Discrete Mathematics

Prof. Sourav Chakraborty
Department of Mathematics
Indian Institute of Technology - Madras

Lecture-39
Solving Recurrence Relations (Part 2)

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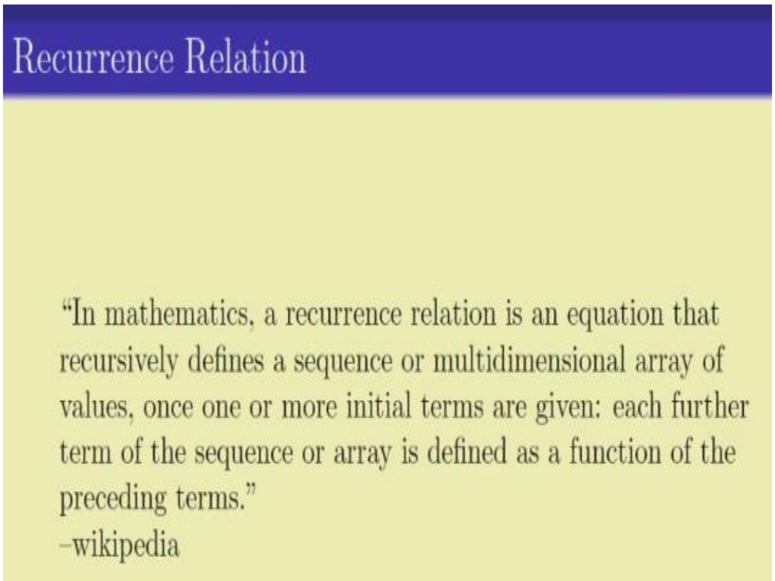

Welcome back, in the last few videos, we have been studying recurrence relations. So recurrent relations is basically an equation that recursively defines a sequence of values. There are some initial terms and the nth is defined as a function of the preceding terms. So, recurrence relations had been used extensively for combinatorics, analysis of algorithms, in computational biology, in theoretical economics and in various other subjects.

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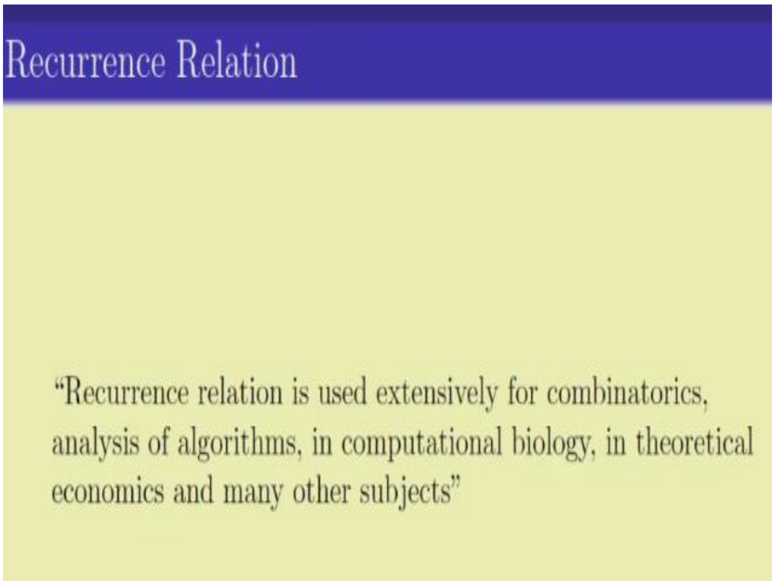

In the last week, sorry, couple of videos, earlier we saw how to use the recurrence relations for modelling, some of the counting problems.

