Discrete Mathematics
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Lecture - 33
Counting for Distribution

Welcome back. So, we have been studying how to count or we have been dealing with the subject of combinatorics.

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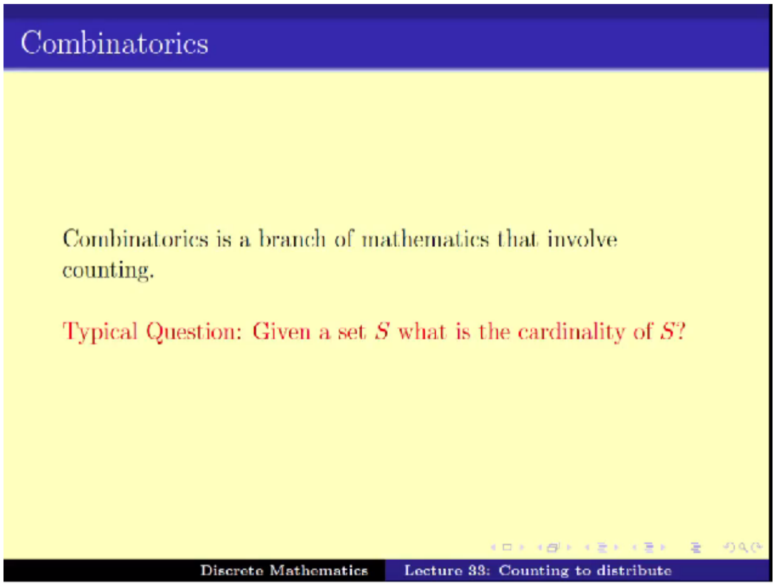

So, combinatorics is the subject of branch of mathematics that involve in counting. And a typical question in this regard is given the set S what is the size of effect or what is the cardinality of S? Or how many elements are there in S?

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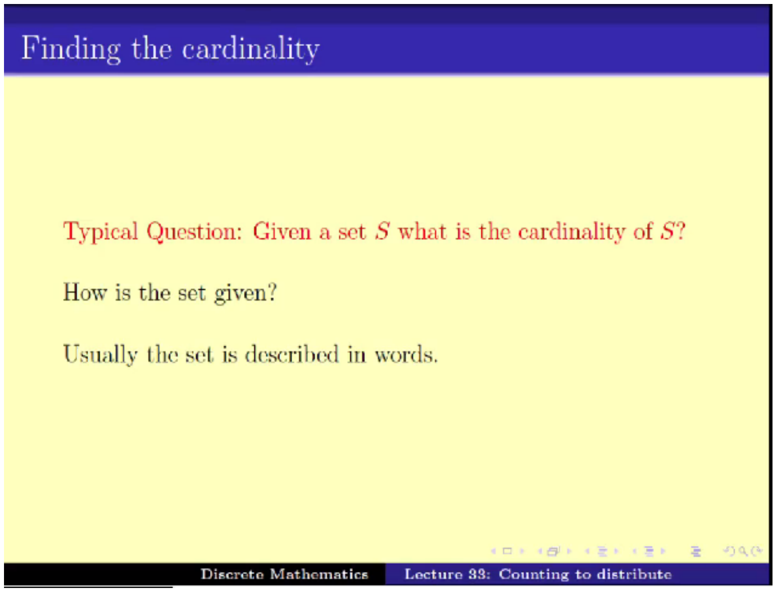

Now, the question to be asked is how is the set given? In most of the case, the set is given implicitly by describing the set and not explicit and that is what makes the problem hard. So, the set is usually described in words and we have to count the numbers of element in this set.

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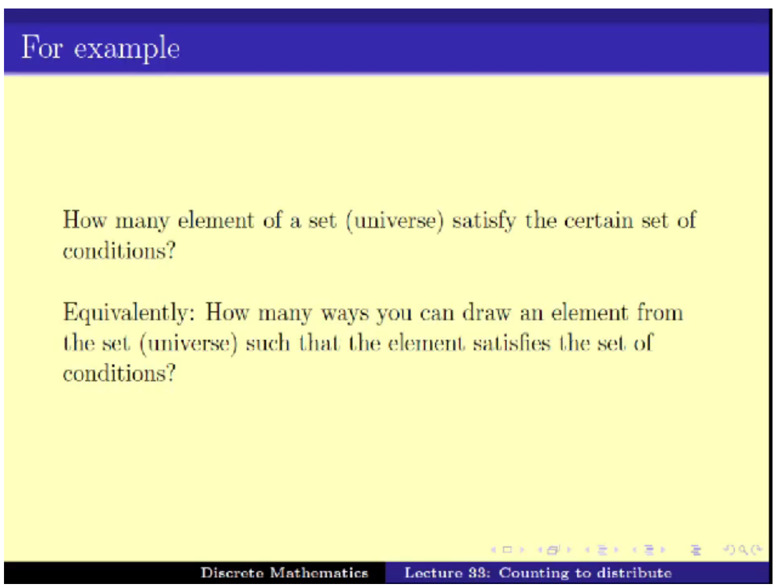

