Discrete Mathematics
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Lecture - 34
Counting for Distribution (Part 2)

Welcome back. So we have been looking at how to count the cardinality of sets.

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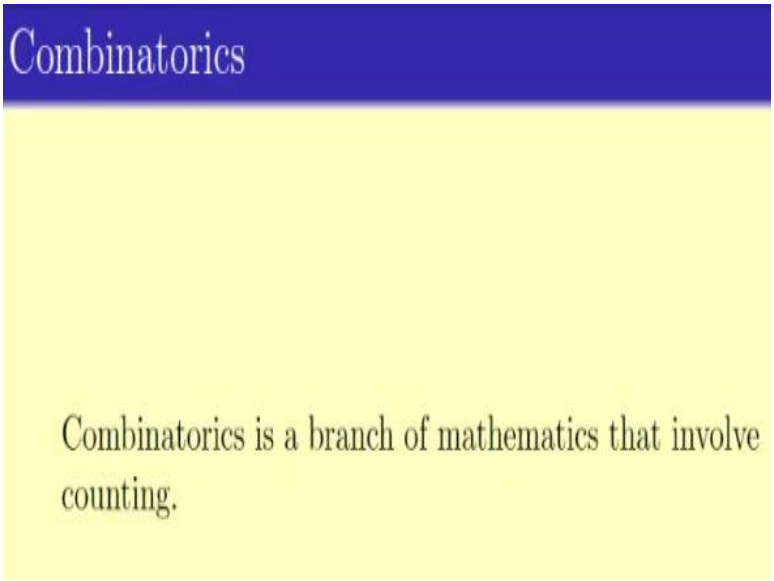

So this is what we call as Combinatorics, so which is the branch of maths that involves counting. So most of the time the typical question is, given the set S what is the cardinality of S, or in other words what is the size of the set S, or how many elements are there in the set S.

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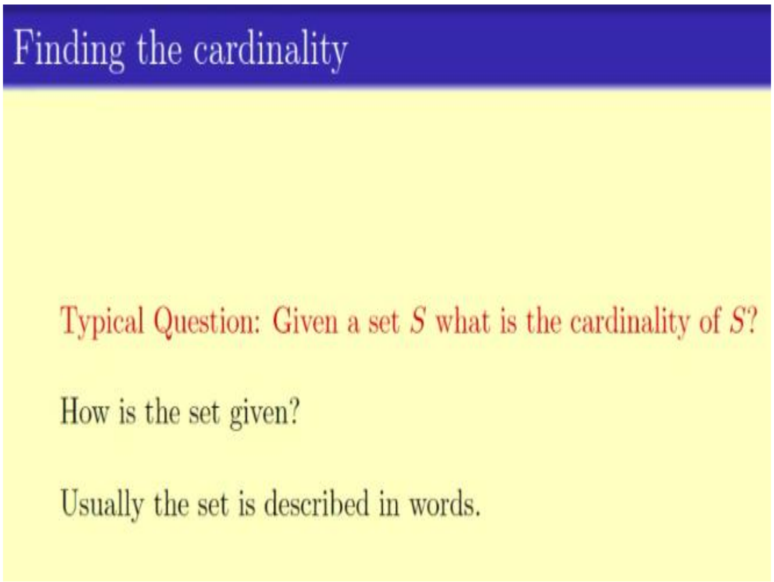

Now the question to be asked is that how is the set given, and most of the time when we talk about this kind of problem, we said that the set is given implicitly. It is not given explicitly, it has been described.

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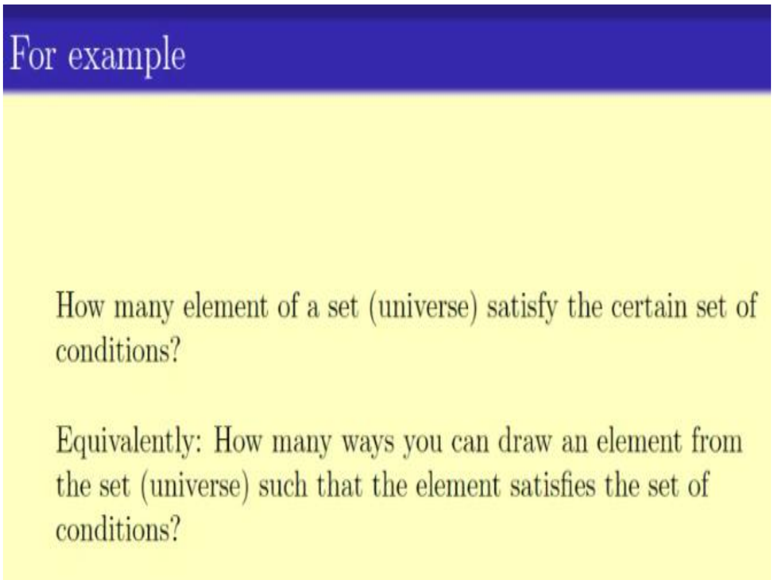

For example, it can be something like, how many elements of a particular universe set satisfy certain conditions. Or Equivalently put, how many ways can you draw an element from the set, such that the element satisfies a set of conditions.