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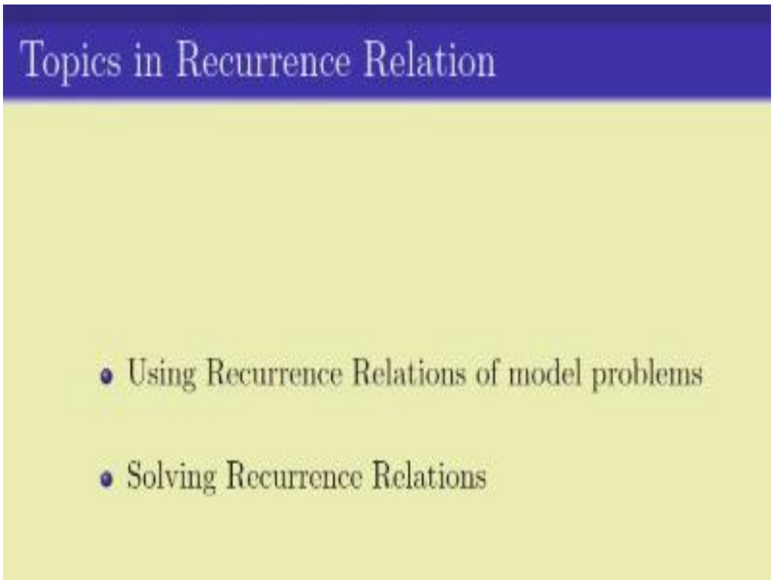

Now once you solve the, when you model some of the counting problems, you have to solve the recurrence relations in some way. So you have a some of the examples that appears in real life, say for example, T(1) =1 and T(n)=2+T(n-1) or T(1)=2 and T(2)=3 and T(n)= T(n-1) + T(n-2) or this is the one that came from the Tower of Hanoi problem.

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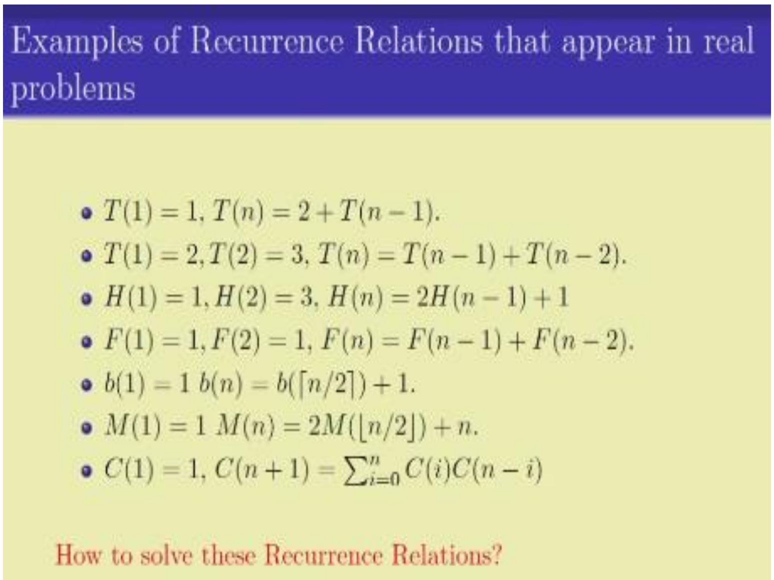

T1=1, sorry, H1=1, H(2)=3 and H(n)= 2 times H (n-1) +1, or this is the one that come from the Fibonacci sequence, F(1)=1, F(2)=1 and F(n) = F(n-1) + F(n-2), or this one that come from the binary search algorithm, b(1)=1 and b(n)= b n over 2 + 1, or this one that comes from the merge sort algorithm M(1)=1 and M(n)= 2 times M and n over 2 + n, or this one comes from, what is known as Catalan number C(1)=1 and C(n+1) = summation i= 0 to n C(i) C(n-i).

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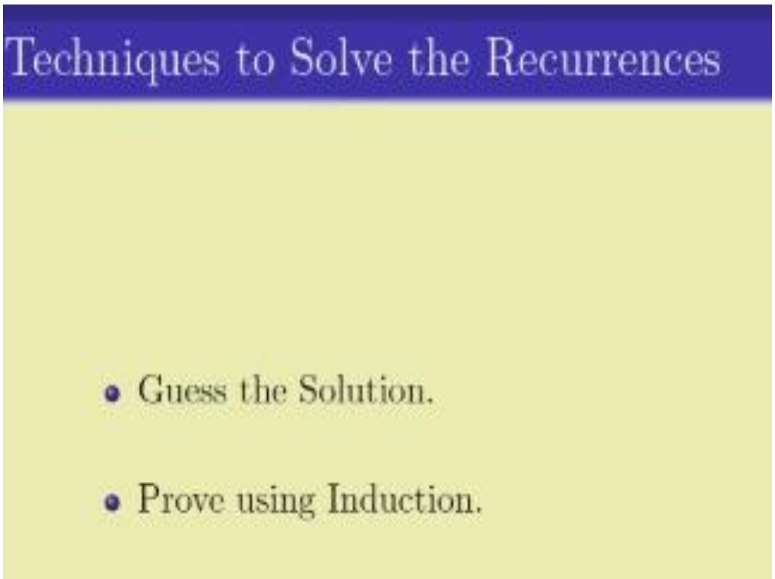

