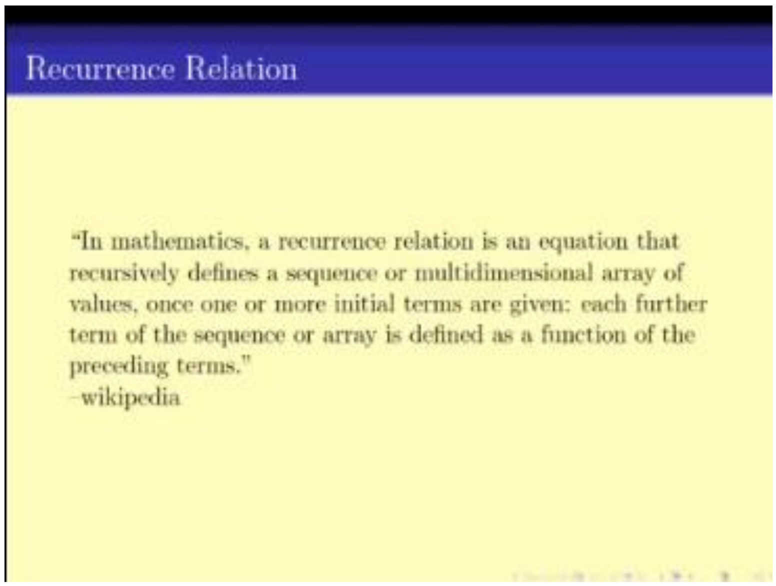
Discrete Mathematics
Prof. Sourav Chakraborty
Department of Mathematics
Indian Institute of Technology - Madras

Lecture - 38
Solving Recurrence Relations (Part 1)

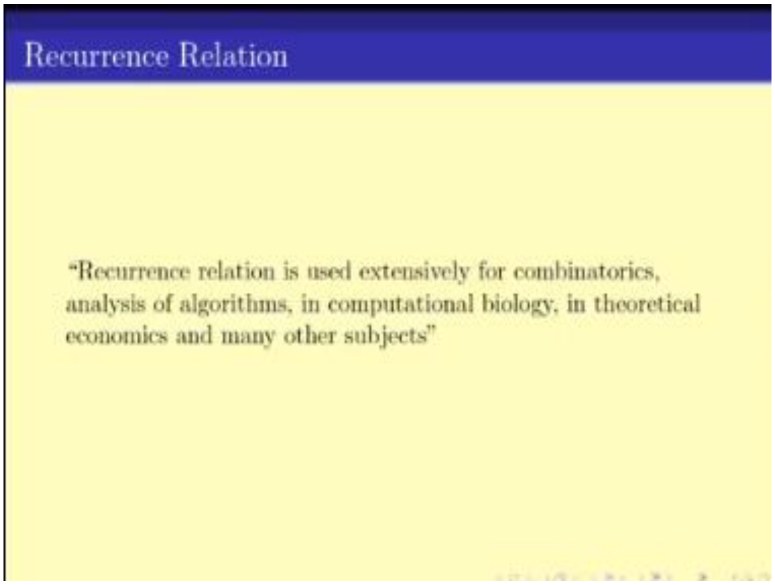
Welcome back. So in the last few lectures, we have seen how to use the recurrence relations to models, various counting problems.

(Refer Slide Time: 00:24)