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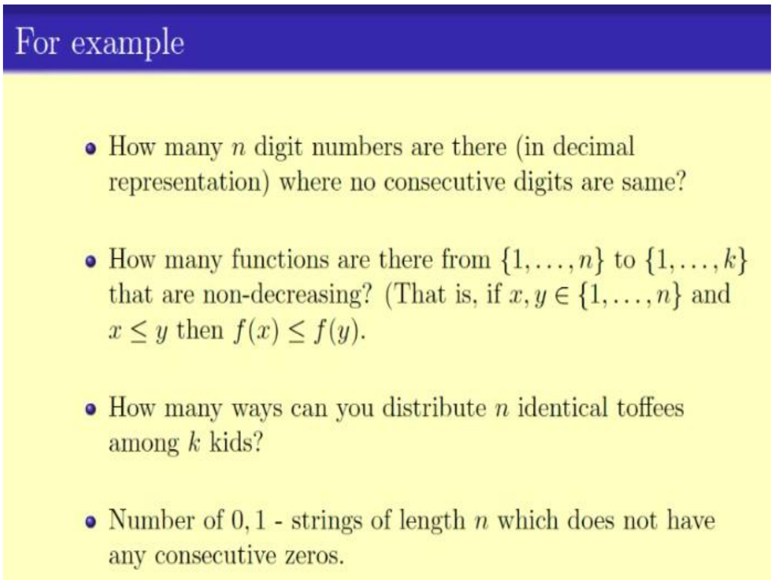

Now we have been, we have some problem that we should keep in our mind before we what we should like to solve. The first one being, how many n digit numbers are there, in decimal representation with no consecutive digits same. The second, how many functions are there from the set one to n to one to k, that are not decreasing. The third, how many ways to distribute n identical toffees among k kids.

And the last one is the number of 0.1 string of length n, which does not have any consecutive zero.

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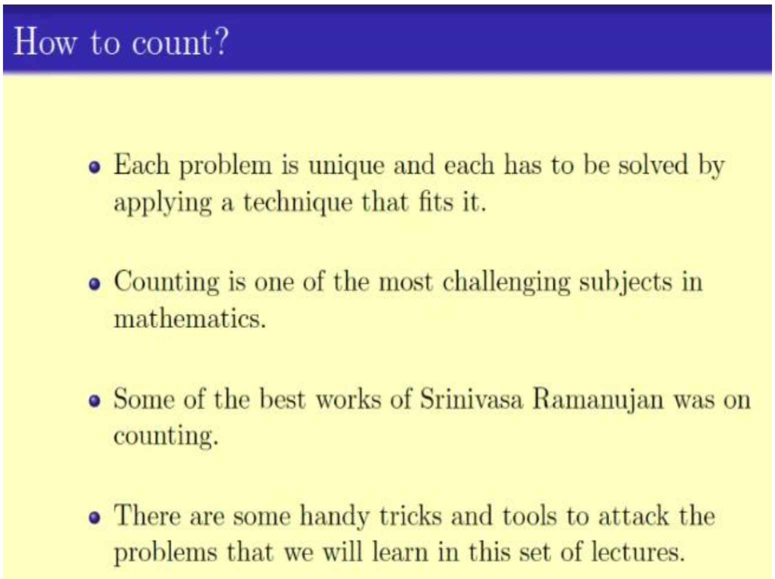

Now, question is that, how to count is something an extremely challenging problem. Every problem, every set that we have to count has its own complications and should be tackled differently. Even, some of the greats of maths like, Srinivasa Ramanujam have worked on counting for quite a big part of his life. And, here in this few lectures we will be giving some tricks and tools, which will help you to attack problems.

But again, as I told you, as I tell in every class, you should use your creative mind to understand, which problem can could be solved with which trick, every problem is different, every problem is unique.