Now these are some of the recurrence relations that appears in a real life, these are the very small sample of them. Now the main question is how do you solve this recurrence relation? So this recurrence relation are supposed used to module various problems, but once you model them into a recurrence relations, the next step is to solves it. In the last video, we saw a technique of solving them.

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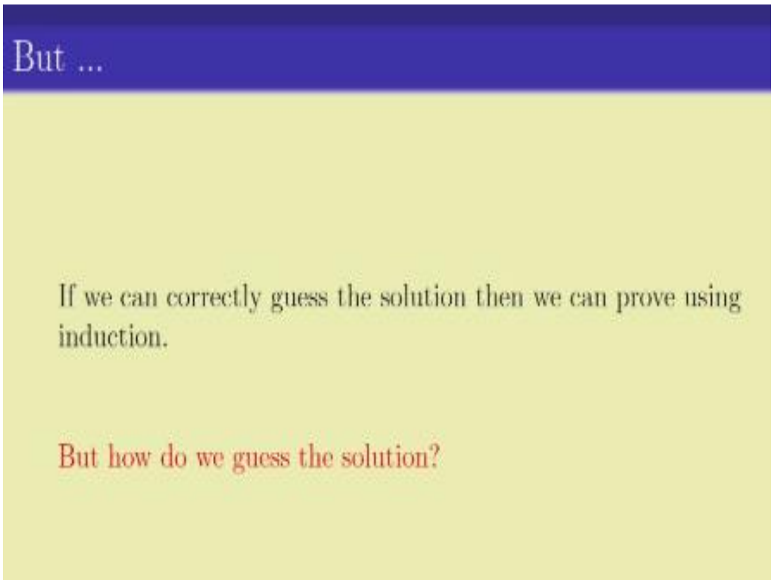

We told these are the techniques, that first of all guess the solution and then proves using induction. We saw that if we can guess the solution correct, then proving it by induction possibly not to hard a problem, it is like a difficult induction problem. The main issue is how do we would guess the solution, now guessing the solution can really be a challenging problem. So we will be dedicating quiet number of lectures on guessing the solution.

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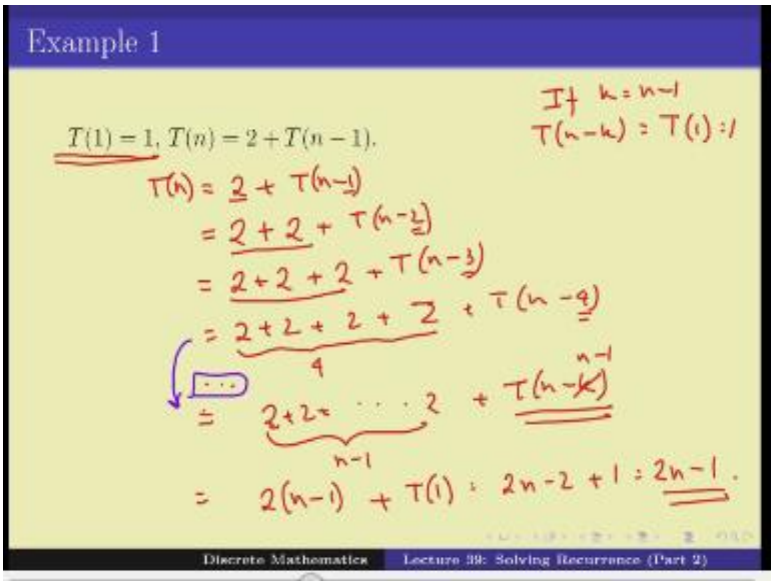

Today we will be looking at the first and the simplest technique of guessing the solution. So here is it, so technique one, the idea is just infolding the definitions, but what we mean by unfolding the technicians? So let us look at some of the examples and you will understand what really mean by that? Note that these are not a formal proofs, these are near guessing which might work or might not work and whether it works or not, of course you have to go back to the induction and prove it and only then we get a formal proof.