So, for example one can ask, how many element of a universe set satisfy a certain set of conditions? Or equivalently, you can ask things like how many ways can you draw an element from universe such that the element satisfies certain set of conditions. So, these are post counting problems or these are equivalent questions. And these are the kind of questions that we would like to answer in deep subject of combinatorics.

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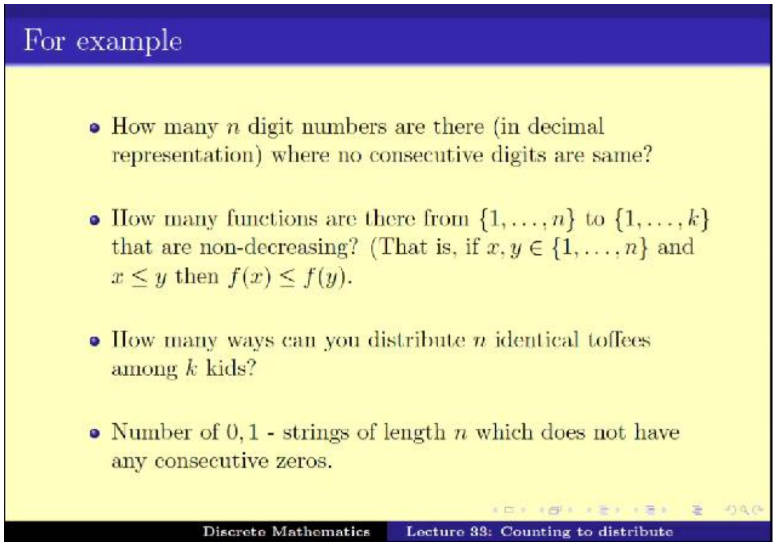

So, here are some examples that we discussed last class or let me repeat it all over again. So, how many end digit numbers are there where there are no consecutive digits are same? Similarly, the next question is how many functions are there from the set 1 to n, to 1 to k that are non-decreasing? That means if x is less then y then f of x is less than or equal to f of y. Or how many ways can you distribute n identical toffees among k kids?

And the last one is how many strings of length n, I mean zero on strings of length n are there which does not have any consecutive zeros. So, these are some of the problems that we should keep in our mind and we have been looking at some of the general theories of attaching counting problems.

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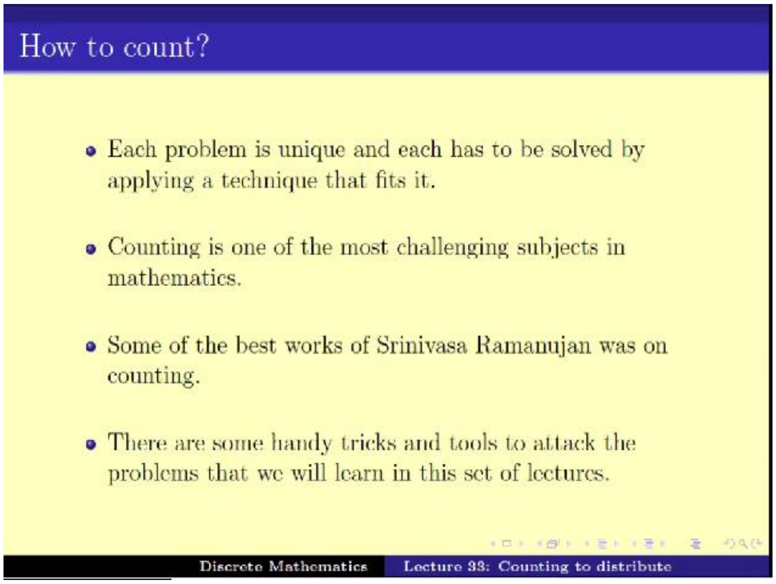

Now, while we can do a lot of theories about counting problems one thing to remember always is that each problem is unique and each has to be solved applying the technique that fits it. So, different problems require different technique and that you have to solve that you have to find out yourself. Counting is in fact one of the most challenging subject in mathematic in fact big names like Srinivasa Ramanujan spent a lot of his time solving problems in counting.

