Discrete Mathematics
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Lecture - 32
Counting for Selection

Welcome. So, today we will be starting a new subject in discrete math is called combinatorics. Now what is combinatorics?

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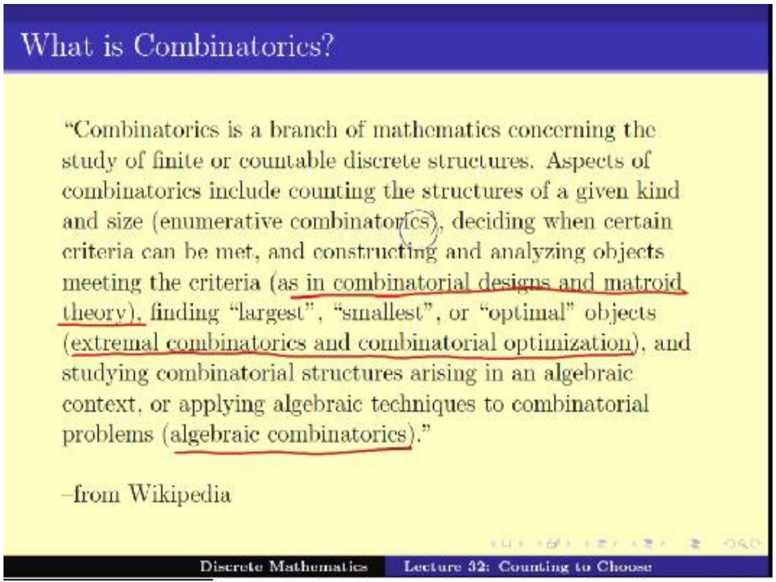

So, taking a combinatorics is a branch of math concerning studies of finite or countable discrete structures. Aspect of combinatorics include counting structures of a given kind and size, deciding whether certain criteria can be met, and constructing and analyzing objects meeting the criteria, finding "largest", "smallest", or "optimal" objects and studying combinatorial structures arising in an algebraic context, or applying algebraic technique to combinatorial problems.

Now, this is what Wikipedia says about combinatorics and as you can see in this quotation there are (()) (01:09) of combinatorics for example, enumerative combinatorics, combinatorial design and matroid theory, extremal combinatorics and combinatorics optimization, algebraic combinatorics and so on. This is one of the most interesting and challenging part of discrete math and we will see some of the basic aspects of combinatorics .

Of course, we will not be able to do a full in depth coverage of combinatorics in fact each of the subject itself requires a whole course for advance course for understanding, even understanding the context of this subject. Now, why do we need combinatorics? Again as Wikipedia says

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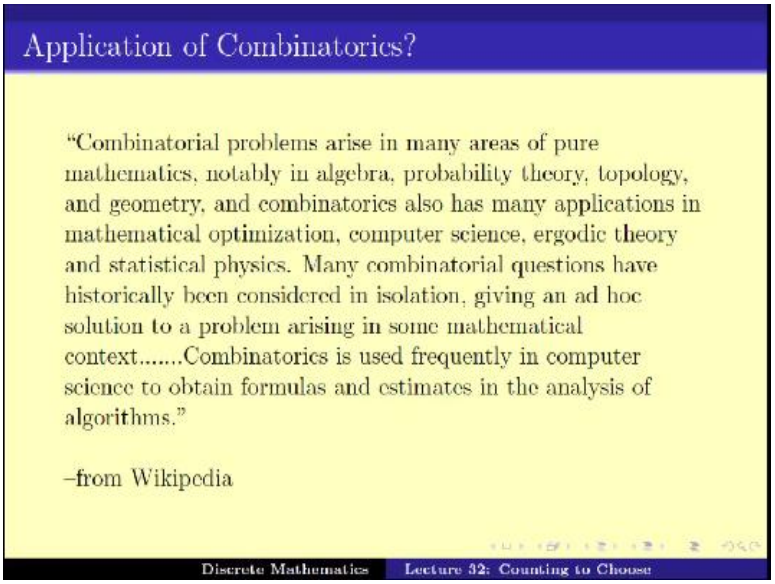

Combinatorial problems arise in many areas of pure mathematics, notably in algebra, probability theory, topology, and geometry, and combinatorics also has many applications in mathematical optimization, computer science, ergodic theory and statical physics. Many combinatorial questions has historically been considered in isolation, given an ad hoc solution to a problem arising in some mathematical context.

Combinatorics is used frequently in computer science to obtain formulas and estimates in the analysis of algorithms. So, in short combinatory is kind of the back bone of mathematics or applied mathematics or many (()) (03:12) mathematics that actually. So, in the next few weeks we will concentrating only on this subject of combinatorics and let us see how to go about it?