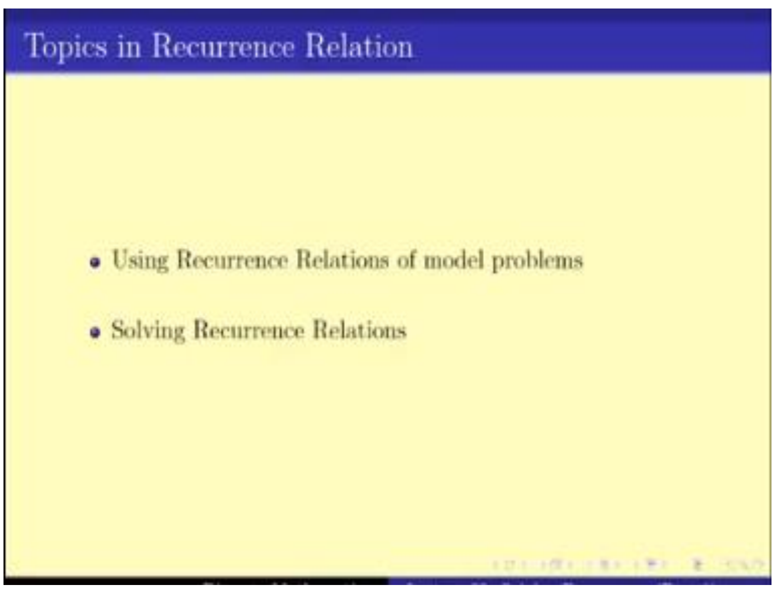
Now, as we have told recurrence relations are very essential part of mathematics or particularly in counting. So in such, recurrence relation is an equation that recursively defines a sequences or multi-dimensional array of the values where they are some use initial terms and the nth term is defined as a the (()) (00:44) terms.

(Refer Slide Time: 00:47)



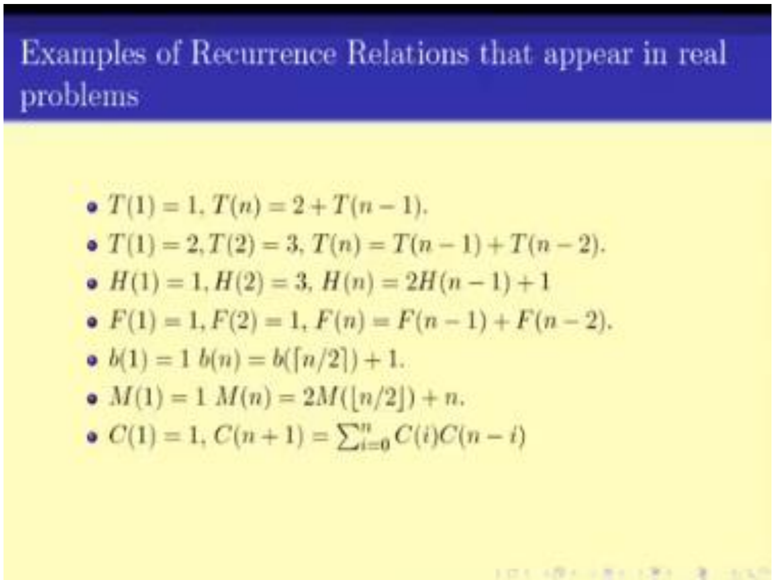
Recurrence relation is extensively used for combinatories, analysis of algorithms, in computational biology, in theoretical economics and many other subjects. We have already seen some of recurrence relations.

(Refer Slide Time: 01:05)



And we have seen how recurrence relations can be used to model various problems particularly combinatories problems. We have not seen how to solve the recurrence relations. In this video, we will be focusing on how to solve recurrence relations. So here some of the recurrence relations that do appear in the real life problems.

(Refer Slide Time: 01:36)