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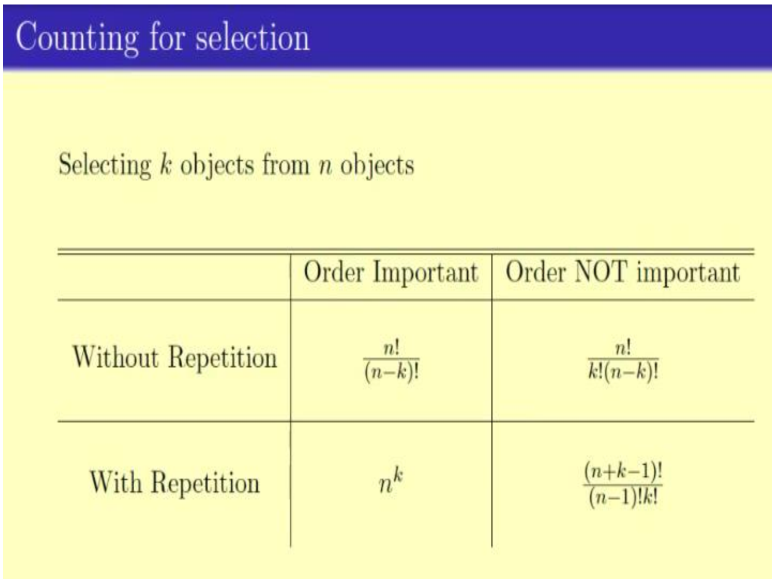

Till now, we have been looking at, we have already looked at this problem of how many ways can I select k objects from n objects, and we have looked at the two cases, namely what happens whether, are we allowed to pick a same object multiple times, is repetition allowed, among the k selected objects. And, is the ordering of the objects in this k objects matter. And we have seen how to solve all the four cases.

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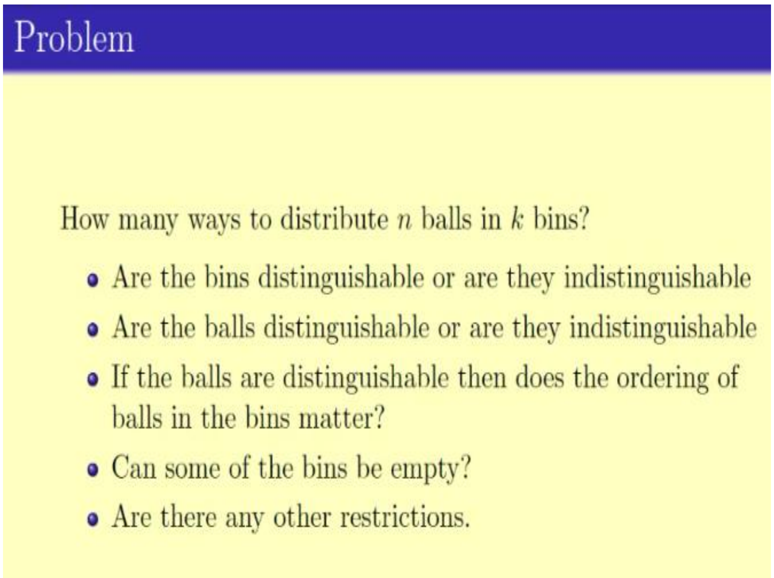

The next one that we looked at was, how many ways can we distribute n balls into k bins. And there are few cases to be studied, particularly whether the bins are distinguishable or not, whether the balls are distinguishable or not. If the balls are distinguishable, is the ordering in the bins matter, can the bins be empty, and is there any other restrictions that can be asked. And, in the last video, we looked at this problem and we solved some of the parts of this problem.

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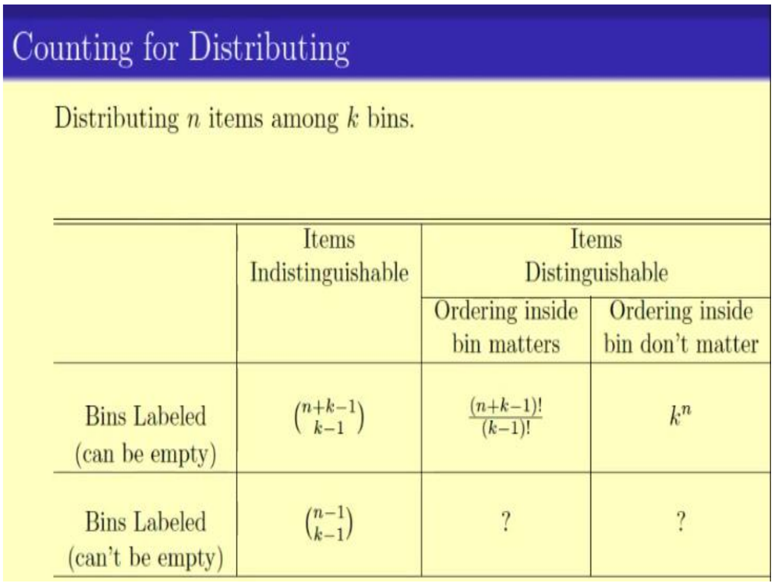

Then we will trick solve the case when the bins are labelled, and items are indistinguishable, and in case bins are labelled, but bins can be empty. We also solve, when the items are distinguishable and ordering matters or does not matter, either can be solved that. But we are left with two of the problems, which we will be tackling in this video, namely if bins are labelled but cannot be empty, then how do you solve it. And also we were left with the case when bins are unlabelled.

