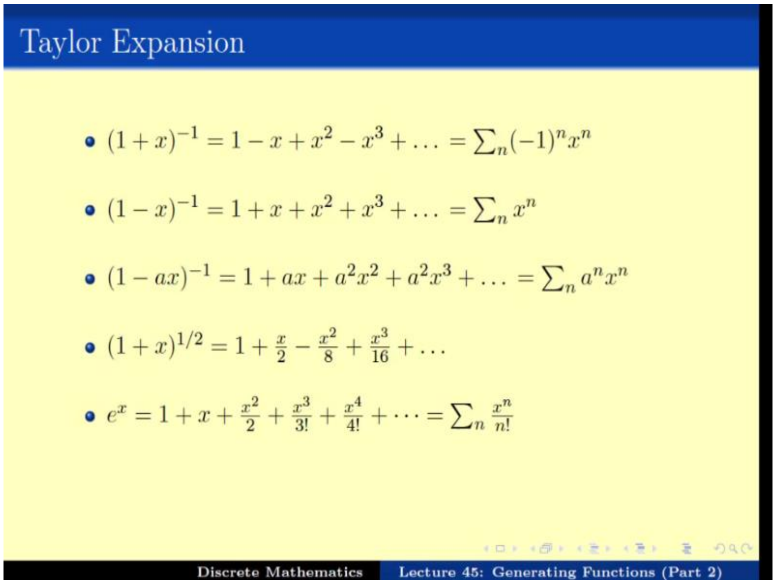

to get some Taylor expansion for some of the, for functions like 1 plus x for minus 1 or 1 plus x, 1 minus a power minus 1 and so on and we will be needing it in due course for this thing.

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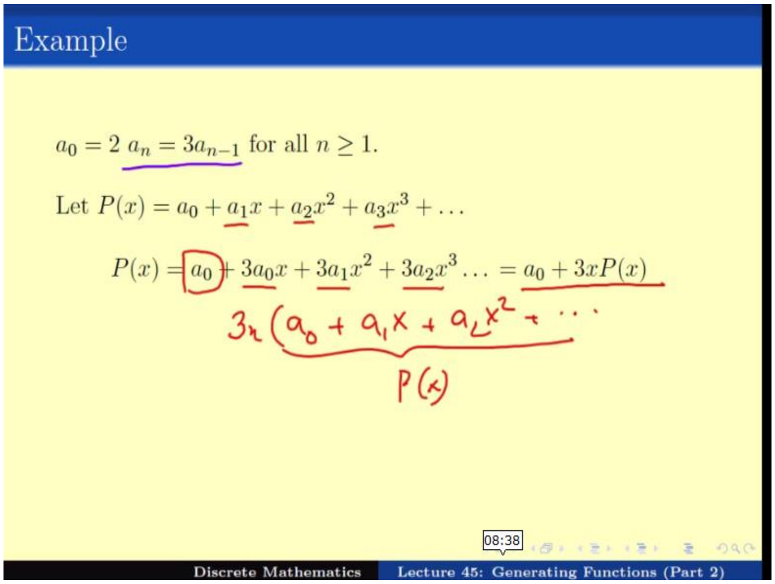

Now let us see how do we use the formula for this exercise, say this is a, this is a simple one, we start with a simple one and slowly we will handle bit more and more difficult recurrences. So this is a simple recurrence, a 0 equals to 2, a 1 equals through 3 times a n minus 1. So if I write the polynomial P of x as a 0 plus a 1 x plus a 2 x square plus a 3 x cube and so on.

Then since I know for all n greater than equal to 1, I can write it as a equals to 3 a n minus 1. So, I can write a 1 as three times a 0, a 2 as three times a 1, a 3 as three times a 2 and so on. (()) (07:45), now let us look at this first one, so this is a 0 and now if I take the 3 x common, what do I get? I get 3 x times a 0 plus a 1 x plus a 2 x square plus and so on, right.