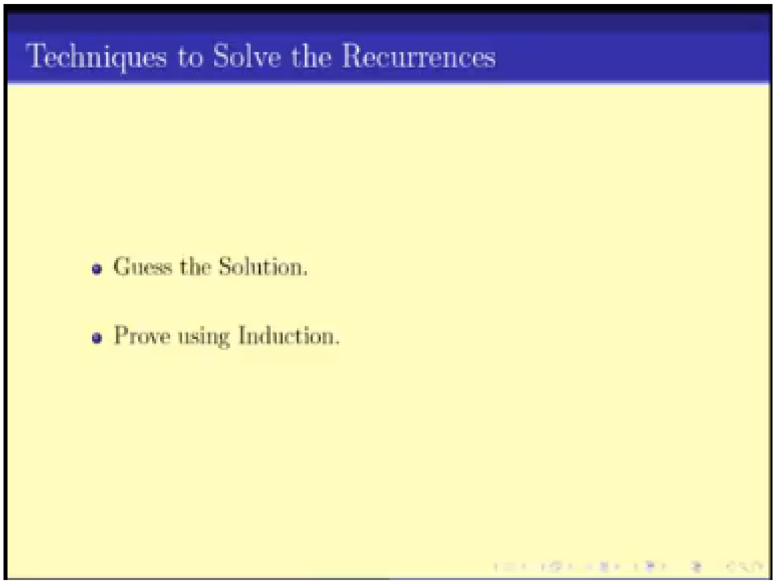


The first one says, T1 equals to 1 and Tn equals to 2 plus Tn minus 1. The second one say, T1 equals to 1. T2 equals to 2 and Tn is equals to Tn minus 1 plus Tn minus 2. Or this one is what we got from the Tower of Hanoi problem but H1 equals to 1, H2 equals to 2, Hn equals to 2 times Hn minus 1 plus 1. This one is basically what we have the Fibonacci series but F1 equals 1, F2 equals to 1 and Fn equals to Fn minus 1 plus Fn minus 2.

This flow is known as the Fibonacci series quite as same as series that we have this again and again in real life. This one b1 equals to 1 and bn equals to b n over 2 plus 1. Then we have M1 equals to 1 and Mn equals to 2 times Mn over 2 plus n. So this 2 appears in various algorithms. Particularly, the binary search algorithm and the merge sort algorithms that are very popular in the algorithm's literature.

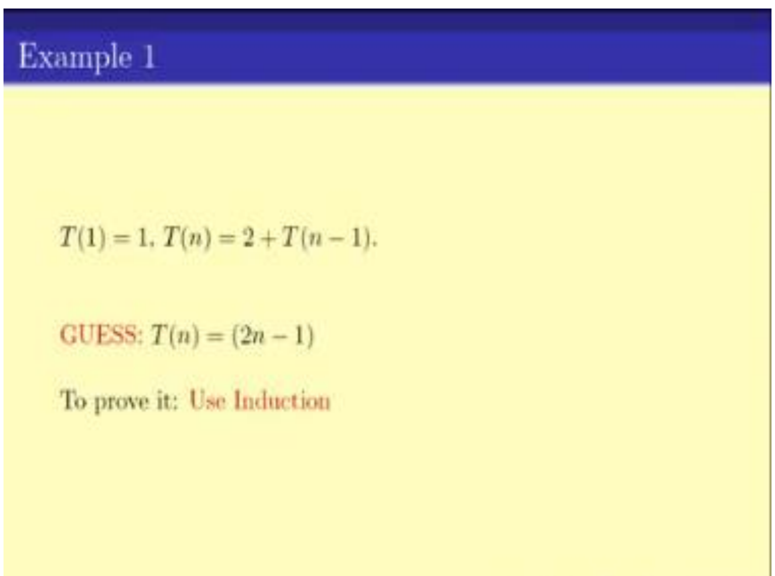
And then we have applied some of complicated one. C1 equals to 1 and Cn plus 1 is equals to submission of i equals to 1 to, 0 to n. C(i) C n minus i. Now for all of them, we have to now understand how one can solve them. So what is the technique for solving any these recurrences. So how to solve these recurrences.

(Refer Slide Time: 03:46)



Now, the first technique that we have going to look at is the simple thing of guess the solution and prove using induction. In this video, we will see how this technique is useful and then we will - In next video, we will see how one can guess the solution. So say here we have this example.

