Discrete Mathematics
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Lecture 42
Asymptotic Relations (Part 3)

Welcome back. So we have to looking at recurrence relations.

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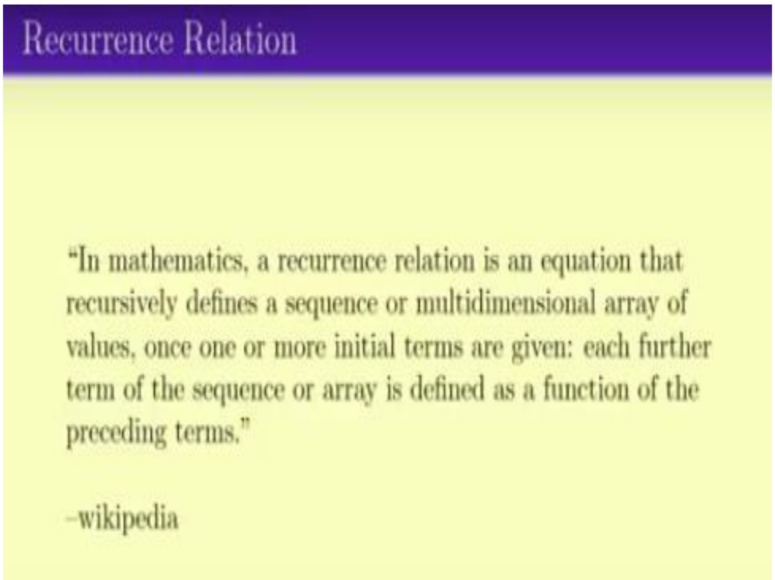

So, recurrence relation is basically a sequence of numbers where the initial set of numbers are given and the nth term is defined as a function of the earlier terms.

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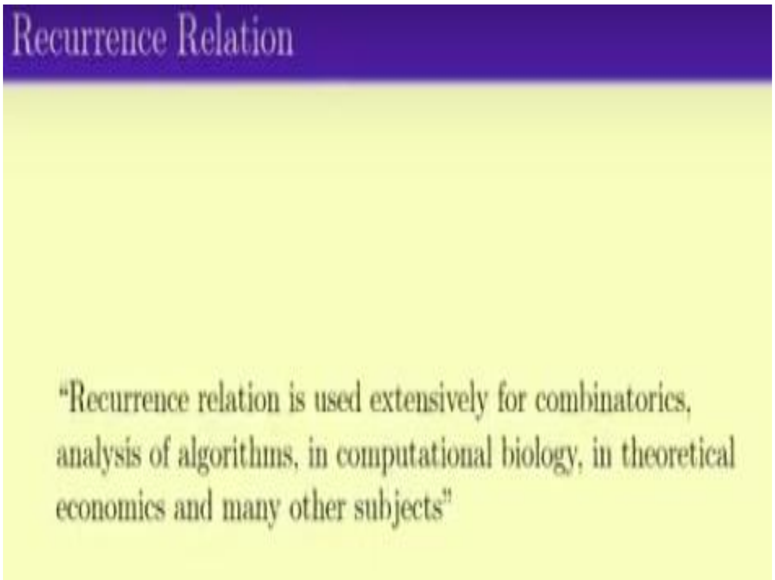

And we have seen that recurrence relation is used extensively in combinatories, analysis of algorithms and various other subjects.

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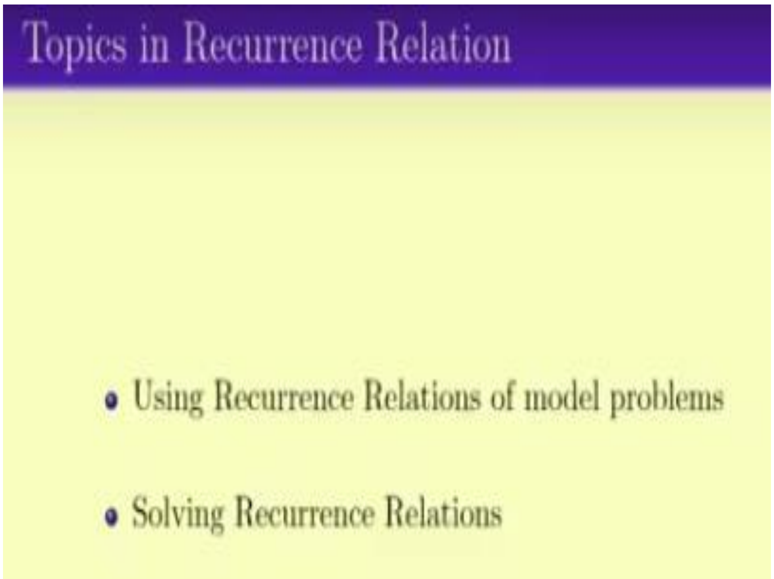

So there are two things. First of all, recurrence relations can be used to model various counting problems and once we model the problem using the recurrence relation, we need to solve the recurrence relation. Now, question is how to solve the recurrence relation.