So this is of course the polynomial that we are looking at. So I get a 0 plus 3 x times P x. So in fact what I am doing is that I can write down the polynomial in an equation where I am using the given difference to write this one, right. So this is very crucial. So, you first write down the polynomial and then I write the polynomial using recurrence, but using some nice trick.

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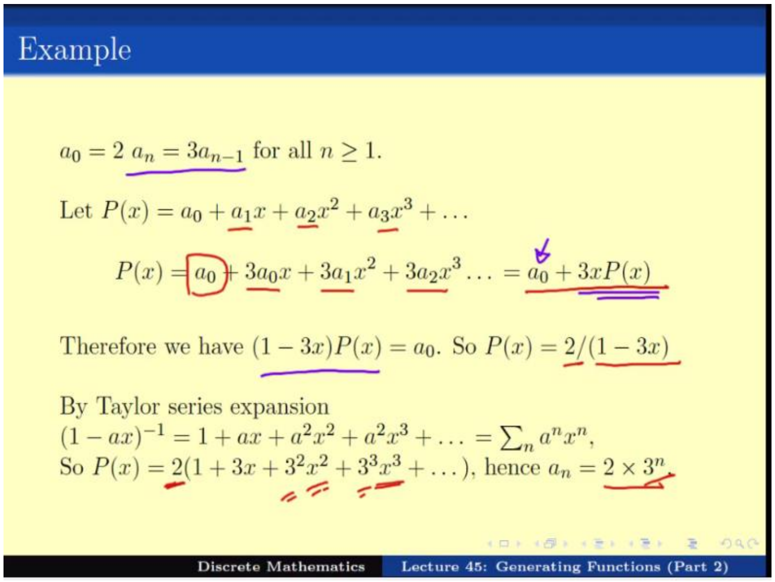

So by doing so, I have, I get a formula they are namely, I can take this minus 3 x, 3 x P x in the right-hand side, I get 1 minus 3x times P x is equal to a 0, right. Or in other words, P x equals to a 0 which is in this case 2 by 1 minus 3 x. So P x was some polynomial, which was written abstractly using the recurrences or the sequence a 0 to a infinity. But now I have written P x as a polynomial that does not have anything to do with the recurrence any longer.

Now you remember this Taylor expansion that we did. We have 1 minus a x whole power minus 1, if 1 plus a x plus x square plus, a cube x cube and so on. So by putting the value of 3 x here, what do I get. So we get that, so this is 1 minus x 3 x power minus 1, right. 1 by this 1, so this is 2, which is this 2 and 1 by 1 minus 3 x is 1 plus 3 x plus 3 square x square plus 3 cube x cube and so on, right.

You can see that here I have used the Taylor series expansion of 1 minus 3 x power minus 1 and by doing so I now know what is the coefficient of x power n? Here the coefficient of x power 3 is 3 power 3 cube, coefficient of x square is 3 square, so coefficient of x power n is 3 power n, so in other words coefficient of x power n is 3 power n times 2 which is 2 times 3 power n and hence this helps us to solve the whole generic currents relation.

So we have used an abstract form known as the generating functions for first writing a polynomial then trying to use the recurrence to get an equation for the polynomial and then writing the polynomial using some Taylor series expansion, which helps us to identify the inert coefficient, which is of my desired thing that I want to do, right.