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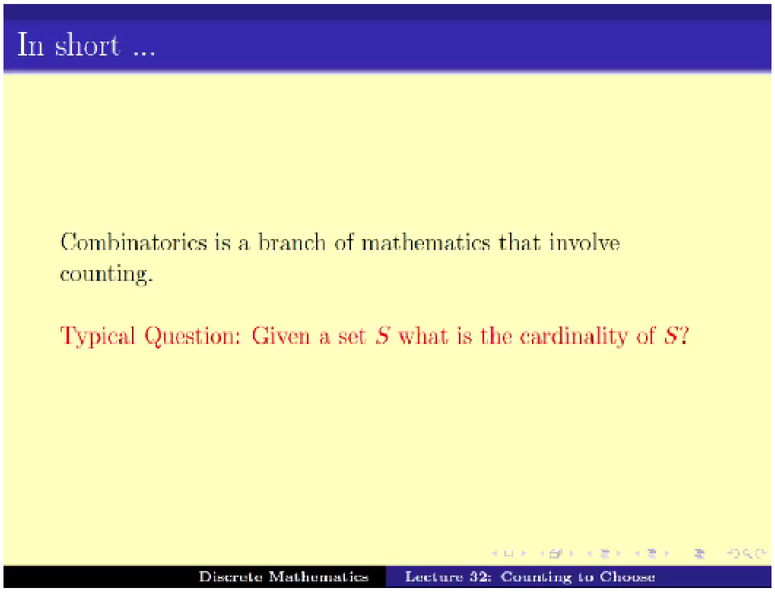

So, in short combinatorics is a branch of subject that involves counting. So, in fact that is what we will be doing for most of our time in combinatorics. We will not have time to deal with the other aspects of combinatorics but we will be talking about counting. Now, the typical question is something like I give you a set and I ask you cardinality of set S. What is cardinality? Cardinality means the number of element in the set S.

Now, you might be wondering what is there to ask in this question. I might give you a set you just count the number of elements in set S. Unfortunately, sometimes the set A is not given in explicitly. It is given you kind of implicitly in other word something like okay all the solutions satisfying this form. This is a set and it is not a set that I have given you explicitly and now I ask you how many for specific cardinality of set S.

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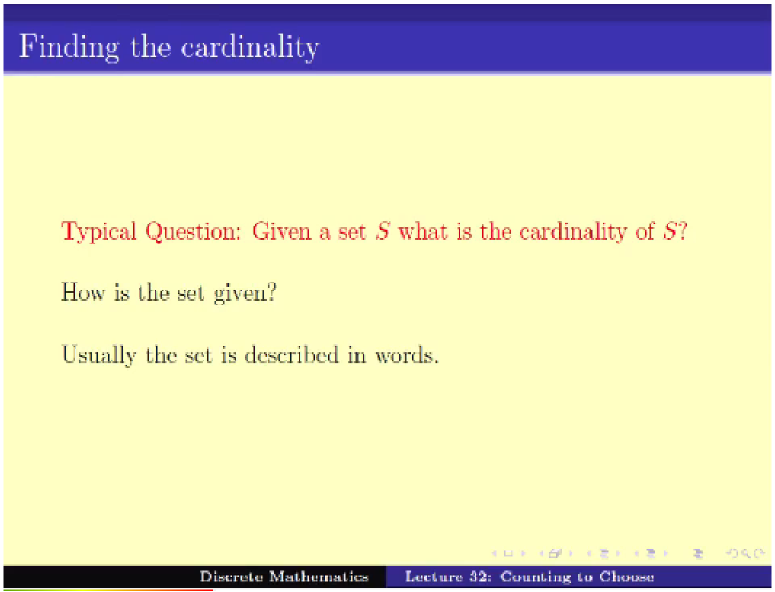

Now, the main question is how is the set given as I told you that if the set is given explicitly means set S, if I say set S is the set of 1, 2, 4, 6, 9, 10. Now, you can we just go and counting and say the set has six elements. But what happens if I say set S is the set of prime numbers less than two thousand. Okay, already it is realized that counting the set is not the easiest job and you do not know how to exactly to simplify your way to get a simple solution other than just listing out all the possible prime numbers.

Now, in general we can ask you something like what is the if S is the set of all prime numbers less than integers n then what will be the cardinality of the set S? So, that means I am asking the number of prime numbers less then n, can you give me a formula for that? And again you can hear, you can understand that this is a very hard problem. If not serious problem, it is not a serious problem it is not just counting the number of primes.

So, as I just told here the set that usually describing in words like I just now did

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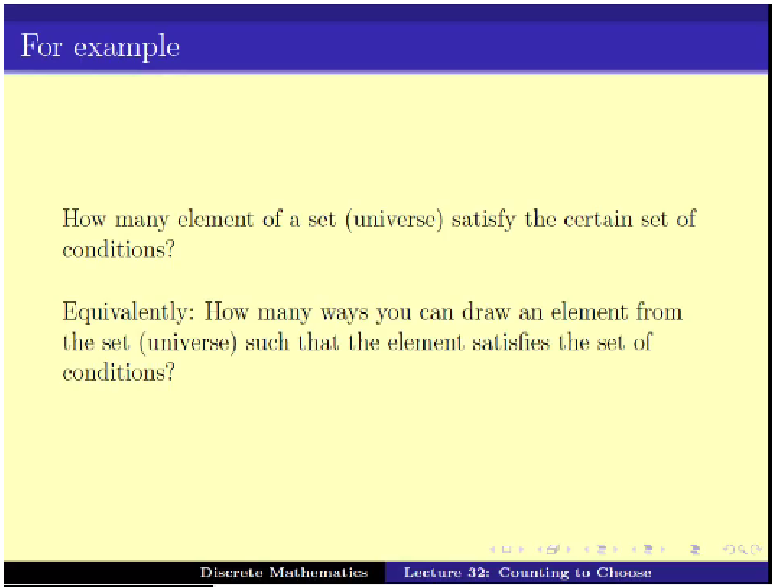