So this is whatever I am going to say it now is how to guess this step? Say if T(1)=1 and T(n)=2+ T(n-1), so let us write down here, T(n)= 2+ T(n-1). Now what is Tn-1, I can (()) (05:04) that now open up, right? so Tn-1 is 2+T(n-2), Okay, let us write down once again, 2+ T(n-3), okay, let us write down once again, 2 + T(n-4). So this is where comes up the big leap of faith, your kind of say that okay when there are some 4 of the 2’s here, I have a 4 here.

When I have 3 of the 2’s here, I have 3 here. When I have 2 of the 2 here, I have 2 here. When I have one 2 here, I have 1 here. So if I have keep on doing this way, I will get a 2+2+, for some k of them + T(n-k), of this is a leap of faith. Okay, again as I told it is a guessing work, right. Now this number is somehow to vanish. The idea is that, initial things, here T1=1, gives as a hint, so we have to set, so here if k=n-1, then T(n-k) = T(1), this is =1. So I have to put this one as n-1.

So instead if I remove this k here and instead I write here as n-1, what will I get? Here I should get 2 times n-1, because I have 2(n-1) + T(1), which is 2n-2+1, which is 2n-1, okay. So this by doing so we have guess that Tn=2n-1, again also it might seem very formal way of proving that T(1) =2(n-1), the fact is that it still not are correct proof or complete proof, because we have these dot, dot, dots here. There was a big leap of faith from this to this.

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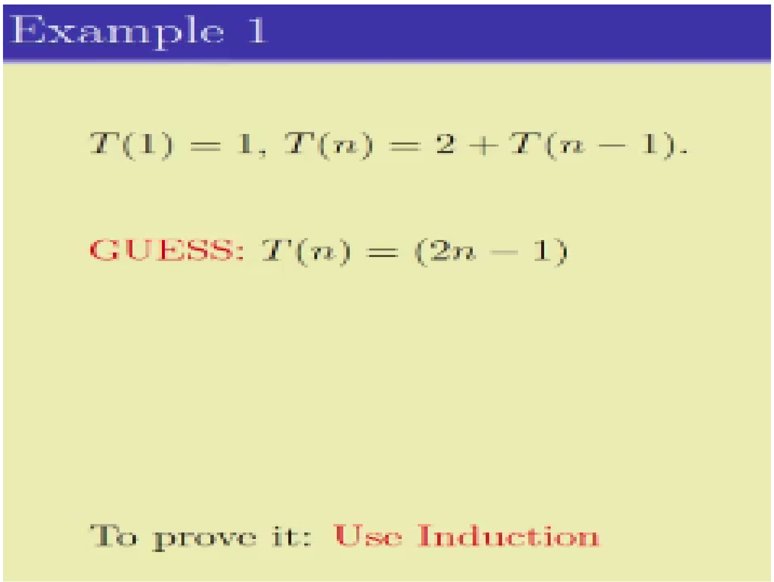

May be our intuitions are correct and we get the right answer and in which case we go ahead and prove it using induction, right; and there are times, there are examples where this leap of faith may not be exactly correct. But this is the one way of kind of guessing what the number is. So in this case, we have Tn= 2n-1, is the guess and you prove it by induction. You had seen in the last video that this is indeed right way of guessing it and we have or in right guess by proving it by induction.

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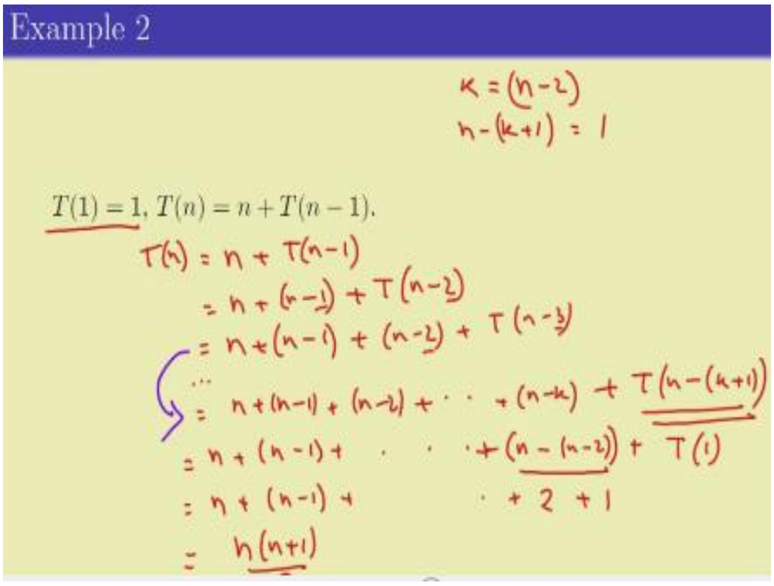