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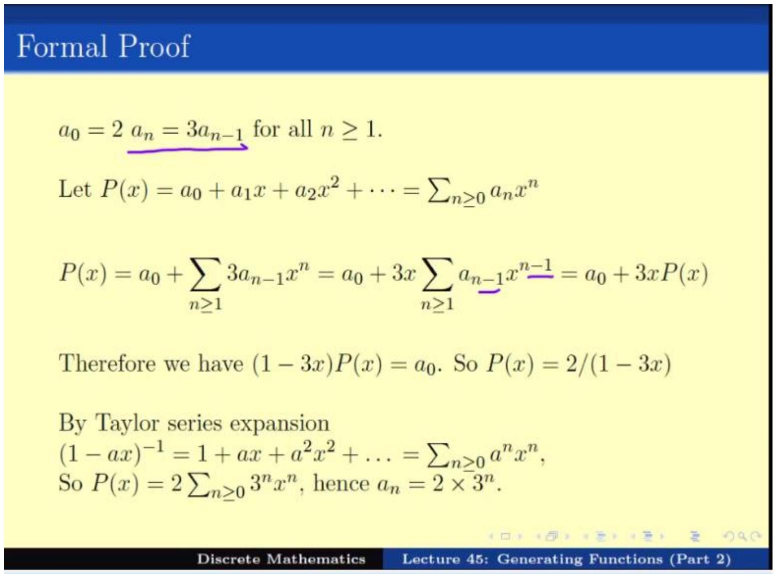

So we will see how to write it down formally once. So a formal proof, you have this one, so the P of x is summation of a n x power n. So P x is equals to a 0 plus, n is greater than equals to 1 in that can I guess a n, which is three times a n minus 1, this is where I have used the recurrence relation, which means I can take 3 n and out, 3 x out, so I get a 0 plus 3 x into summation a n minus 1 x power n minus 1, n is greater than 1.

Now I can change the base for the sum, so n is greater than one and so I have both n minus 1 and n x minus 1. So that means this is exactly n equal to 0 a n x power n, which means this one is polynomial. So this is of the same a 0 plus 3 x times P x and so we have 1 minus 3 x equals P x times a 0 that means P x equals to 2 by 1 minus 3 x and by Taylor series expansion, we can see that the P x equals to 2 times 3 power n x power n, which means a n equals to 2 times 3 power n.

So this is the formal proof for solving this recurrence relation a n equals 3 n minus 1. Now this is not a recurrence relations, I mean you could have solve this recurrence relation in some other way. They are different ways of solving this recurrence relation, but I wanted to use this recurrence relation to show you how generating functions can be used to solving it. In the next two video lectures, we will see how this generating function technique can use be used to solve way more complicated recurrence relations.

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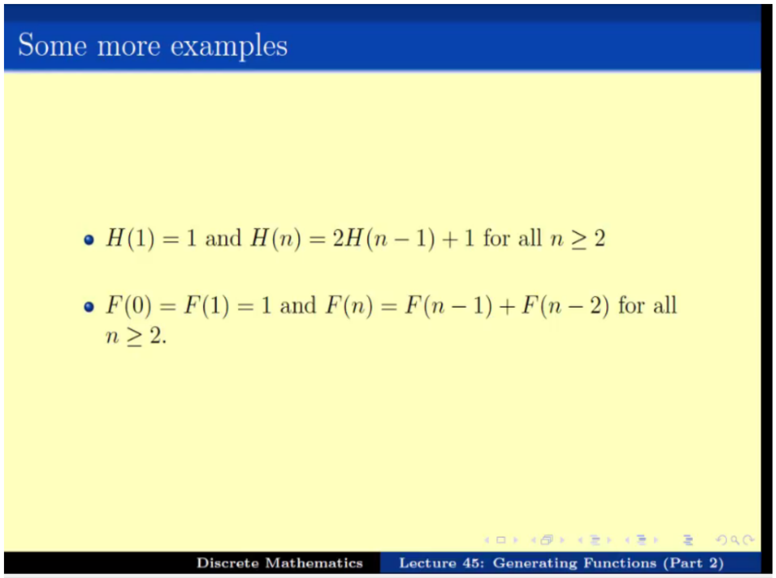

So in fact, in the next video we will see another example, which is the Tower of Hanoi example, how can we solve the Tower of Hanoi example and in the video after that we will see how we can solve the Fibonacci series expansion. In the meantime, if you want to try out please try out your hand on using generating function technique to solve these two things. Thank you.