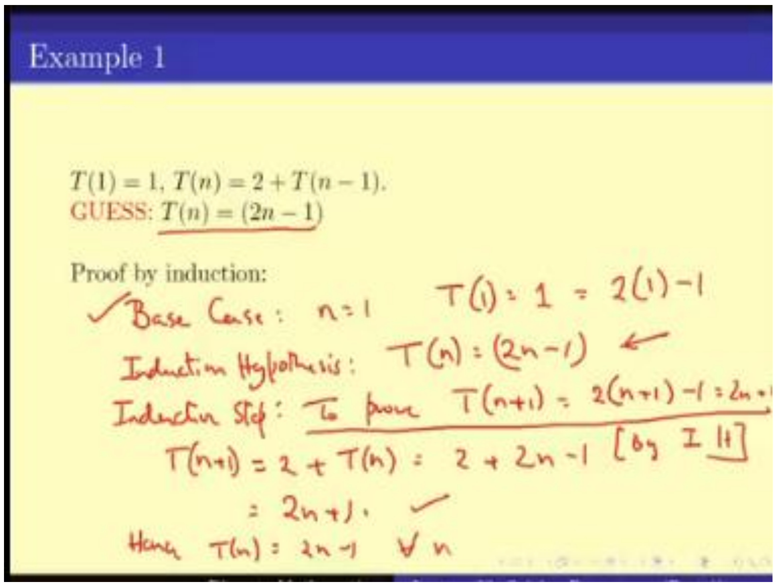
(Refer Slide Time: 04:14)



T1 equals to 1, Tn equals to 2 plus Tn minus 1. Now how we solve this particular problem. Now first of all, is somehow magically you can guess this number, then you play. For example, if I tell you, that all guess that Tn equals to 2n minus 1. Now If this is the guess that we made, then we can try to prove this statement using induction. So the technique is first to guess and they are improved by induction. I have kept a big jump of how to guess this number.

We will see that one – see the technique of guessing in the next video as well as in the next whole video. Guessing the solution for the recurrence relation is possibly the most challenging part of solving recurrence relation. But in this video, we will be focusing on how to solve the guess if we have the induction If we have the guess right. If we guess the thing right, how do you prove? How do you prove by induction?

(Refer Slide Time: 05:39)



Now If you remember, though we should have a base case. In this case, base case is say n equals to 1 and we have, T1 is equals to 1, this is a nothing that is (()) (05:58) which is of course same as T2 times 1 minus 1 right? So this value is correct for T1 right? And now, we have the induction hypothesis, induction hypothesis which says that say for some n Tn equals to 2 times n minus 1.

And in that case, what is the inductive step? Inductive step is to prove the same statement for T(n) plus 1 which is 2 times n plus 1 minus 1 which is 2n plus 1. How do you prove it? Now, of course by the thing that is given to us T of n plus 1 equals to 2 times 2 plus T of n which n equals to 2 plus 2n minus 1 by the - so this is by induction hypothesis which is equals to of course 2n plus 1 and that is what we had to prove. Right? Hence, we are done.

We are proved the inductive step that means Tn equals to 2n minus 1 which is a for all n greater than or equal to 1. Right? So this is proof by induction for how once we have the guess right. Correct? So let us go over the next one. One more example say, So this example two says that T1 equals to 1 and Tn equals to n plus Tn minus 1.