Let us move on to the next example, so here T(n)= n+ T (n-1), again we have to let us keep on unfolding a definition, so T(n)= n+ T (n-1), unfold this Tn-1, this is (n+1) + T(n-2). Now again if I unfold it more, this is (n-2) + T(n-3) and here again now let us do a leap of faith, as you see here, when I have 3 here, I keep on doing it till 2, when if 2 here, I keep on doing it to 1, so may be if I keep on doing this thing till n-k, I have Tn-k+1, right. Sorry, T(n-(k+1)).

Now again we have to somehow disappears this term, so the idea is again that we have to get this n-k+1 as T as 1, so in other words, if I take k to be =n-2, right, what we had? n-(k+1) =1, so in that case what should we get is that, I keep on going it, +n-k, which is n-2 + T(1), now T(1) =1 and this keeps on going and this n-(n-2) is nothing but 2 and the T1=1, so in fact we get the sum over the first n integers, which is n(n+1)/2.

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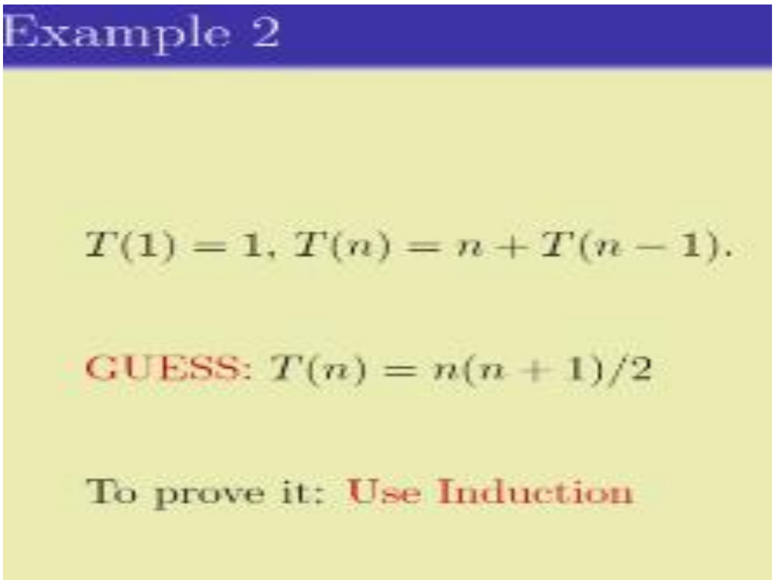

So by doing so, we have guess that T(n)=n(n+1)/2, now again as I told you this is a leap of faith, because there was a leap of faith here and hence this is just a case, so formally proving it we have to solve it by induction and verify that or guess it indeed, right. So again the simple idea is, keep on unfolding the definition and it will be possibly we able to guess the value and in this case we did guess Tn=n(n+1)/2 and we then prove it by induction.

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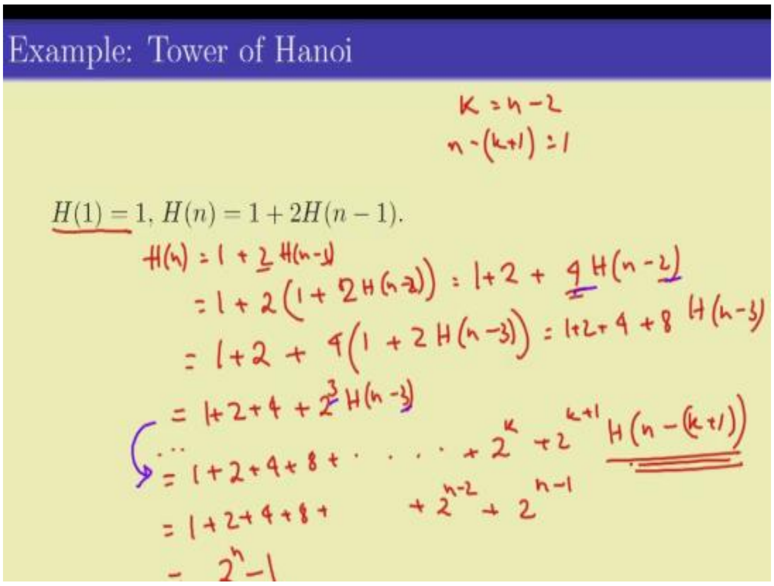