There are some handy tricks and tools to attack the problems and that is what we have been looking at in this set of lectures.

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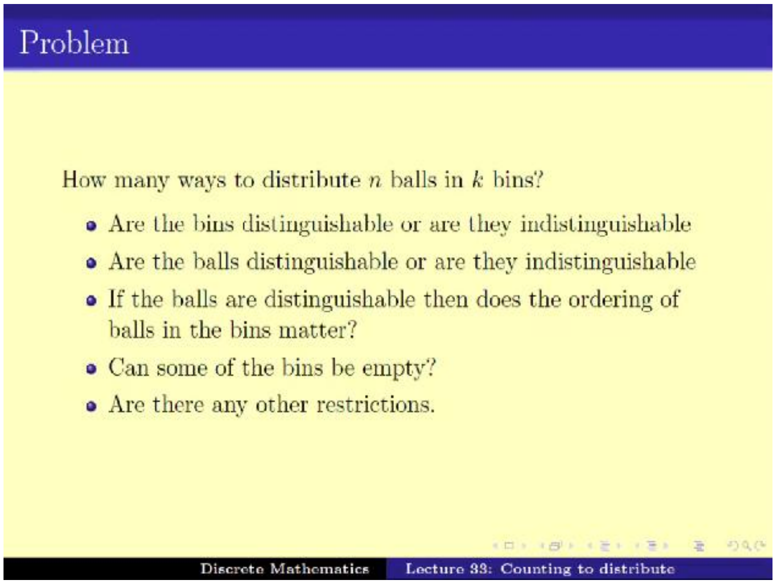

In the last video we looked at the problem of how many ways can be select k objects from n objects and there are four cases that we have to look at question is that almost two cases that we have to look at. Case one, are we allowed to pick objects multiple times meaning, is repetition allowed? And the second one is inside the selected set of objects does ordering matter or does it not matter? Is it just a group or is it a kind of an ordered set and depending on that we have got whole different counts.

So, in this video we will start looking at a kind of similar problem but slightly different it is called the distribution problem or in other words, how many ways can you distribute n ball in k bin, in k baskets? So you have the n balls and you have to distribute in k baskets. Now, just like last time here there are few questions to be asked. First question, are the bins distinguishable or are they indistinguishable? Or in other word is it like said okay I have the bucket 1, basket 2, basket 3, basket 4 or are we just asking them to club some of balls into groups and I do not care which is group 1, which group 2.

Similarly, are the balls distinguishable or indistinguishable? Meaning are the balls same identical, it does not matter whether I put the first ball in the first basket or the second ball in the first basket as long as one of the ball goes in the first basket set. And in case if both are distinguishable the inside a particular bin does the order matter or not? So, inside a bin do we care about how the balls pick placed or which ball gone there first and which ball has gone there second and so on or it does not matter it is just another group of balls.

Next question is that are we allowed to keep some of the things empty? Are we allowed to keep some of the baskets empty? Just ignore some of the baskets. And are there any other restrictions missing. So, we will be looking at this problem and this is in fact a very general class of problems. How to distribute any ball into k bins. We call it as a ball and bins problem and we will be through various cases and see how to attach each of the cases. Some of them is easy. Some of them are notorious method.

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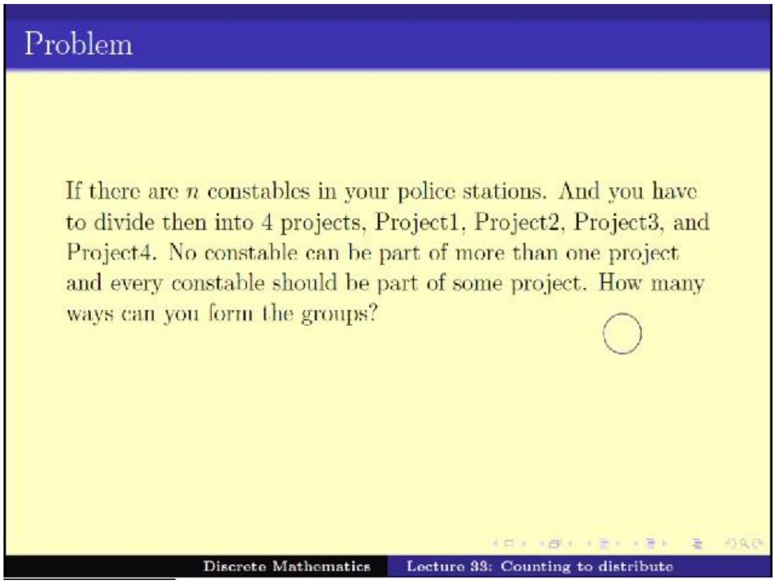

Still let us start with the first problem that we have. So, this is the same kind of problem that we were doing in the last video. So, here we have n constables in your police station and you have to divide the n constable into 4 projects, Project 1, Project 2, Project 3, Project 4. No, constable can be in part of more than one project and every constable should be part of some project. How many ways can you form the groups? Now, this is basically a balls and games problem.