So, typical example is how many elements of a set satisfy a certain set of conditions? So, as an example if universe that we are looking at is the set of integers, I am asking how many integers satisfies the condition that which is less then n and which is prime and that is what we want to ask. Equivalently, the same question can be framed in a different way. How many ways can you draw an element from the set such that the elements satisfy the set of conditions?

So, these are two exactly same things. I mean, the first one says that how many elements are there. Second one says that if I am asking to draw a particular number say between 1 to n such that the number is prime. How many different ways can you will draw it? Now, of course the numbers of ways they draw it is number of prime numbers that is n that is there. So, you can understand that these two numbers are exactly same these two questions.

And we would be kind of jumping between the first way of setting and the second way of setting it.

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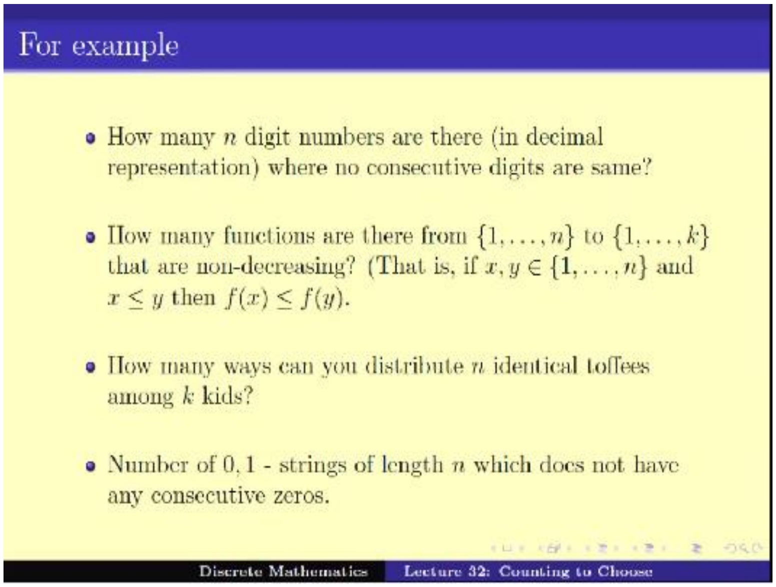

So, let us look at some examples. So, here are some examples. First one is how many end digits numbers in decimal are there when no consecutive digits are same? For example, something like if n equals to 1, I am sorry n 1 is a serial number if n equals to 2 then what I am asking is that okay you can have 1, you cannot have 1 1 but you can have 1 2 you can have 1 3 you can have 1 4 you can have 1 9 you can have 2 1 you cannot have 2 2 but you can have 2 3 you can have 2 4 you can have 2 9 and similarly it goes on.

Now, of course if n equals to 2 you can almost write down the actual number. Or, actual number of such options but if n is a general parameter then how many n digits integers are there where there are no two consecutive digits. I mean no consecutive digits I mean like 3 4 4 5 8 9 has two consecutive digits same because they are 4 followed a 4. But for example 3 4 5 4 8 9 is fine. This is an integer with single digit when no two consecutive digits are same 3 it is followed by 4, 4 is followed by 5, 5 is followed by 4, 4 is followed by 8, 8 is followed by 9.

No, integer so there is no two digits next to each other as a set. Fine, so this is a problem we will be getting back to this set of problem after we do some theory on this particular subject of Combinatorics. But let me first explain you all the various problems that we have or some of the problems that we had. The next one is consider functions from the set 1 to n to set 1 to k. The 1 2 before 2 n to 1 2 3 4 to k. How many non-decreasing functions are there?

Now, what do I mean by non-decreasing function? Meaning, if I take x y from this set 1 to n and x is less then y then f x must also be less than or equal to 1, it should not go down. So, in some set if I end up plotting this graph so this is f of x. Of course it has integer value, it takes and if this is x. Now f x so of course this will kind of a dot size so may be f of 1 of 1 is 1, f of 2 is 3, f of 3 is 4 and so on. If I draw this one it should not end up suddenly jump and dropping off.

It is fine if it remains in the same level. For example, this is okay but there is no, the function curve does not go down. Function curve is allowed to remain for a little with the acceptance. How many functions are there from 1 to n 2, 1 to k? The third question is if you have n identical toffees and you have k kids, how many ways can you give it to the k kids? So, for example if I have 5 toffees and 2 kids, how many ways can you give it?

I can give first kid I can give 3, I can give 2 to the other one. Or I give first kid 2, 3 to the other one. Or I can give 1 and 4 to other one or I can give 4 and 1 to the other one. Or, I may also can give 0 and 5, 5 and 0. So, here for example you can see that there are six ways of giving this 5 toffees to this two kids. Now, how many ways if I have a general n numbers of toffees k kids. How many ways can I distribute the n toffees to k kids?