As I told you, most of the time the guess does what correct if we can unfold it in a like way. Let us look at one more example, is the tower of Hanoi problem and where H(1) =1 and H(n)= 1+ 2 times H(n-1), here H1=1+2H(n-1), so this is actually quite interesting, so this one is 2 times now here, 1+ 2times H(n -1), sorry (n-2), which is 1+2+2 times H(n-2), sorry not 2 times, this is in fact 4 times. Let us I open it again, 1+2+ 4 times, what is H(n-2)?, is 1 + twice H(n-3), which is 1+2+4+8H(n-3).

Now this where you really have to take a leap of faith, so let us write this one here, 1+2+4+, what is H? H is 2 power 3, H(n-3), note that here it was also n-2 power 2 and I had 2 here, if 2 power 1, and I had 1 here. So again by the complete leap of faith, we can write it as 1+2+4+8+2 power k+ 2 power k+1 and when k+1, I had 8n-k+1. Now here again we need to make this one disappear, so again I have 8n+1, then again take k to be = n-2.

If I take k=n-2, then n-k+1 =1, so when this becomes t1, this becomes t1, which is 1, so I get 1+2+4+8+2 power K which is n-2+2 power n-1 times T(1)and since T(1)=1, so I can forget this statement, and so I get this number, which is a GP series and the GP series at the 2 power n-1. So I guess that 8n=2 power n-1, so this one clearly was slightly more complicated than the earlier ones, but again here there is a massive leap of faith here.

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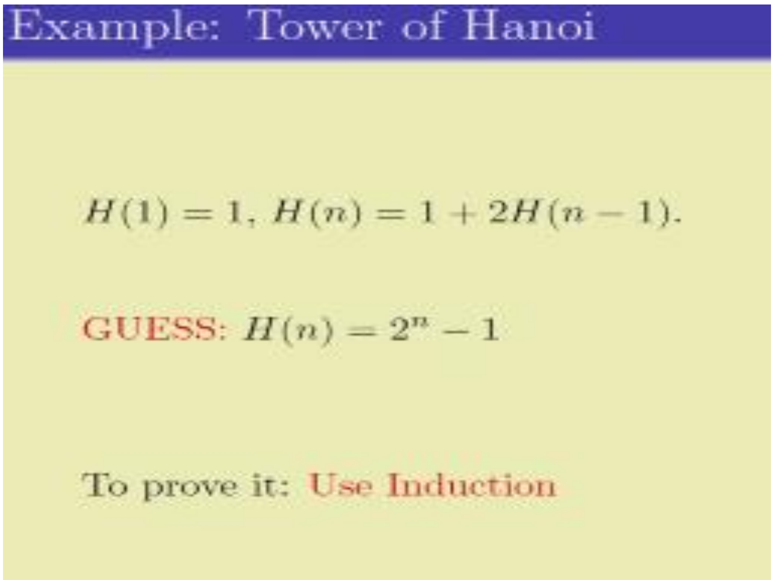

When we guess that, again this 3,3 and similarly here 2, 2 and so on exist and so doing so we have manners to guess it, but we need to proves that the guess is right again by induction. It so happens that in this case, we guess it indeed right and we saw it last time that we can guess this one and we do get the induction, by induction we can prove this. So the basic idea that we learned from this video is that if you are given a particular recurrence relation like this and if you can unfold it, maybe you can try to guess the number.