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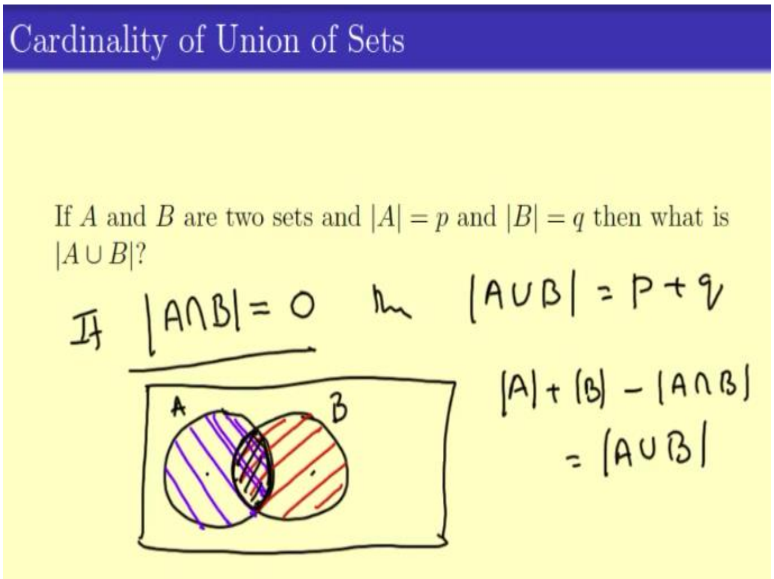

Now, before we start working on those things, let us do a little bit study of the theory behind it. The important thing is that, if A and B are two sets and size of A is p and size of B is q, what is the size of A union B. Note that, this is very similar to the additive law that we studied two videos ago. It told that if A and B does not have anything in common or other words, what told was that if A intersection B is empty, if the cardinality of this one is zero, then the cardinality of A union B, either A happens or B happens is p plus q.

But what happens if this is not zero, for example, if I have this as the set, so if this is universal set Z, where we are working on, if this is A and this is B, and A intersection B thus have some place here, how do you solve it. Now as you can see, from the picture inferences, what happens if I look at size of A plus size of B. So the case that, okay, let me just complete this line, size of B is this set, and A is this set.

Now note that every point here inside A union B has been selected once, if it is somewhere here, and once if it is somewhere here. But if it in the intersection, it has been counted twice, right. So it is like, if you have to place two round paper cuts, one over the other, you need to cut away, one to cut away this chunk out once. So, if you subtract A intersection B, you should be getting A union B.

And that is a very crucial thing, that A union B is size of A plus size of B minus intersection of A and B. Now, so what is the answer, answer is of course that A union B is size of A plus sign of B minus A intersection B.

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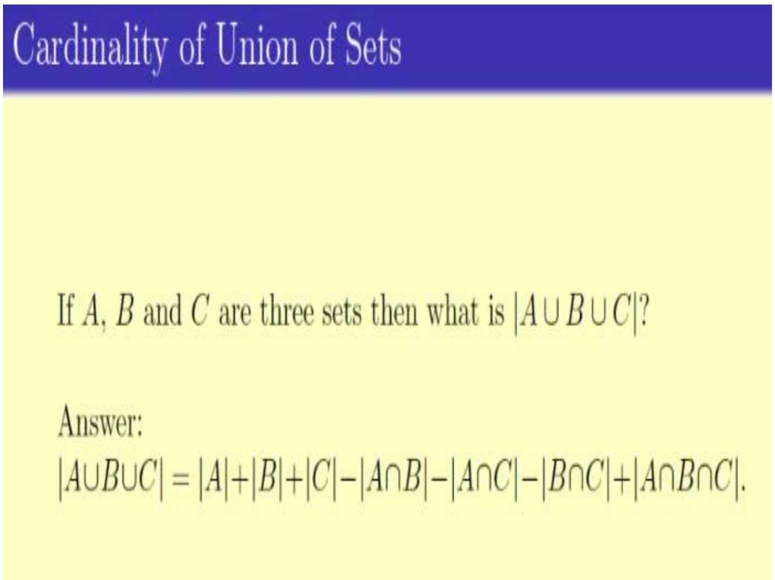

Now, what about if I have three sets, we will go back to the same Venn diagram for set we always look at for such kind of situations, if we have A, B and C, then you can first pick up all of them, you can cover all of them. But then, say this is A, this is B and this is C. As you can see, that this space of A intersection C has been counted twice, first by A and then by C. So i need to subtract at least one of this away, right.