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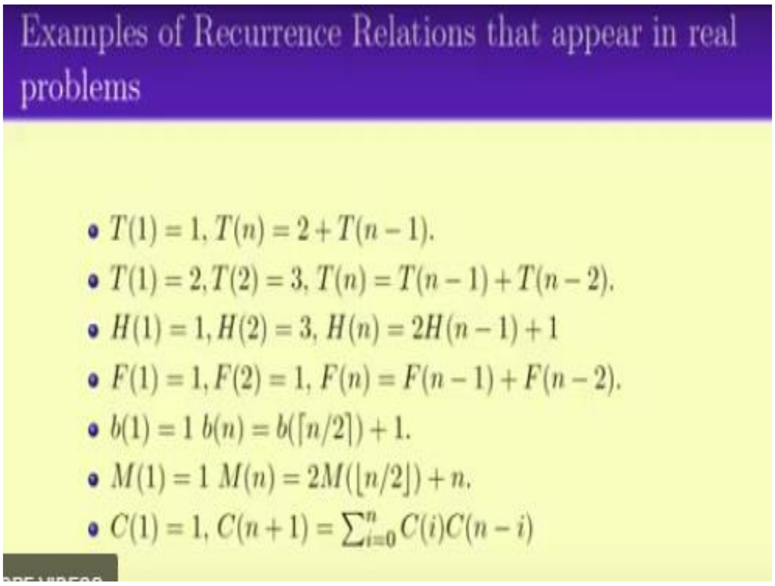

So, here are some of the recurrence relations that we have looked into in the last few videos.

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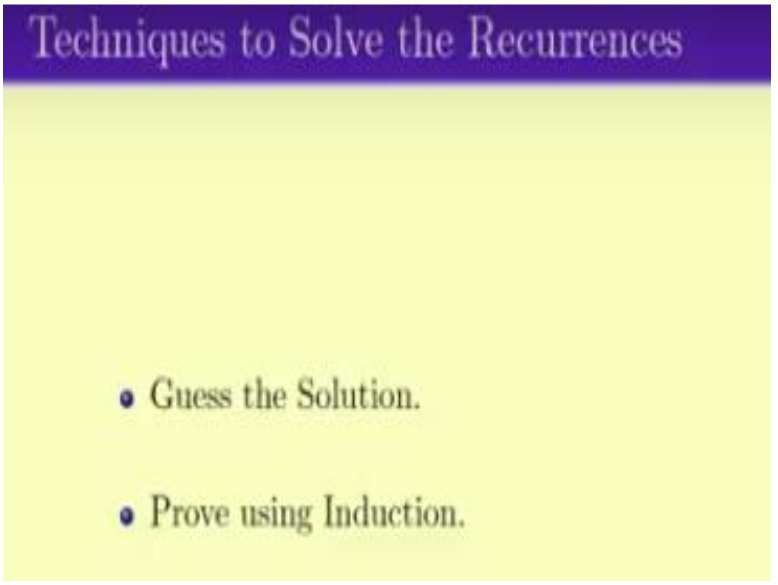

And the question is how to solve the recurrence relations.

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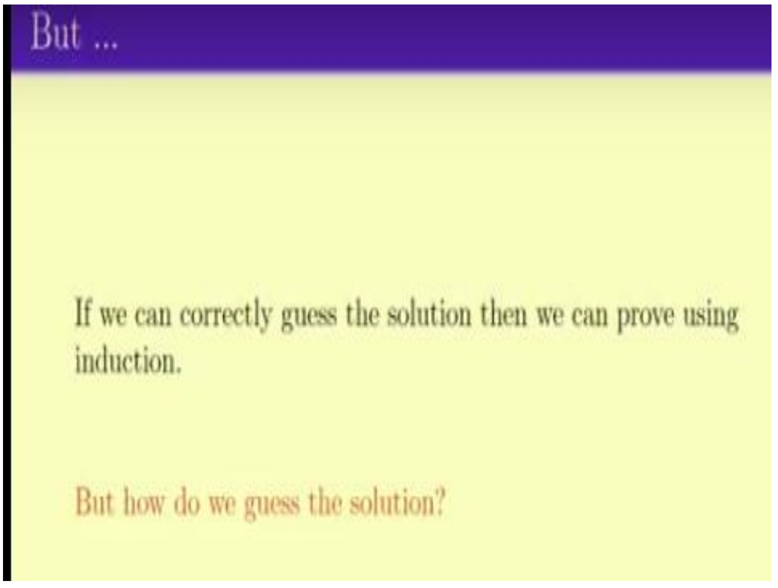

So, the first one that we learned was that we can guess the solution and then prove by induction. Now, this is fine if you can correctly guess it, then you can use induction to prove it and that is pretty standard induction proof. But how do you guess the solution? Now, there is no one way of guessing the solution.