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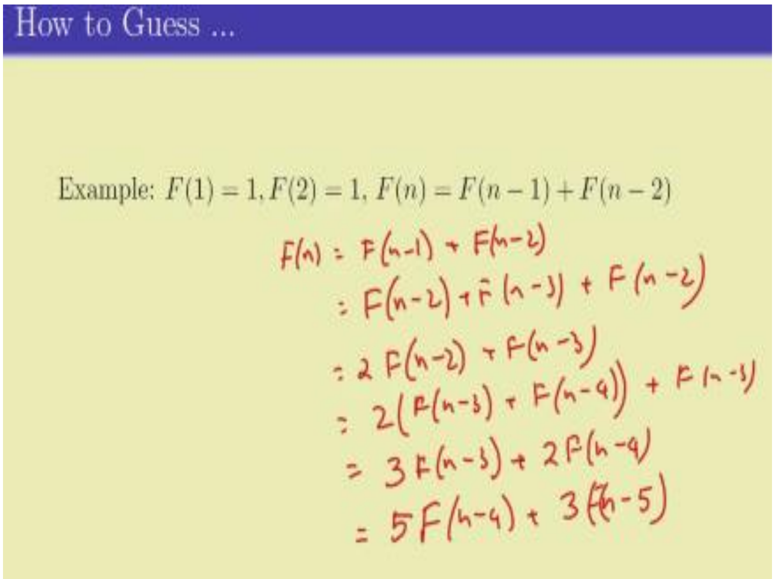

The problem is that they are all complicated ones like this, F(n)=F(n-1) + F(n-2), now if we try to unfold it by thing locate, F(n)=F(n-1) + F(n-2), where F(n-1)= F(n-2)+ F(n-3) + F(n-2), so I get 2 times F(n-2) + F(n-3) which will remain 2 times F(n-3)+F(n-4)+F(n-3), which is = 3 times F(n-3)+ 2 times F(n-4)and of course it will clear to keep on doing it and but then as you can see I can write down the next step directly and this will become something like 5F(n-4)+3 F(n-5).

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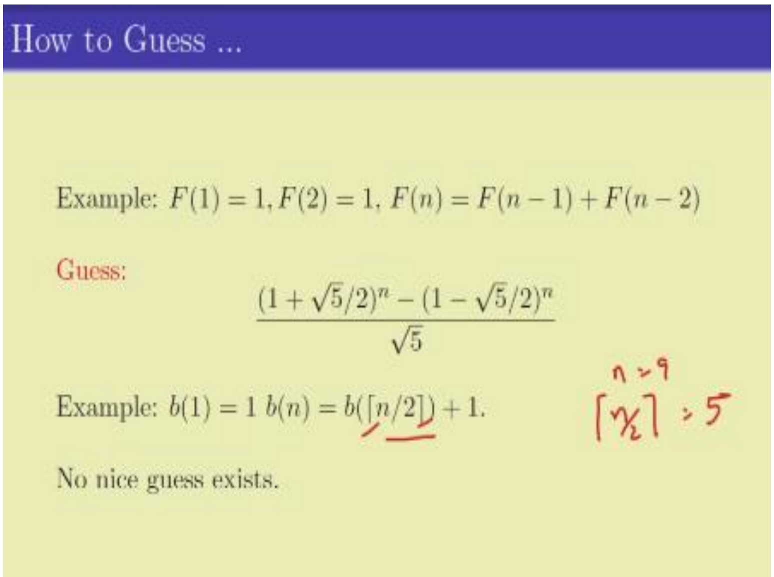

Since, particularly no pattern coming out in this recurrence relations. So instead for this kind of recurrence relations, unfolding will not help. I leave you guys to check and verify and convince yourselves that here by unfolding you will not be able to guess the actual value, in fact guessing the actual value is quiet complicated, here is the actual guess and you can see by looking at this expression, it is not something that easy to guess, right.

Also we have other kind of formulas like this one, when b1=1 and b(n)=b(n/2)+1, but this is, this one will denotes the integer that is bigger than or equal to n/2, so if n=9, then n/2, with these 2 things here is 5, right. So with this kind of an expression, unfortunately there is no clean guess can be made because of this extra bit of weird thing that are there, this what we called as ceilings right.

So there are expression of this form where either the guessing is too hard or we do not have a very clean guessing for this, how to attack this particular kind of recurrence relation, we will be doing in the next couple of weeks. Thank you.