So subtract A intersection C, and also I have to similarly subtract A intersection B and I have to subtract A intersection B, sorry I have to subtract B intersection C. Now, what happens in this case of this small area, this was counted once for A, once by B, once by C. It was subtracted once by A intersection C, subtracted once by A intersection B, subtracted once by B intersection C, that means, right now this one has not contributed anything.

So I need to now add it up, so I have to add this part, A intersection B intersection C. So in other words what I will get is that, I will get that A union B union C is equal to first adding the sets individually, then subtracting the pair of intersection by adding the triple intersection. Now, question is that what happens in the case of 4 case, if I have four sets A, B, C, D, in that case what is the size A union B union C union D.

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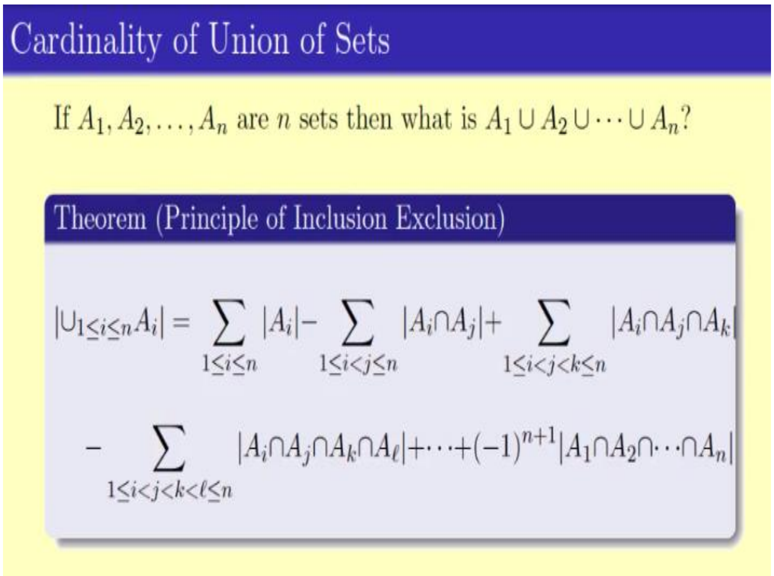

Is there a formula for it and we do have one and that is called the principle of inclusion/exclusion and it basically talks about that if I have n sets A1 to An, then what the size of A1 union A2 union till An. So the size of the union is first add the individual sets, then subtract all the pairs of intersections, then add all the triple intersections, then subtract all the four intersections.

In sometimes you are adding, subtracting, adding, subtracting and it keeps on going on till the last one where you add or subtract depending on what is the size of n, if n is odd or even, it is minus n power n plus 1. You add or subtract the intersection of all the sets. Now this is a theorem. I am not going to prove this theorem. I leave this to you guys to check that this theorem is indeed correct. You can prove this theorem using induction.

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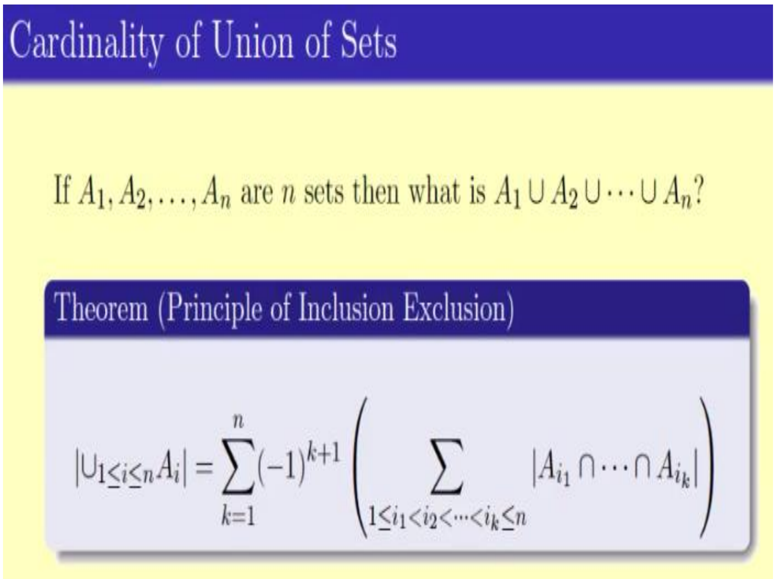

This particular theorem is an extremely powerful tool for counting. So writing is in a nice way, we do get this particular expression. So once we have this principle of inclusion/exclusion, we can use to solve a many of our problems. So let us start with this problem.