Again, we will go over this problem after we do some theory on this subject. The last one that we are looking at is look at this string of length n made by zeros and 1. How many of them are there such that there is no consecutive zeros. Meaning, if I have a string 110101011 is fine. This is an accepted string but something like 11000111 is not an accepted string because there are two consecutive zero all right here actually there are three consecutive zero.

So, how many strings are there of length n which does not has any consecutive zero? Now, this is the four problems and let us try to take a step back and see some theory behind this subject of combinatorics and we will get back to this problems after a couple of videos.

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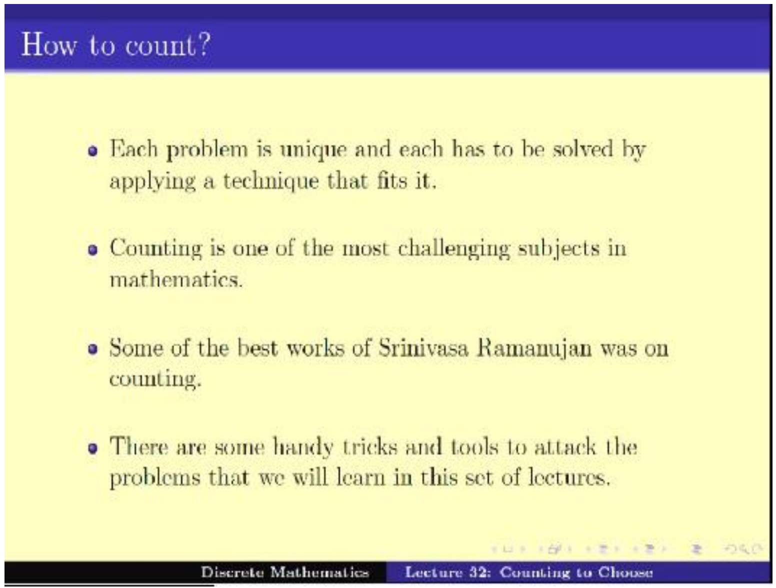

Now, how to count now this is an important question. Note, that each problem is unique and has to be solved by applying a technique that fix it. Which technique, fix which problem is not known that's per se? In fact, counting is one of the most challenging subjects in mathematics. Some of the best minds have worked on it in fact some of best work of Srinivasa Ramanujan was on counting. We will be talking briefly on this particular aspect in one of the videos.

Now, there are some handy tricks and tools to attack the problems and that we will be learning in this set of lectures. But again that we reiterate this problem, every problem is unique and you should try to be creative in trying to solve any of the problems. Tricks and tools are just handy gadgets that will have to attack this problem.

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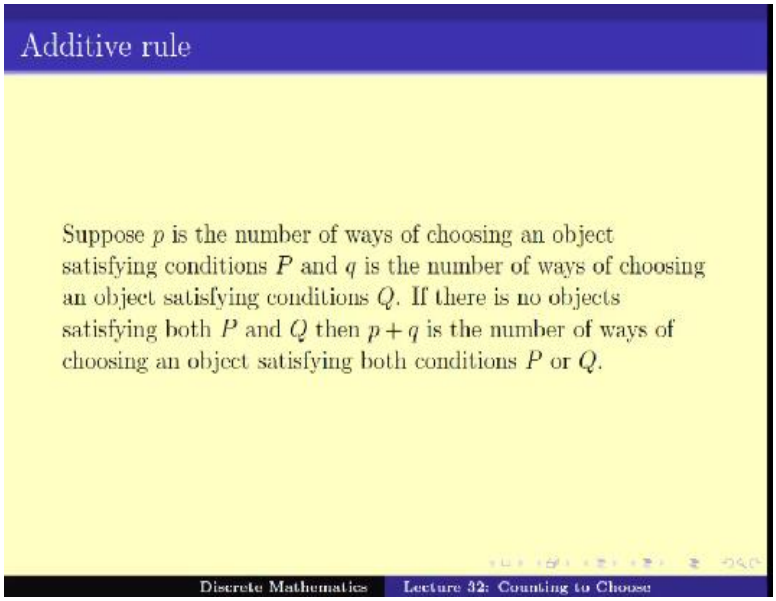

Let us start with the simplest of tricks or rules that is there. The rules says that suppose P is the number of ways of choosing an object satisfying some set of conditions say P and small q is the number of ways of choosing an object satisfying the condition Q and if there is no object satisfying both P and Q then the number of object that satisfy either the condition P or conditions Q is p + q. So, this is a very simple additive rule that we have.