So, you have say Project 1, Project 2, Project 3, Project 4. And you have the constables C 1, to C n. Now this constable you can put, you have to divide them into this baskets basically. I do not care at this point about what are that ordering of the things in the Project 1 of the constables that goes in Project 1 and so on. So, I just have to divide into groups. How many ways can I do it? So, the idea is that the constable number 1 can be either we put into Project 1 or Project 2, or Project 3 or Project 4.

So, there are four possibilities for constable 1. Still constable 2 can be put into Project 1 or Project 2 or Project 3 or Project 4. So, there are four possibilities of constable 2 also. Similarly, for every of the constable there are four possibilities and that means the total number of possibility that is there is 4 times 4 n times which is 4 power n. So, the number of project of placing this or dividing this n constable to four projects is 4 power n. Speaking of a general problem

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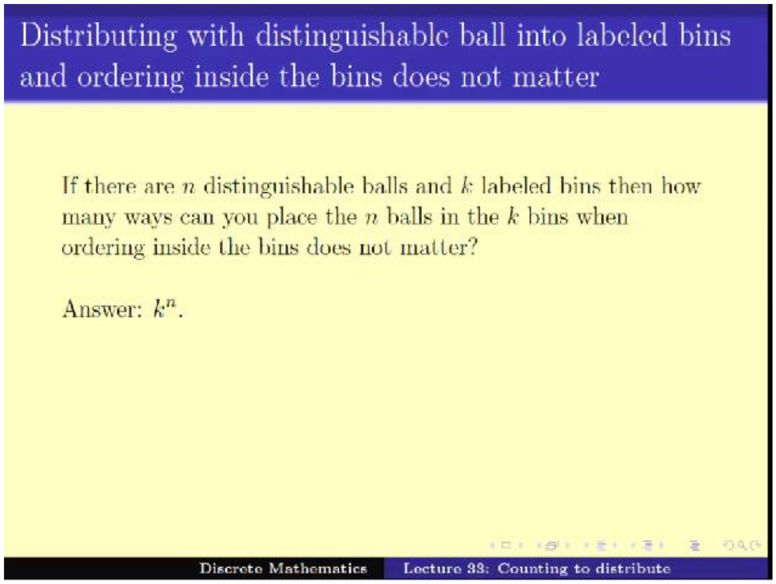

If you have n distinguishable balls now distinguishable balls what were we make distinguishable balls? If the balls are different like in the constables, constable 1 is different from constable 2 and you have k labeled bin in Project 1, Project 2, Project 3, Project 4. Constable 1 going to Project 1 is different from constable 1 going for Project 2 and I do not care about this ordering inside the bins. So, in that case the total number of ways of taking n balls into k bins by the same argument the first ball can go to any of the k bin, second ball can go any of the k bins.

Third ball can go to any of the k bins. So, all total it is k power n. Now, couple of things to note here it is very much possible that if say I have this four buckets and I have this n constable and I might also put all of the constables into one basket and not in the other basket. This is a valid possibility. So, that means it is possible that some of the bins are left empty. The problem as such does not stop us, does not restrict us from doing. So, we are fine but it is something to note that in this particular way of dividing it the bins can be made empty.

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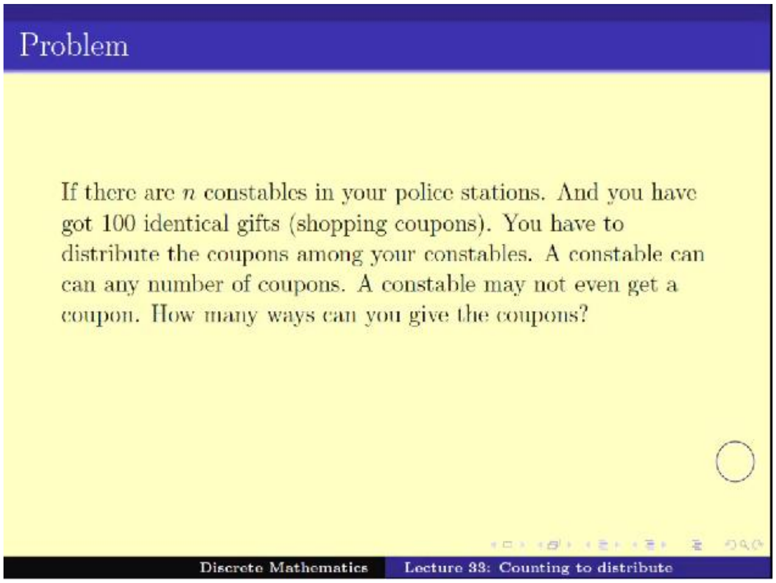