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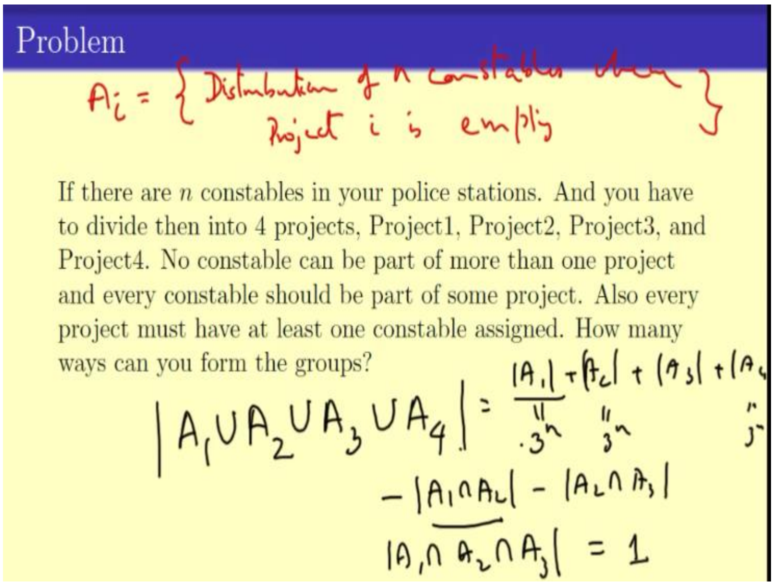

We say that there are n constables in a police station. You have to divide it into four projects, project 1, project 2, project 3, project 4. No constable can be part of more than one project and every constable should be part of some project and also at least one constable should be assigned to every project. None of the project should be empty. Now, how do you solve it?

If you remember, if we do not have this particular thing that every project has at least one constable assigned, then we saw that the first constable can be put in either of the four projects, the second constable can be put in either of the four projects, and so on, so the total number of ways of assigning the constable in the four projects is four power n, right. But here unfortunately, it is a possibility that some of the projects might be empty.

So we need to get rid of them. If we want to ensure that every project have some constable, then we have to subtract all the possible, subtract all distributions where at least one project is empty. Now, how do you do it. So let me define this new clause Ai. Ai is the set of all distributions. Distribution meaning of n constables into projects where project i is empty or it does not have any constable.

Now if I have defined this Ai in this way, then how do I define the set of all distributions that are empty. So the set of all distributions that are empty is nothing but union over Ai or rather in this case it will be A1, all the distribution for the first project is empty or all the distribution for the second project is empty, or all the distribution for the third project is empty, all the distribution of the fourth project is empty and abstract them out, right.

So we can now subtract, so this is the set of all assignments, or all distributions of constables in to four projects where some of or at least one of the project is empty and when I subtract that one from four power n, I get my required answer. Now by the principle of inclusion/exclusion, I can write this one as, so all I have to do is that I have to write this particular term, right, the union. So let me just do not worry about this.

This one, what is this term? This is summation of A1 plus A2 plus A3 plus A4. Now what is A1 plus A2 plus A3 plus A4? What is A1? A1 is the number of ways in which the i-th project is empty. That means nobody goes to the i-th project. How many such ways can you distribute n constable into four projects, where the first project is empty. That means it is the same way as distributing n constable into the project 2, 3, or 4.

So this one equals to 3 power n. Now, principle of inclusion/exclusion, I have to subtract A1 intersection A2, I have to subtract A2 intersection A3 and so on. Now what is A1 intersection A2? A1 intersection A2 is the set of all distributions where the first and second project are empty, right and this can of course happen with that both the projects are empty, that means it is same as dividing n constable into only project 3 and project 4, which is 2 power n.

And all of them are same and similarly when we have to do it for A1 intersection A2 intersection A3, these are the number of distributions where project 1, project 2, and project 3 are all empty and this can only happen with one way where all of them goes to project 4 and so on. Note that this is what, so the main idea is that intersection of Ai can be sometimes ____ ((19:18)) as in this case.

And if intersection of Ai is equal to q, then by the principle of inclusion/exclusion, I can write cardinality of the union of the sets and by doing so, we can get the number. It is not to be a very compact number, but it is still something, which has an expression. So if we can write down the expression of that exactly.