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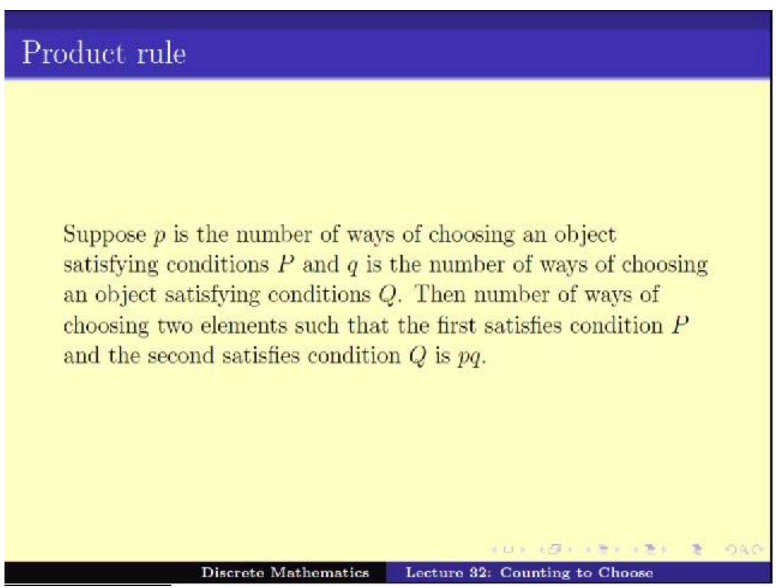

And we also have another rule called the product rule and this says that if say we have to choose two objects and if P is the number of ways of choosing an object satisfying P and q is the number of ways of choosing an object satisfying the condition Q. Then the number of ways of choosing two elements such that the first segment satisfies condition P and second element satisfies the condition Q is P times Q. So let us quickly go over some examples to see both the schools.

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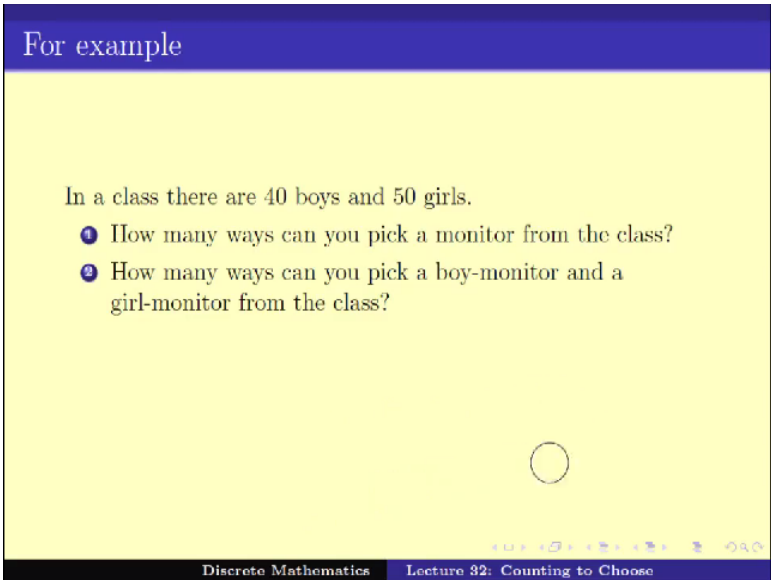

Say in a class there are 40 boys and 50 girls. How many ways can you pick a monitor from the class? Now, here let me first define two conditions P and Q, P being the number of ways or rather I shouldn't say number of ways but P is the case that monitor is a boy. Q is the case that monitor is a girl and now if this is the case, then what are you asking? In this problem, you are asking how many ways can be pick up a monitor who either satisfies P, meaning that monitor is the boy or satisfies Q that is the monitor is a girl.

Now, number of ways I can pick a monitor with a boy is still there are 40 boys I can pick any one of the 40 boys that means this is 40. So, in our definition P is 40 and here the Q is 50. So, if there are 50 girls number of ways I can pick up or monitor who is the girl is again any one of the 50 girls and I get 50. So, the total numbers of ways I can pick a monitor with either a boy or a girl is p + q which is 90.

Now, you might think that this is a very stupid kind of example I could have just straight away gone and prove it directly that the number of ways the picking a monitor from 90 student is 90. You are right it is just a way example of our swing is the additive rule. Now, the additive rule is something really simple but also we do apply it all the times and we will be coming back to the additive rules sometimes in the next video.

Actually, next to next video when we will see some more complicated example for the additive rule. Now, let me ask the second question, how many ways can you fix boy monitor and a girl monitor for the class? So, I want to fix two monitors first of them will be a boy monitor and second one will be a girl monitor. Now, again the same way that I did the last time here. P if the monitor is a boy and Q the monitor is a girl and therefore that here p was equal to 40 and q equals to 50.

Now, the number of ways I can pick first a boy monitor and then a girl monitor, there is by the product rule p times q which is 2000 for them 50 is to (()) (21:58). So, there are 2000 ways of picking one boy and one girl such that one of them is a monitor, boy monitor and one of them is girl. So, this is some very simple application of the product rule and the addition rule. Now, do you think this two simple rules let us move ahead and try to see some more interesting problems.