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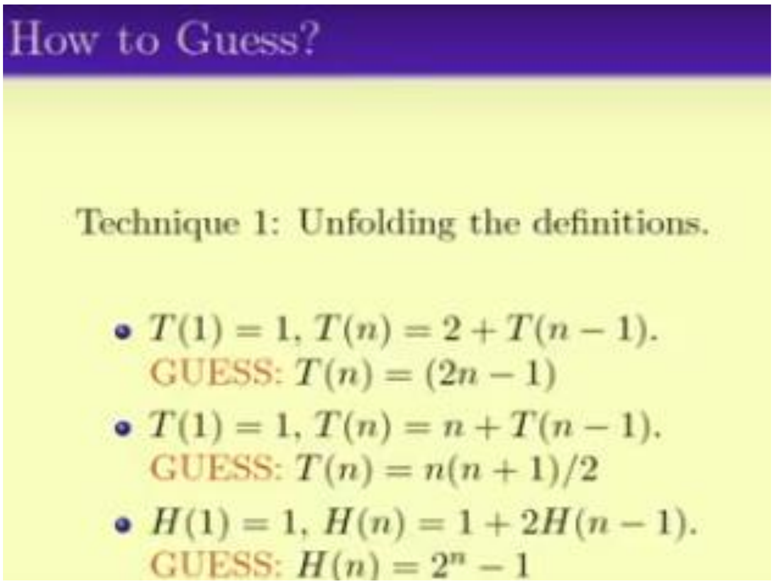

One simple one is what we called as unfolding the definitions. So, by unfolding the definitions, we can possibly guess the solution and we saw how this can be done for these set of recurrence relations.

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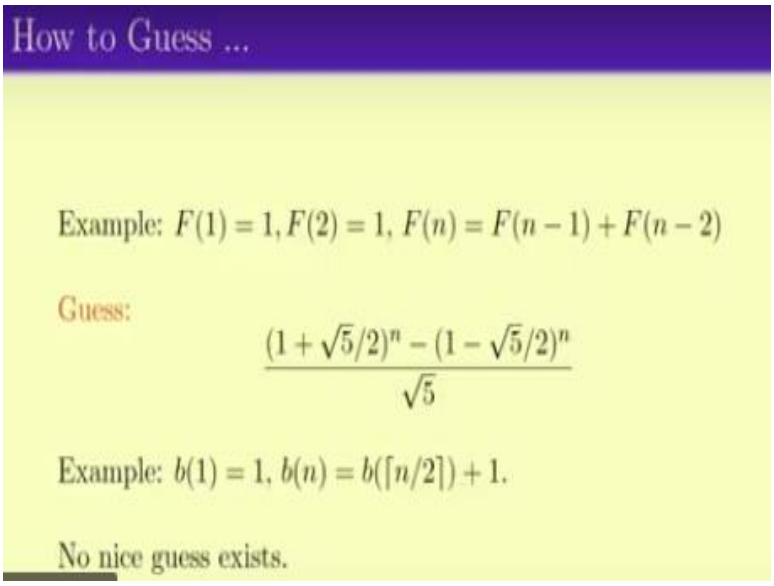

But, then there are recurrence relations which are very hard to guess. One of the reason can be that is the recurrence relation itself is very complicated. What an example, the Fibonacci number, this is a pretty complicated thing. Though, you do not expect to guess this one easily and secondly, there are examples where there are things like a floor and ceiling in the expression and which are pretty complicated to work with also. There is no nice case that can be there.