Now, one more problem and this time we have talked about giving gift to constables. So, we have to say n constable in your police station and there are 100 identical gifts think of shopping coupons. Your 100 shopping coupons and you want to distribute the coupons among your constables. A constable can get any number of coupons and a constable may or may not get a coupon also. How many ways can you give a coupon?

Now, this is very similar to the technic that we will apply is very similar to the one that we apply when we talked about choosing k% from n, from a group of size n where repetitions are allowed and ordering matters. So, let us quickly just see this technique again. So, here we have 100 coupons. So, let us we say that they are 100 gifts and let me draw them as square as these are the coupons. Now, all of these coupons are identical so it does not matter what type it is?

How I place them? So, we lay the coupons in a straight line and now we have to decide how to divide the coupons amongst the n constables. So one thing that we can do or let us think what we will do next? So, as soon as the first constable comes up we will tick, okay let us see. This three of them are yours. The next constable comes, he says okay. This two of them now are yours. So, in some case think of them that at constable comes arise. I just take that first three and gives its to, the constable, first constable then I am left with things from here to here.

So, next constable comes I again said okay everything here bit all yours. The next constable comes said everything –this three of them are yours. If fourth constable comes, okay I do not like you. I do not give you any coupon. so I was okay that is your bar. So, what am I doing? So, in other words, I am basically placing some kind of bars between these coupons. So, there are 100 coupons and I am placing some kind of bars between these coupons and how many bars should I put.

Now, if I put three bars or four bars I say that everything left of it goes to the constable 1, everything between these two bars goes to constable 2, everything between this one goes to constable 3, everything between this two goes to constable 4 and everything right of this goes to constable 5. It is like walls we have created right? So, four walls basically is a way of dividing coupons into 5 constables. Similarly, if I have n constables I can use n minus 1 walls like this and I can place this wall anywhere I want.

So, any placement the square and bars give me a partition of this n identical gifts into they can sorry, 100 identical gifts into n constables. For example, if say I suddenly so I look at the (()) (16:16) already. So, say this is some ordering of the square and bars and what does this say? This says that constable 1 get 2 coupons, constable 2 gets of course there is nothing here zero coupons. Constable 3 gets zero coupons, constable 4 gets 2 coupons, constable 5 gets 4 coupons, constable 6 gets 3 coupons and constable 7 gets zero coupons.

So, this is a way of distributing this 11 coupons into 7 constables. So, in other words what we are talking about is that every distribution of this 100 identical objects into n constables can be represented as an ordering of 100 square and n minus 1 bars. At any ordering of this 100 square and n minus 1 walls givens me a distribution of 100 coupons to n constables. So, the answer to this question of how many way can be distribute 100 coupons to n constable is basically the number of ways we can order 100 square and n minus 1 parts.

In how many ways can that we done? So how many objects are there, 100 square and n minus 1 bars. It is n plus 100 minus 1 right. So, this many objects are there so I ordered this many objects, how many ways can I do it? I can do it in this many number of ways, factorial number of ways. But again since all the coupons are all identical so we have to factor them out you should not consider the fact the permutation between them.

So, you should factor that by dividing by n factorial, sorry by dividing by 100 factorial and we should also factor out that fact that all the walls are identical, they are just the walls are nothing written on them especially. So, we should factor that out also and we get n minus 1. So, that is the answer to this question. It is very similar to the technique that we did last time in the last video for selecting k people from n, to a group like n by repetition are allowed and ordering matters.