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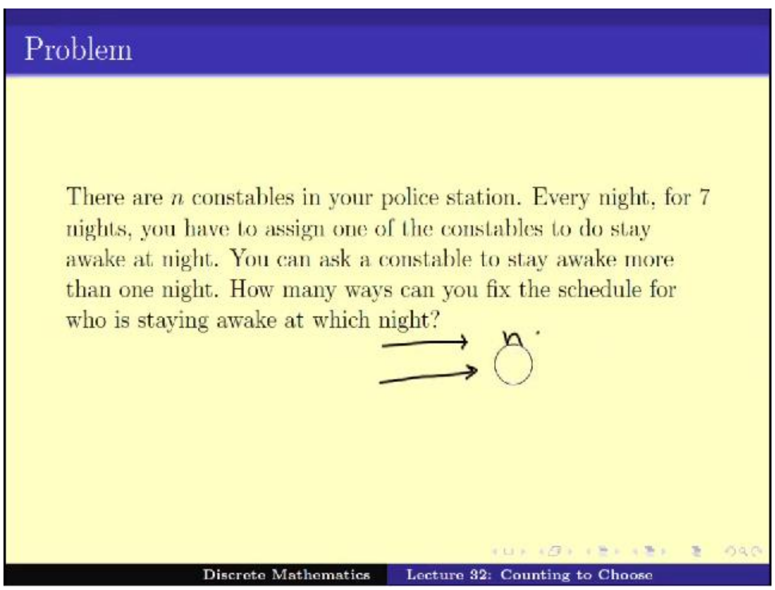

So here is the problem set. Imagine you are a police office and there are n constables in your police station now every night for seven nights you have to assign one of the constables to stay awake at night. Now, you can ask the constable to stay awake for more than one night. May be for one particular constable is a very –if someone you will depend on much more. So, you want him to stay awake on two of the nights on Monday and Wednesday.

So, let us think of this way that there are seven night, namely Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, and Sunday. Now, each of the night you have to assign a constable right? so I can for example, I can put the Constable number 1 goes for Monday, Constable number 2 goes for Tuesday, Constable may be again I can again ask constable 1 to take up to do the duty again on Wednesday. Thursday maybe I can ask constable number 3.

On Friday maybe I can again ask constable number 1. On Saturday constable number 2 I ask and on Sunday also I ask constable number 2. Now, this is a possibility. This is a one way of assigning the constable, constable 1, 2, 3 to the seven days. Now, there is other way I can do it, for example, so as the line manager has decided that no I will put constable 3 here and constable 1 here. Now, you as you can see there are different ways of filling up this chart question is that how many ways can you fix this schedule for who is staying away at which night. In how many different charts can you prepare?

Now, the answer is not too far it is again slightly kind of the same product rule application from the first night I can assign, how many ways can I assign a constable. I can assign any of the n constables. So, the number of ways I can assign is n. On Tuesday, again the number of is again assign is n. I can ask any of the n constables again to guard to stay awake at night. On Wednesday again n so except for all the days numbers of ways I can fill up that or assign as constable to that day n.

Now, with product rule the total number of ways in which I can assign constables to this days. So that these are ordered in some sense like there is order going about it is the product of them which is n power 7. First day n, second day n. So, number of ways I can put a constable in the first day and the about second day is n times n, n square and then along with (()) (26:50) n cube and so on. So, this is a very simple observation and in fact this falls under a general problem which basically asked the question.

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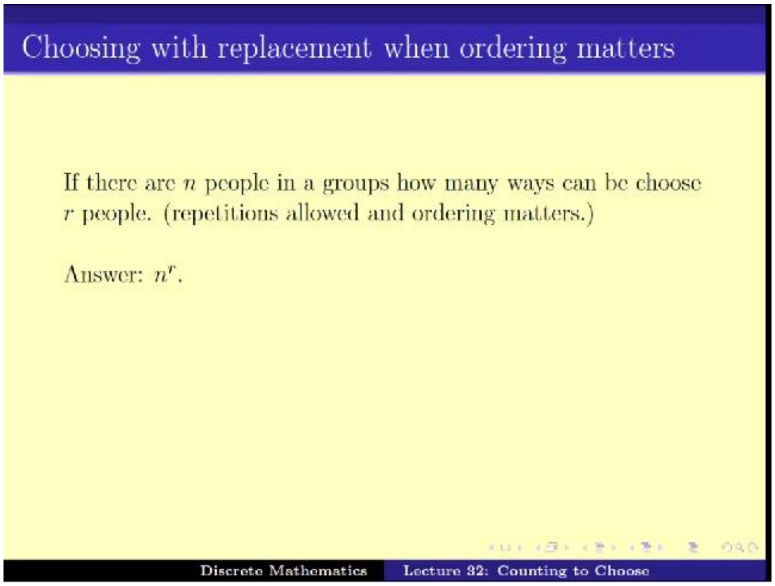

If there are n people in a group how many ways can you choose r people, repetition allowed and ordering matters. Now, let us try to see how does the earlier problem match here? Earlier problem I had n constables and there r was the 7. I have choose 7 people in this way back that I have to choose 7 constables for seven days repetitions allowed in the sense that the same constable can be assigned multiple nights and ordering matters meaning whether the constable wants us the Monday job.

And constable 2 does the Tuesday job is different from constable 2 doing the Monday job and constable 1 doing the Tuesday job. If you flip or reshuffle the assignment, I get a completely different allocation. So, again how to answer this question let us see. First, of all there are positons we have to people r people and they are ordering matters. So, let us we call for Position 1, Position 2, Position r.