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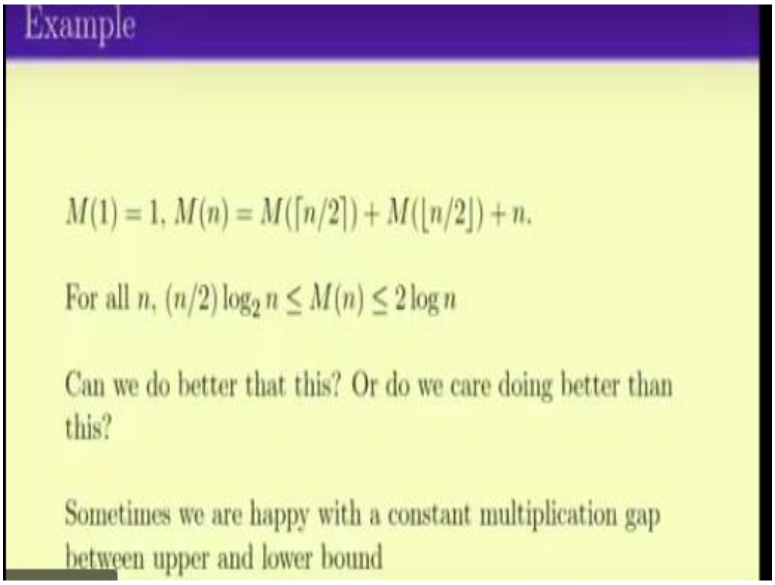

So, we looked at this particular examples, of Mn equals to M ceiling of n by 2 plus M floor of n by 2 plus n and instead of guessing what we could do is we can upper bound and lower bound the Mn by some value and the question was that can we do better than that? So we could not guess the actual value of Mn, but we could upper bound Mn by some value, lower bound Mn by some value and this two values were not too far from each other.

Sometimes, this gap that are there between the upper bound and lower bound which is a 4 times gap. Yes, smaller number that we do not care about improving it.

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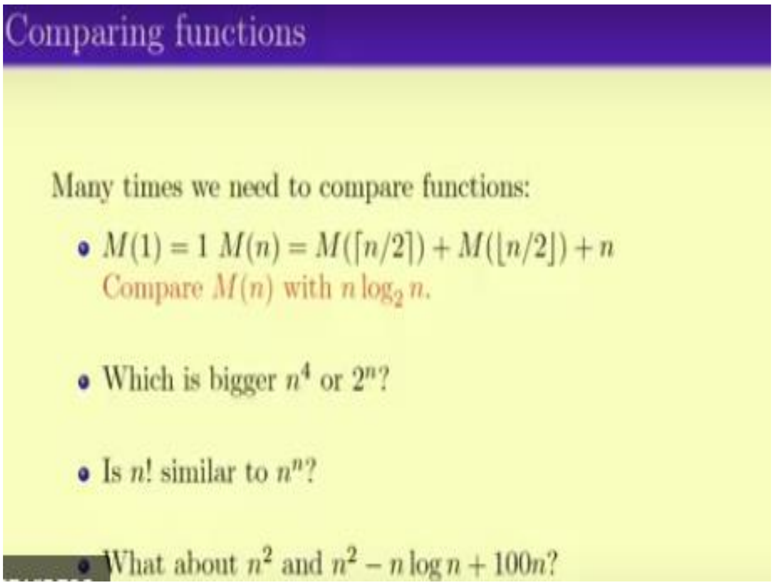

So, how many - how can we compare whether we have to done a good job or not. And for that, we need to work with something of the particular asymptotic notations. So here are some of the questions that are asked, for example, in the earlier case, how can we compare

Mn with n log n or which is bigger n power 4 or 2 power n or is n factorial similar to n power n. And what about these numbers? n square and n square minus n log n plus 100 n.

Now, we know that two functions for - given two functions, we can say these two functions are equal if they are same at all points. We can also say that one is less than the other if it is less than the other and all points and so on. But sometime, if you are not worried about the initial set of numbers, meaning you are only worried about what happens when n is large, when the domain is large and we are only concerned about things up to a constant multiplication.

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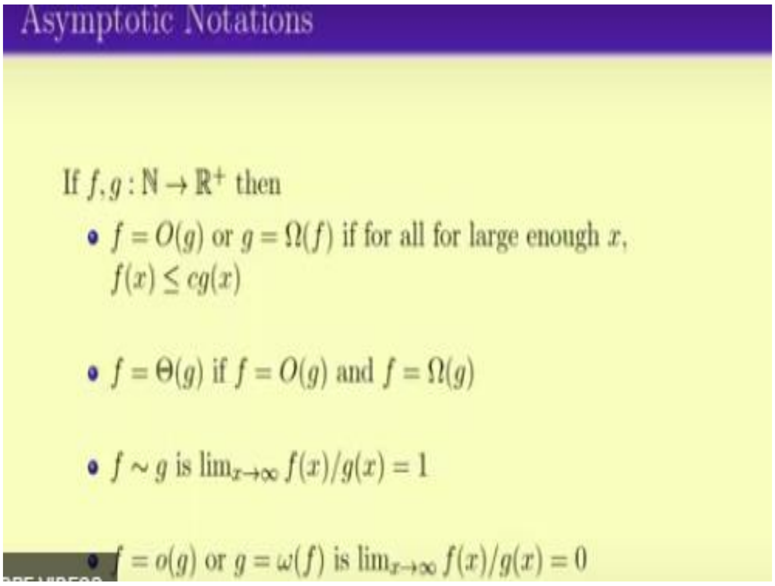