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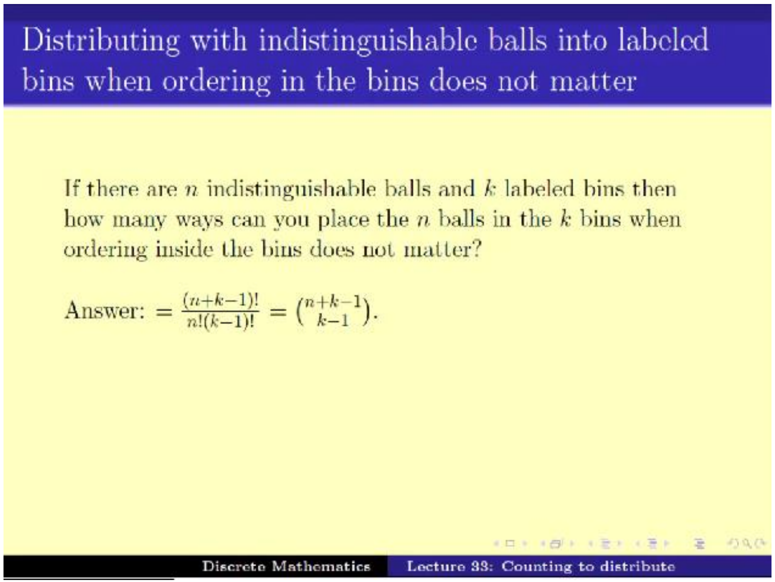

So, in general what we are talking about here? If you have n indistinguishable balls and k labeled bins, they are labeled because once we are doing this kind of parts I have labeled this goes to bin number 1. This goes to bin number 2. So, here I am already talking about some kind of a natural labeling that is coming up because of the placement of the bars. So, if we have n indistinguishable balls like coupons and k labeled bins like the k constable.

In how many ways can be place the n balls in k bins by ordering inside the bins does not matter? It is (()) (21:15) which is n + 1, n + k minus 1 factorial divided by n factorial take care of these balls are around the indistinguishable and k minus 1 factorial to take care of the bars or the balls indistinguishable. Which actually in terms of notation is same as n + k minus 1 choose k minus 1. Note here that since two walls can be next to each other so there can be a possibility that a bin remains completely empty.

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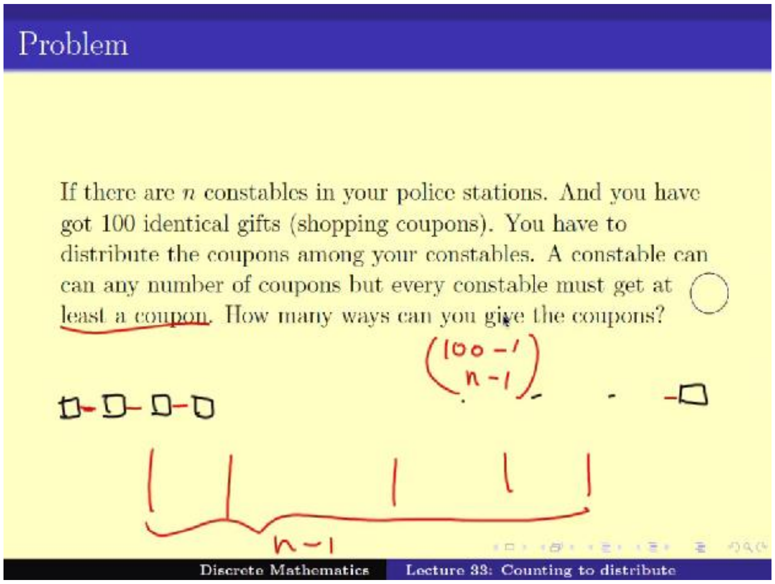

Now, what happens if we add the constant at every constable must get at least one coupon. Now, there are two ways of doing this whole thing two ways of calculating it. Number one thing is that you can first give all the n constable one coupon each. So, in that case I will be left with 100 minus n coupons and now dividing this n coupons in whatever way you want in among the n minus 1 constable which is by the earlier rules.

This factorial by n minus 1 factorial times 100 minus n factorial. Or which does come down to 100 minus 1 factorial by n minus 1 factorial in 100 minus n factorial which is equals to 100 minus n, n minus 1. So, idea is that just keep one coupon to all the constables to start with so everybody now has at least one coupon now whatever is left divide among the constables in whatever way you want and which is this many number of ways.

Now, there is one other way of solving the same or writing the same answer it is like this bars and coupons problem. Again we lay out the coupons in a straight line and again we have to basically put up n minus 1 horizontal bars sorry vertical bars. But where do we put them up since we cannot put two of the vertical bars next to each other. So, that means between any gap between this two consecutive coupons there can be at most one horizontal bar.

So, this n minus 1 horizontal bars should be placed among this set of all the possible positions. How many positions are there? How many gaps are between two consecutive coupons since there are 100 coupons the number of position between the coupons is 100 minus 1. I can choose any one, n minus 1 of them and place the bars there and I get affirmed. I get a distribution of the coupons into n constables so that every constable gets at least one coupon.