Now, Position 1, number of ways I can fill up position 1 is I can choose any of the n people so n. Number of ways, I can fill Position 2, again since I am allowed to use this people and the repetitions are allowed we can choose it with n also and similarly for all the r positions each of them can be filled in n ways. So, the number of ways I can fill up the all of them if n times, n times, n all the way which basically is n to the power r is the answer.

Now, this is also sometimes called choosing with replacement because here what we are doing is that we are drawing a person from the list and then we have – so in some case, imagine that you have n people and their names have been written into piece of paper and put into a basket. Now, what I can do is that I can pick up one person name from that one paper from the basket. I see someone's name assign that person the first position.

If it put back the slip into the basket so that I can again have an option of picking that person out and I can draw this one more chit and assign the next one, next chit to the next position. So, this way, we are basically choosing with replacement. We are replacing the chit every time. So, this is very generic thing that we did very general statement that for what we just show the constable problem. Now, let us look at some slightly different problem.

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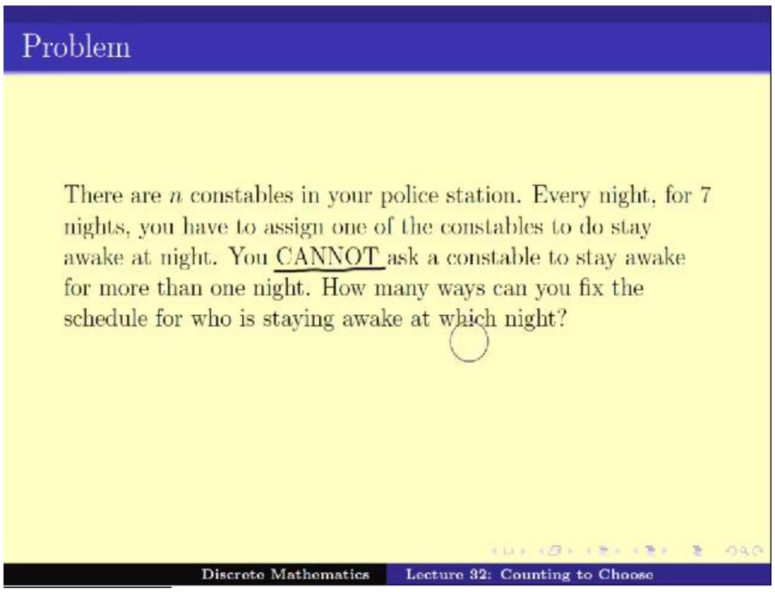

Here, it has the same thing except that again you have to assign some n constables for seven nights but you cannot ask a constable to be awake more than one night. Let us try to see how to go about it. Again I have Monday, Tuesday, Wednesday, Thursday, Friday, Saturday and Sunday. Now, how many ways can I pick a person serve the Monday night? Of course, all the n people are there and hence I can pick up in n ways.

Now, how many ways can I pick up a constable to solve the Tuesday night. Now, the question we have already done the Monday night cannot be asked because I have been told that a constable cannot be asked to do two nights. So, he is out of question. So, I am left with n minus 1 possibility which means that this on Tuesday I can fill up this position in n minus 1 ways. What about Wednesday night? Wednesday night.

I cannot use the person who has served the Monday night. I cannot use the person who has served Tuesday night. So, I am left with n minus 2 constables and we can pick any one of them. So, the n minus 2 and goes on like this so I have n minus 3 over here, n minus 4, n minus 5, and n minus 6. So, at the end how many ways can I select the 7 constables? I can n times n minus 1 times, n minus 2 times, n minus 3 times, n minus 4 times, n minus 5 times, n minus 6 times, n minus do not know that is all n minus 6.

So, it is like some product of all of this is what my answer is. So, again this can be form in a slightly more general setting.

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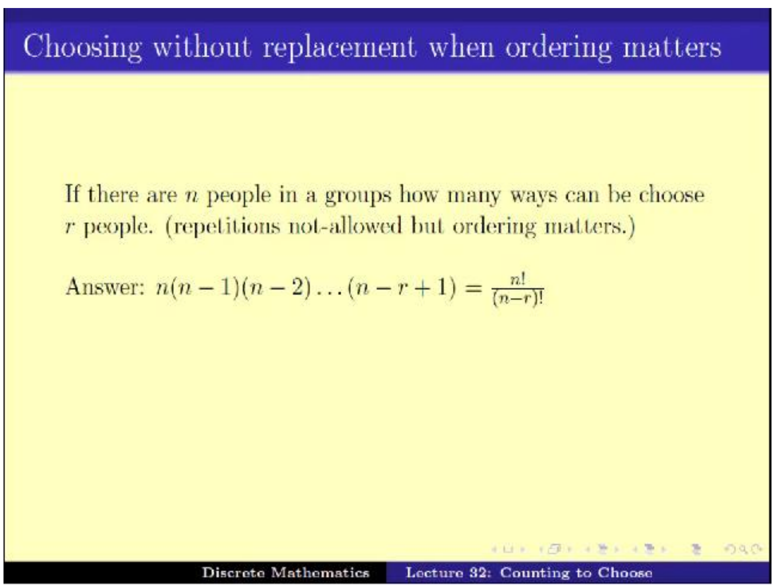

Namely, there are n people in a group in how many ways you can choose r people with repetitions. Sorry, repetition is not allowed but ordering matters. Meaning, I have the position p 1 p 2 till p r. The first one I can assign is n ways, second one I can assign it n minus 1 way, third one is n minus 2 ways just like in case of the constable problem and the last one will be n minus r + 1. So, the answer would be product of this set n times n mins 1, n minus 2 to n minus r + 1 which is by definition n factorial by n minus r factorial.