We use what we known as the asymptotic notations. In the last class, last video, we explained the asymptotic notations carefully. For example, we say f is big-O of g, if fx is less than some constant times gx. Similarly, we say f is theta of g, f is big-O of g and f is omega of g, big-omega of g which means, so this big-omega of g means that f is bigger than constant times gx. We have something called f is asymptotically similar to g if fx by g seems to 1.

And with the f is smaller of g if limit as extends to infinity fx by gx is equals to zero. Since that with this set of asymptotic notations in our hand, we can now try to solve the recurrence relations

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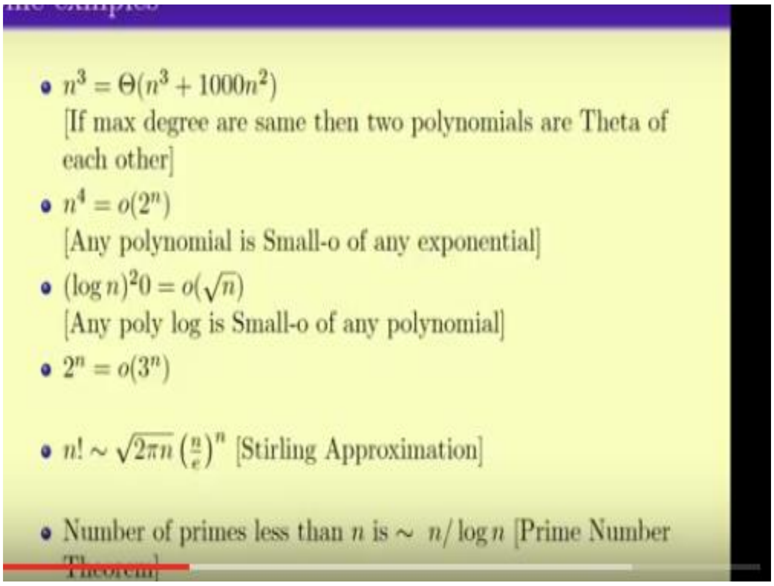

Now, the first thing to notice that, so sorry, there are some examples here that we did last time namely say n - if when we are comparing two polynomials, all we have to look at is the maximum degree. Similarly, a polynomial is always small-o of the exponent, exponential. If a poly log is always small-o of any polynomial, 2 power n is Small-o of three power n and we also looked at school asymptotic approximations to do useful thing.

Particularly in this first one, the n factorial is asymptotically equal to square root 2 pi n times n by e power n.

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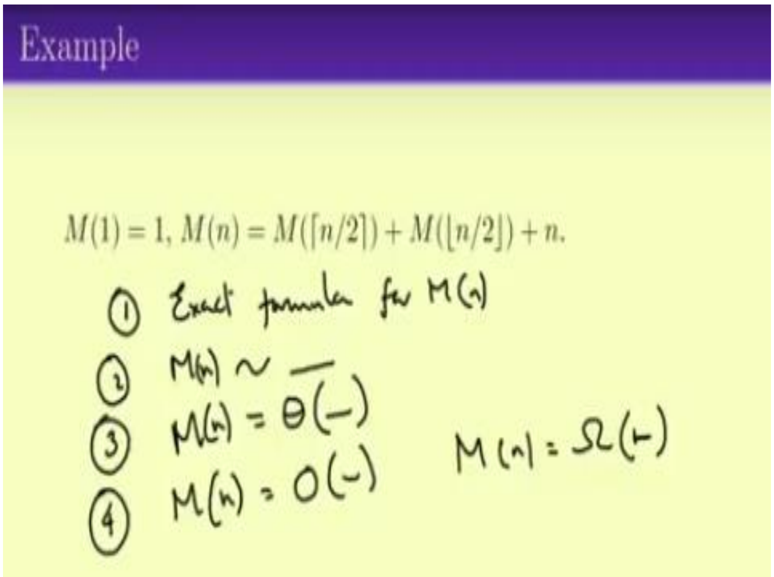

Now, once we have these asymptotic notations in our hand and we have someone has given us this kind of a recurrence relation, we should try to do this solving when we are guessing it, the first thing, if we can is try to give up exact formula for Mn. Now, this is as good as you guess. If you can get an exact formula, that is wonderful.