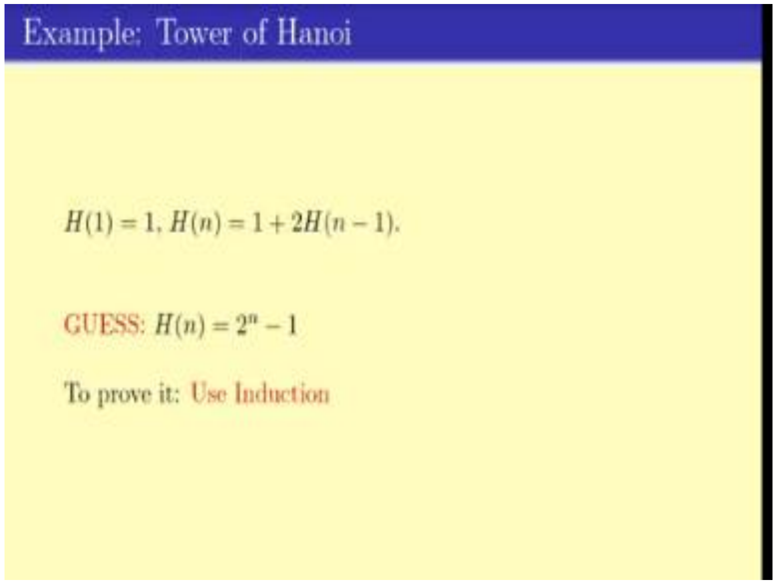
Again first of all, you have to guess it and let us imagine that somebody just concept and manages to guess it correctly and say that somebody concept says that Tn equals to n into n plus 1 by 2. Now once someone has guessed it, we have to prove it, we have to ensure that the guess is right and to get that is true, we have to again use induction. So like in the earlier case, we have taking prove this on by induction and let us see how we prove it again.

See base case, n equals to 1, of course T1 equals to 1 which is 1 times 1 plus 1 by 2 right? Which is what so the thing is correct one the guess was the getting equals to 1. Now we have the induction hypothesis, what is this says that Tn equals to Tn sorry Tn equals to n into n plus 1 by 2. Now inductive step, we have to prove that T of n plus 1 is equals to n plus into n plus 2 by 2. Now how do you prove it? Now T of n plus 1 is given as n plus Tn-1 which is n plus, see n into n plus 1 by 2.

This is by induction hypothesis which is secondly - sorry I made a mistake here. This is not n, this should be n plus 1 right? So, this is also (()) (11:08) Tn equals to n plus Tn minus 1, Tn plus 1, has to be n plus 1 plus Tn. And this is equals to n plus 1 plus the given induction hypothesis which is n into n plus 1 by 2. Now I can take n plus 1 common in that case, I get 1 plus n by 2, so this is 2 plus n by 2 which is of course n plus 1 into n plus 2 by 2 which is what we had to prove.

So we have T of n equals to n into n plus 1 by 2 for all n greater than equal to 1. Again the idea is simple if you can guess the value correctly for Tn, then you can prove what Tn is by induction. Right? Let us see one more example, what can be the various guesses?

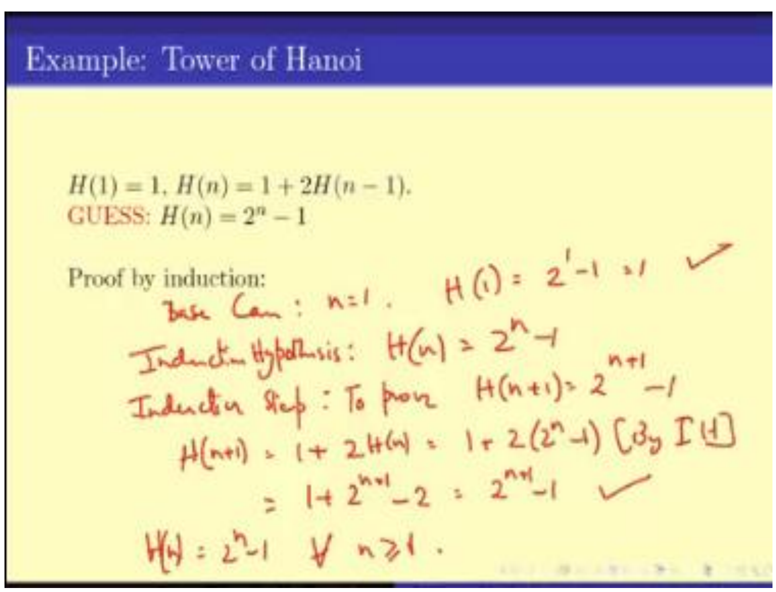
(Refer Slide Time: 12:48)



So, this is a Tower of Hanoi problem, right, so T1 H1 equals to 1 and Hn equals to 1 plus Hn minus 1. Again, we first have to guess it. Now, what is the guess here? The guess is Hn equals to 2 power n minus 1 and again we have to prove this one by induction. Note here that if you guess it wrong, we will not be able to prove it by induction or if you learn, able to prove it periodic.