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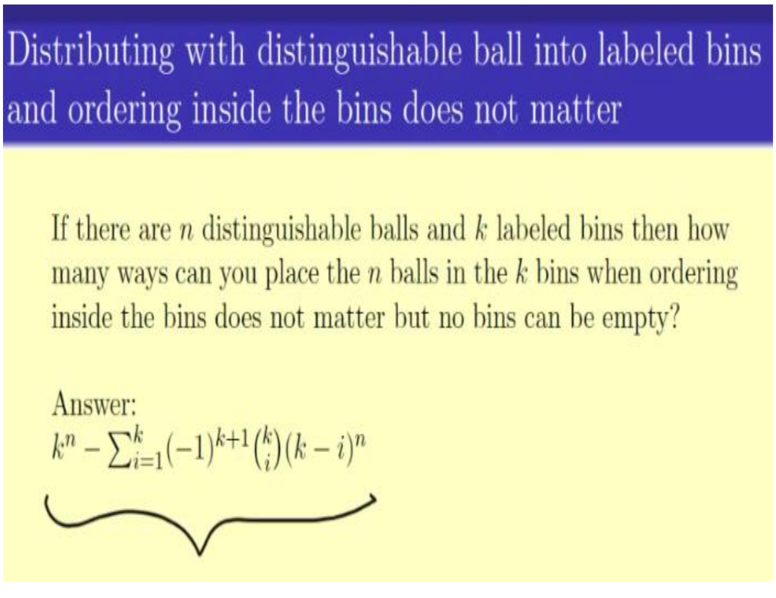

So if there are n objects and distinguishable ball and k labelled bins, how many ways can we place n balls in k bins when already inside the bin does not matter, but no bins can be left empty, it is k power n minus this way. Now why this k 2s i because k 2s i is the number of ways in which I can choose i of the sets and take intersection of them, right. I have to for example it should be like k power n minus intersection of all of the A1, A2 is right.

So all the A1, A2 is k 2s 2 times k minus 1 power n plus all the triples k to 3 and all the triples with value Ai intersection Aa intersection Ak is k minus 2 power n and so on. So this gives us a kind of a close form of expression, not exactly a close form expression, but an expression of this kind for getting the number of ways in distributing n ball with k bins with no empty bin.

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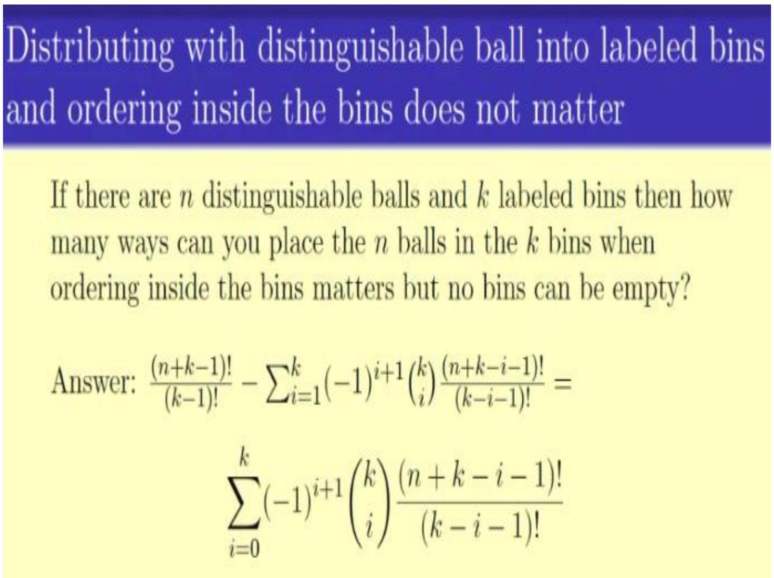

So we can use the same technique, when ranking inside the bins matter. If you remember, ranking inside bin without when they are empty bins were allowed, was this one. Now, we can do the identically same thing with the principle of inclusion and exclusion and can get this started.

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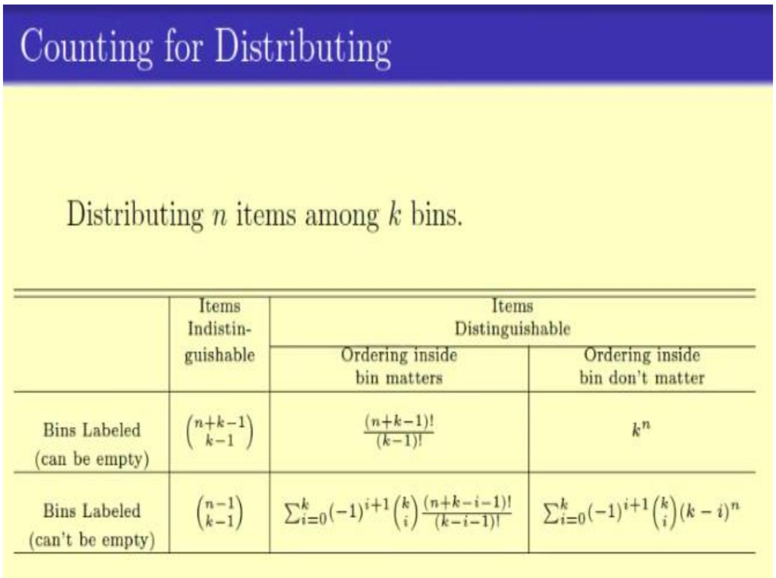

So by doing so, we have been able to fill up this particular matrix. Now let us try to see if we can solve the other three possibilities, namely when bins are unlabelled. Now, when bins are unlabelled for example.