So, the number we get again is same 100 minus 1 choose n minus 1. So, this of course we can generalize again too.

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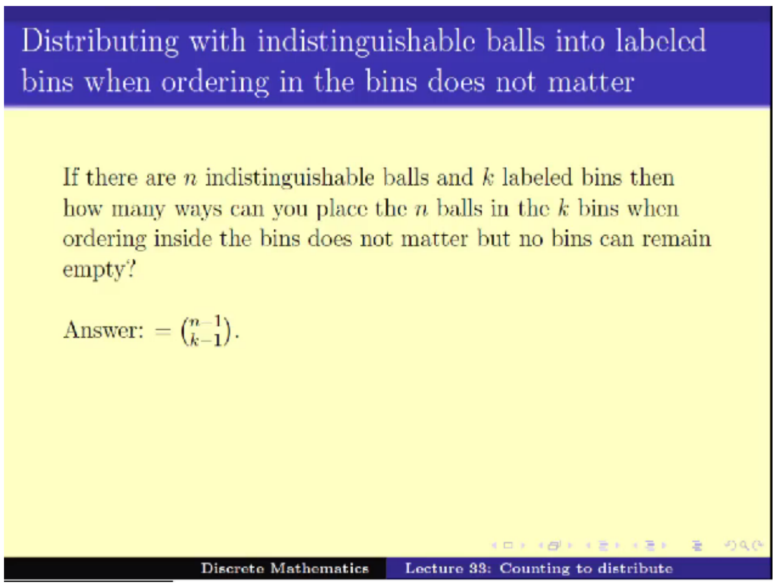

If we have n indistinguishable balls and labeled bins how many ways can it place the n balls in k bins with ordering in the bin does not matter but no bins can be made empty it is arrange the ball in the straight line there are n minus 1 gaps between them. How many ways can you place the k minus 1 ball in between among this n minus 1 spaces which is n minus 1 to k minus 1. So, till now we were not looking at the ordering in the bins at all. What if we look at the ordering inside the bin?

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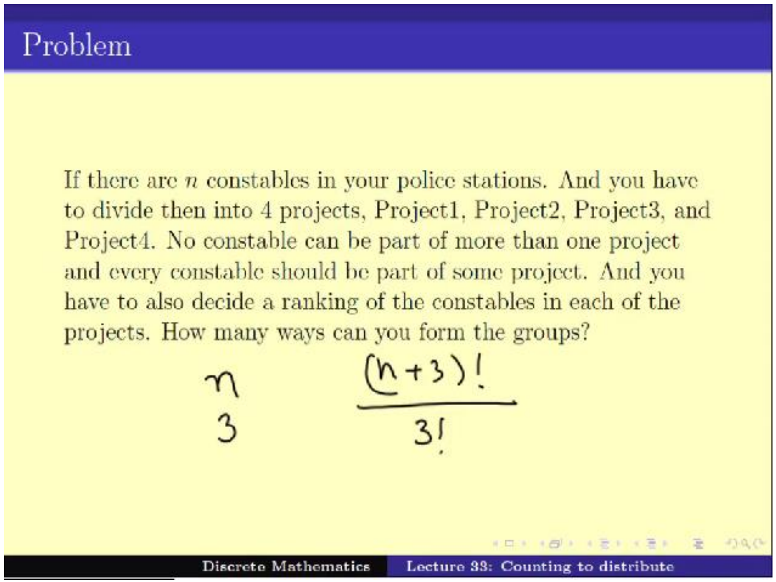

For example, say, in the constable problem we want to divide this n constable into 4 projects, Project 1, Project 2, Project 3, Project 4. No, constable can be part of more than project every constable should be part of some project but you also have to decide the ranking of the constable in each of the project. So, note that I can do the same feed that I did which is –now I can place the constable in a particular ordering so I have Constable 1, Constable 2, till Constable n.

Take any ordering of the constables and if I can party, if they can draw this since the forth project if I can draw three bars along them and I say that okay these set of constable is in project 1, each of them are in project 2, these in project 3 and these are in project 4 till be get a one way of partitioning it and any permutation of C 1 to C n will give me a different way of partitioning them into this 4 Project and clearly for example C 1 C 2 as this thing is same as sorry different as C 2 C 1 and this thing.

So, a different permutation is near different –all though the group might be same but a different ordering of or different ranking of constable in each of the projects. So, we can do the same idea about like the balls and bars issue and I get –what do I get? Okay, number of ways I can place this –so I have the n constables and I have 3 bars. So, I have n + 3 object. I factorize them I do the –these are the numbers of ways I can just order them and I just have to just take care of the fact that if 3 bars are identical the three are the three object.