So thus, only if you guess it right we will be able to prove this statement. There are people who actually come up with these cases by some intuitions of their brain but and there are some techniques also which will help to come up with the correct cases which we will study in next few lectures. But again it for this particular problem, how do we prove this statement?

(Refer Slide Time: 14:10)

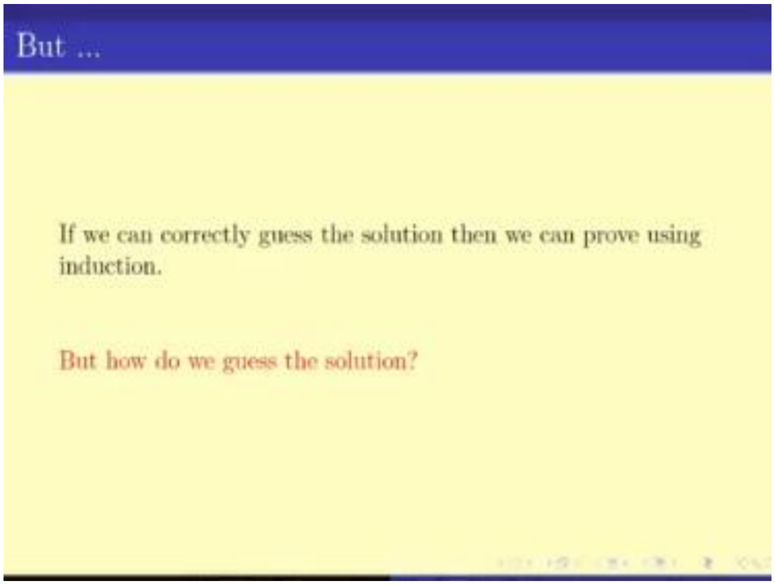


Again and again, we have to look at the base case, so base case n equals to 1. So here, H1 equals to 1 which is 2 power 1 minus 1 which is 1 which is right. So base is correct. So induction hypothesis says Hn equals to 2 power n minus 1, inductive step, so we have to prove, so to prove, H of n plus 1 equals to 2 power n plus 1 minus 1. Now let us see, Hn equals to sorry, H of n plus 1 equals to by (()) (15:04) 2 times Hn which is 1 plus 2 times 2 power n minus 1.

This is again by induction hypothesis which is 1 plus 2 times n plus 1 minus 2 which is 2 times n plus 1 minus 1 and this is what we had to prove. So H of n equals to 2 power n minus 1 for all n greater than or equal to 1. Note that, this is not only a way to proving the recurrence, this also if you go back to our previous video, this gives a compact form for the number of moves required for the Tower of Hanoi problem.

So the Tower of Hanoi problem, therefore requires 2 power n minus 1 moves and we got by first modelling it as a recurrence relation and then solve the recurrence relation. Now how did you solve the recurrence relation, we first guess the recurrence relation and then we prove that the guess is right.

(Refer Slide Time: 15:04)



This is how most of the counting problems work, you first model it a recurrence relation and then you solve the recurrence relation. But this is all the (()) (16:41) fine, if you can guess the recurrence relations correctly. You first guess the recurrence relation then prove it using induction. The main question is how do you guess the solution? and we will be doing this problem and how to guessing the solution to the recurrence relation in the next video.

We will see one of the techniques and then the next few videos we will see the other techniques. Thank you.