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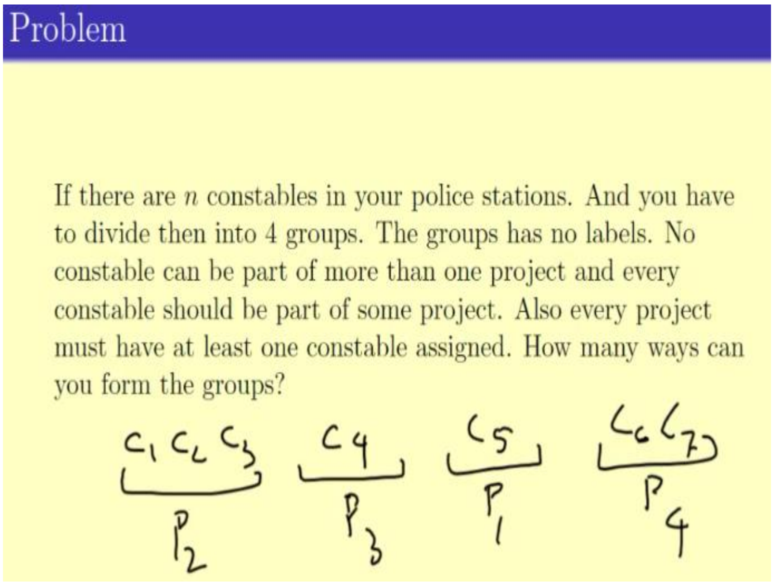

This is a typical example when you have the divide constable into four groups and we do not care about which group is assigned to which project, how many ways can you do it? Now the idea is that if say I assign constable C1, C2, C3 to project 1 and C4 to project 2 and C5 to project 3 and C6, C7 to project 4, note that if I permute this project 1 to project, if you do not have any name for the projects.

In that case if I change the project ordering to P2, P3, P1, P4, any of the ordering are equivalent and they would have been counted when we did not take care of the fact that they are not indistinguishable. By using this technique of the equivalent classes, we can see that every distribution with the labelled projects belongs to the equivalent class of the same size namely the number of ways you can permute these four factors.

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Distributing with distinguishable ball into labeled bins and ordering inside the bins does not matter

If there are n distinguishable balls and k unlabeled bins then how many ways can you place the n balls in the k bins when ordering inside the bins does not matter but no bins can be empty?

Answer: $\frac{1}{k!}\left(\sum_{i=0}^{k}(-1)^{i+1}\binom{k}{i}(k-i)^n\right)$.

We have a special name for this "Sterling Number of Second Kind". Denoted $S(n,k)$.

So by the technique of this equivalence partitioning, we can now just divide. So if I have to solve this thing of n distinguishable balls into k unlabelled bins, we have to take in the case when I have exactly k bins that are filled, no empty, and I divide it by k factorial and I get the answer. Note that, if I have not taken the number here, when everything was not empty, then things would have been more weird.

So by the same argument, by the way, this particular number, so this is the number of way in which one can distribute n distinguishable balls into k unlabelled bins when the ordering in the bin does not matter, but no bins can be left empty. This is called Sterling Number of Second Kind denoted by S (n, k).

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Counting for Distributing

Distributing n items among k bins.

	Items Indistinguishable	Items Distinguishable	
		Ordering inside bin matters	Ordering inside bin don't matter
Bins Labeled (can be empty)	$\binom{n+k-1}{k-1}$	$\frac{(n+k-1)!}{(k-1)!}$	k^n
Bins Labeled (can't be empty)	$\binom{n-1}{k-1}$	$\sum_{i=0}^{k}(-1)^{i+1}\binom{k}{i}\frac{(n+k-i-1)!}{(k-i-1)!}$	$\sum_{i=0}^{k}(-1)^{i+1}\binom{k}{i}(k-i)^n$
Bins Unlabeled	?	$\frac{\left(\Sigma\, i=0^k(-1)^{i+1}\binom{k}{i}\frac{(n+k-i-1)!}{(k-i-1)!}\right)}{k!}$	$\frac{\left(\sum_{i=0}^{k}(-1)^{i+1}\binom{k}{i}(k-i)^n\right)}{k!}$

Similarly, we can do it for the third one, for the last of the same of it is, when ordering inside matters or does not matter and bins are unlabelled. Now with this, we have done basically all the cases except this case namely when items are indistinguishable and bins are unlabelled. How many ways can you do it? This is a very hard problem and we will be talking about this problem in next video. Thank you.