I shouldn't different between them but I do not need to differentiate 55 by any factorial because the every different ordering of the constable give me a different way of putting the constable in different index and ranking. So generalizing this

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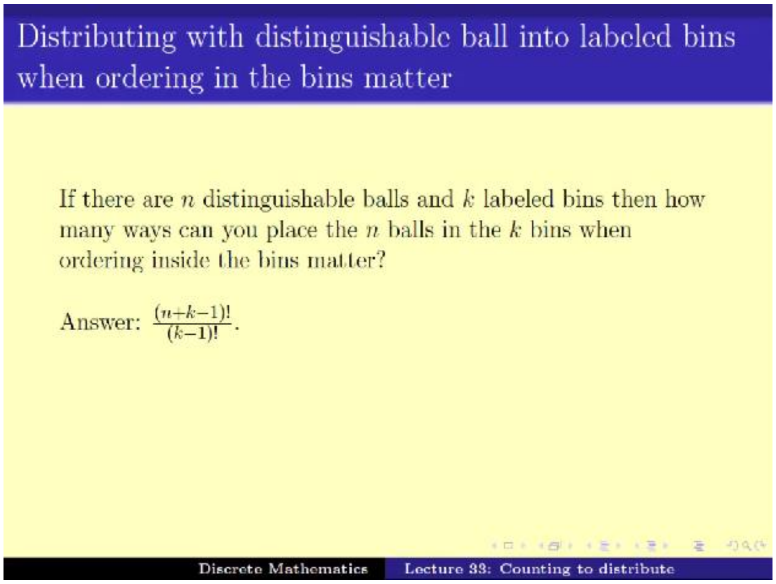

If we have n distinguishable balls and k labeled bins then the numbers of ways we can place n balls in k bins when ordering inside the bins matter is just n + k minus 1 factorial divided by k mins 1 factorial. Do have seen quite different versions of them different version of distributing k involves into k bins

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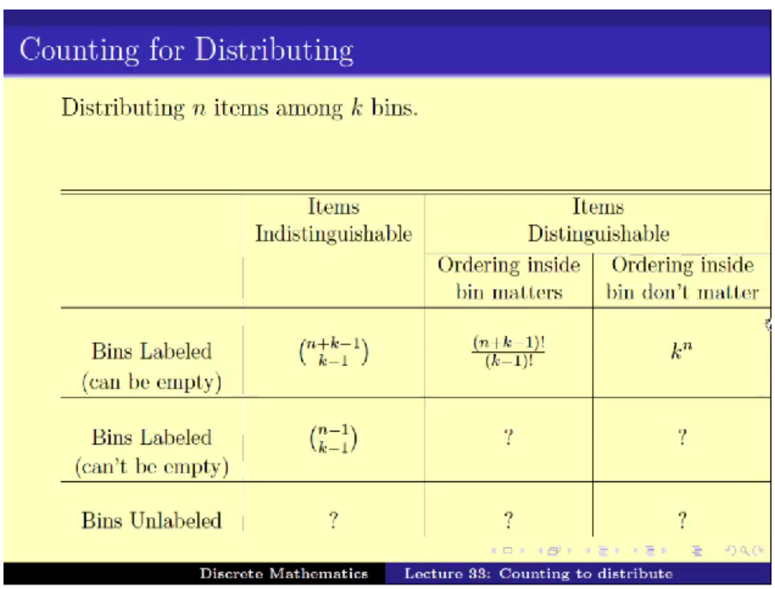

But this is what we have seen is that if the items are indistinguishable and bins are labeled we know how to solve it, n + k minus 1 choose k minus 1 when they can empty and bins are labeled and they cannot be empty then I have n minus 1 to k minus 1. When the items are indistinguishable and ordering inside the bins does not matter in that case when the bins are empty, can be empty then total number is k power n where as other place we just (()) (30:56) it is n + k minus 1 factorial by k minus 1 factorial.

As you can see in this diagram at least two of the boxes are left empty namely what happens when items are distinguishable and ordering inside the bins does not matter but the bins are not be empty and similarly the order inside the bins matters but the bins cannot be empty. There are three things that are also which are in fact what if the bins are unlabeled? What happens when the –I am not at all interested in the labeling of the bins but I just want to know how many groups in which you can form it.

So, we are left with five options, five places to solve and we will continue with our understanding of trying to solve this particular miss fit in the next class. Thank you.