Now, if we cannot second more that we can ask is can we say that Mn is asymptotically similar to some number that is second option. Third option, okay I cannot even prove that Mn is seemed small asymptotically equal to some option. Third option is Mn is theta of something and the fourth one is okay, I cannot prove Mn is theta of something but can we give you an upper bound and can I give you a lower bound.

And now this is basically the way which you should go along. If somebody gives you a recurrence relation, you should first try to see whether you can come up with an exact formula. If you can, if you can get exact formula and prove it by induction. As we have done in earlier some of the example then great. If not, can you prove as asymptotically equal, asymptotically similar notation or can you prove?

If I cannot prove asymptotically similar, can I prove that it is theta of something and if I cannot prove this that it is theta of something. Can I at least give you an upper bound or lower bound. Do this. Okay? So this is the way in which a recurrence relation is false.

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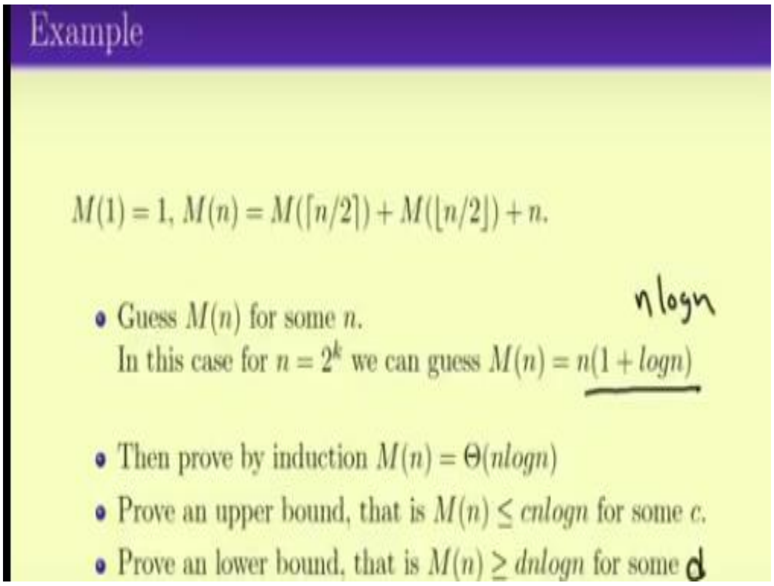

Now, question is that how do you go about solving this one. So, let me give you a couple of examples, we will go through a couple of examples and we will see how one can use asymptotic notations or get solve the recurrence relation using asymptotic notations. Now, once you have been given this one, the first thing to do is can you guess Mn if loss of all n but for some n.

Now, as knew, we did it in this case that when n is the power of 2, we can guess Mn equals to n times 1 plus log n. Now, this is good but what they have is Mn into 1 plus log n, note that it is one plus does not make much difference because the as I told you, it is a polynomial though the (()) (10:17) term only matters. So this is basically nothing but n log n. You can check of course that this value Mn equals to 1 plus log n is not a complete.

It is not the exact solution to Mn for all of them. But, by now, we have got a hand, Mn is somewhat like n log n. The next step to do is by induction proof Mn equals to theta of this number, theta of n log n. What we mean by theta of n log n. Theta of n log n has actually two parts to it. One is an upper bound that we looked to prove that Mn is less than or equal to C n log n for some c.

And then you prove Mn is bigger than some d times n log n for some this is the d, for some d. And once you are done with that, then you get that Mn equals to theta n log n. So I do not want to do this solve these problems right now. I would like you guys to as you are watching this video to solve them by yourselves. Of course, for this particular problem Mn, we have already done it.

We have proved that Mn equals to 2n log n and Mn is thus greater than half n log n. Now that things, important thing know is that how can you choose this c and this d.

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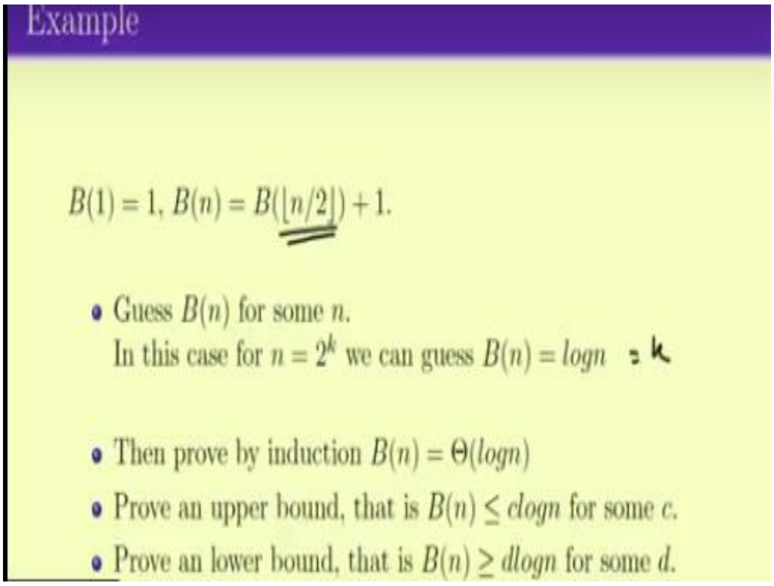

And the way to do it is that assume that Mn is less than equal to c n log n and you go over the whole induction reasoning, the induction hypothesis and inductive step and see if you can come up with some c for which this inductive step goes through. If it can be done then, you have this c log n upper bound. Okay? So here is one more problem, B1 equals to 1 and Bn equals to B floor of n over 2 plus 1.

Again, we need to first guess Bn for some n and as you can see that again since I am getting n over 2 floor. So, we have to somehow choose the n, so that n over 2 is always an integer and I do not have to (()) (13:11) with the floor and so on. And that can happen if n is the power of 2 again. So even here you can convince yourself that and you prove it. I would like to you guys to prove it while you are watching this video that if you put n equals to 2 power k.

Then Bn equals to k which is log, log n actually this is the k and now once you are done again, see that this Bn equals to log n is not an exact answer to begin. So instead of solving it exactly, you prove by induction Bn equals to theta of log n which basically means that you prove that there is an upper bound on Bn which is c times log n and for some c and the lower bound of Bn which is something like d times log n plus some d, for some d.

Now, if actually you get this two things, you get this theta. Okay? If actually you can get the c and d to be arbitrarily close to each other, for example c equals to d plus 1 by n kind of stuff. So as, n goes to infinity, c equals to B. If limit as n goes to infinity, c equals to limit as n goes to infinity. d, then what we have is that Bn will become sim of log n which is as I told you earlier better than getting theta of log n.

An important point here is that you should always reduce when you able to prove this induction thing, you should always reduce into this two parts and prove it individually. When you are proving this induction, do not use any attempt asymptotic notations. Asymptotic notations are just a nice language that helps us to say how the functions going to behave at the in the long run and does not says anything about the exact value of Bn and already (()) (14:45). Okay?

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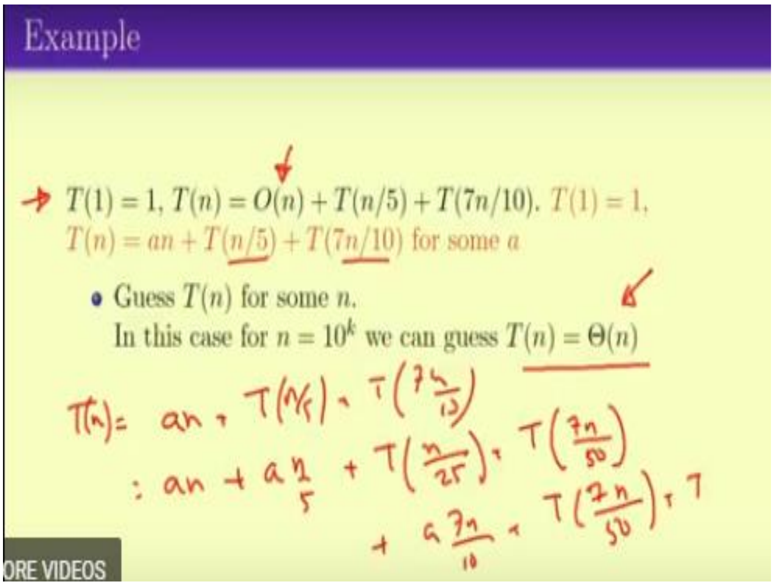

So, here is one more example, here so if Tn equals to Order n plus Tn over 5 plus T 7n over 10. Now what we mean by this statement? When I have something like an order n right in the expression? What it means is that there exists some a for which Tn equals to a times n plus Tn over 5 plus T 7n over 10. Now, this kind of an expression is used a lot in the analysis of algorithms where we and when we count, we are not exactly sure of the constants.

So instead of putting up constants, the assists highly enough Cn or something. We just leave it with a bin with order n. And this is how we are supposed to read it. It is a Tn is equals to an plus Tn over 5 plus T 7n over 10. And now if you have this one, you have to guess again, first thing Tn for some n, now here again as you can see, we have to ensure this, I can guess it by of course ensuring that n divisible by 5, and n is divisible by 10, that means n looks form 10 power k.

And in that case, you would be able to guess that Tn equals to theta n. (()) (17:44) solve it what you will get is that are, so Tn equals to an plus Tn over 5 plus T 7n over 10 which is - if

I now expand Tn, this is a times n over 5 plus Tn by 25 + T 7n by 50 plus a 7n by 10 plus T 7n by 50 + T 7n by 100. Now, the point noticed that by adding this one, I will get a n plus 9 by 10 n plus something which are this term.

Now, if you expand it once more, what you will see is that I am getting some term like 9n by 10 whole square. In fact, I should get exactly that plus something, these are after expanding the second time.

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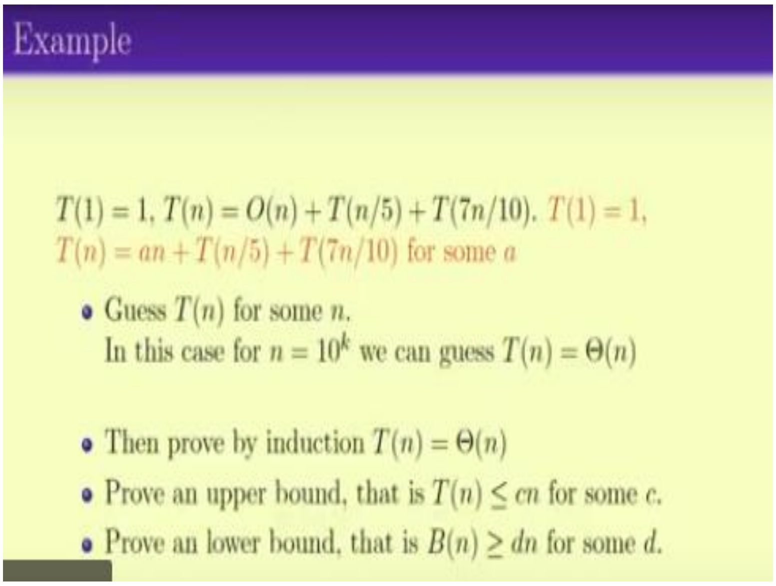

And summation of this an times plus 9n by 10 plus 9n by 10 whole square and so on. It is some number it is a n times something like 10n, 10a or something. So, in other words, you cannot guess this thing exactly again even for this special case of n equals to 10 power 5 and so on. But what you are able to guess is that Tn is something of like theta n. And now prove this one, first of all again, by induction and in which case, you go back to your same technique.

You prove that upper bound of - saying that Tn is less than c times n for some n and then you prove that Tn is greater than B times n for some n and by doing so, you will be able to prove that Tn equals to theta n. Now, this is Tn.

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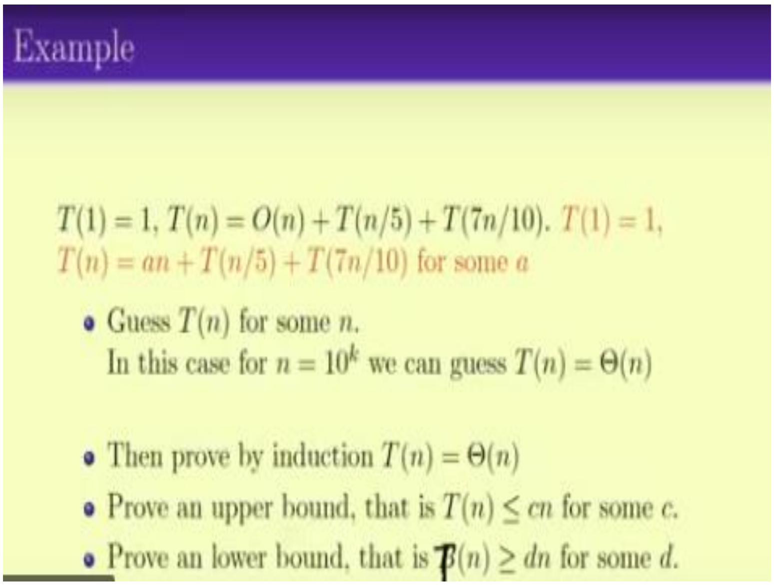

As I told you in the beginning of this course, solving problem is basically the only way of understanding it and here is the template of how to attack this problem and you can go over it. We should follow this template and try to solve these two example that we have given up, given here. In the next class, we will see how to come up with a general technique of solving this problem. You will see a theorem that will help us to forward. Thank you.