Now, note that here ordering matters in the sense that if I pick constable 1 here and constable 2 here and constable 3 here the constable r here that is different from constable 2 here 1 here and 3 to r here. So, this two r two different ways of assigning them because I am assuming that these two things are distinct I mean this two positons are ordered. So p 1 and p 2 couldn't be next to each other.

And as you correctly said since I am not allowing repetitions this will also now called the choosing without replacement. Now this is also a special case. There is a special case of this

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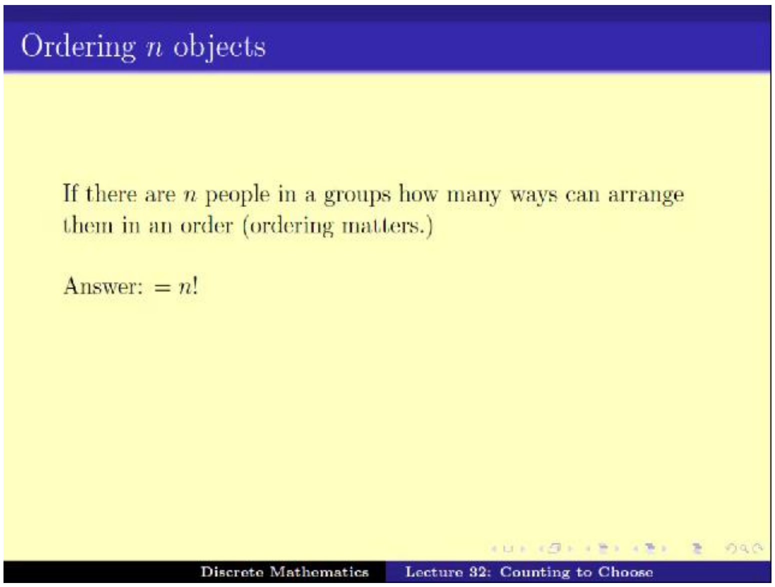

Namely if I give you n people in a group and ask them to arrange in an order, where order matters so in that case what am I asking? I am asking that can you choose n people from this n? How many ways can you choose n people from this group of n people? The first one can be any one of the n people, second one I can any one of the rest of them and goes on like no one left to pick. So, this one is r here is n and if you put plus in the values of r in the original in their earlier problem you get n factorial.

So, does you see that n factorial is in fact the number of ways in which n people in a group can be arranged in a n factor is the number of ways in which you can order n people in a particular order.

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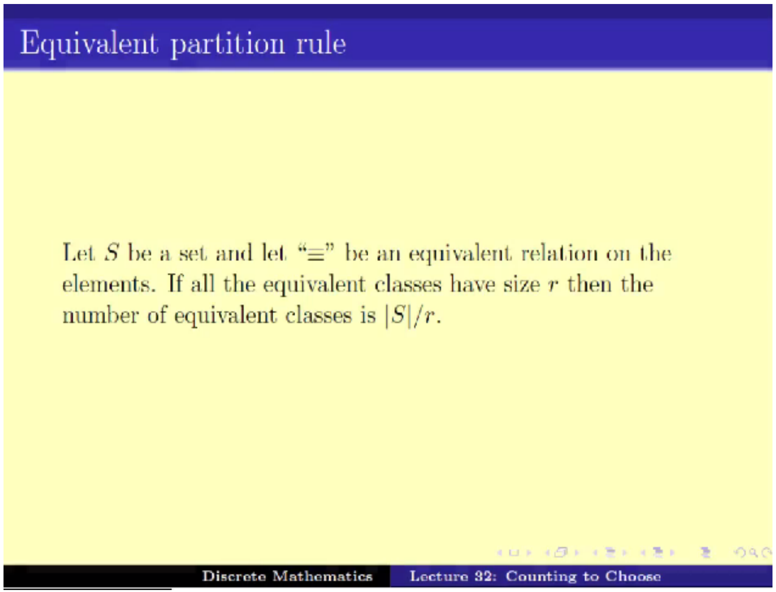

Now, let me refer about the next problem. Let us talk another nice rule because it is equivalent partition rule. Now, let us S be a set and let a defined lesson has this three lines, three bars be an equivalent relation on the on the elements. So, once you have an equivalent relation you know that as equivalent classes and all the set of elements that is equivalent to each other. Now, if all the equivalent classes has same size r then the number of equivalent classes that we have is size by r.

Now, this is not a very hard problem, hard thing to realize. If I have a set here and I have equivalent classes all of them have size r then total number of this kind of same equivalent classes is that S by r. So, we can use this one for counting certain cases.

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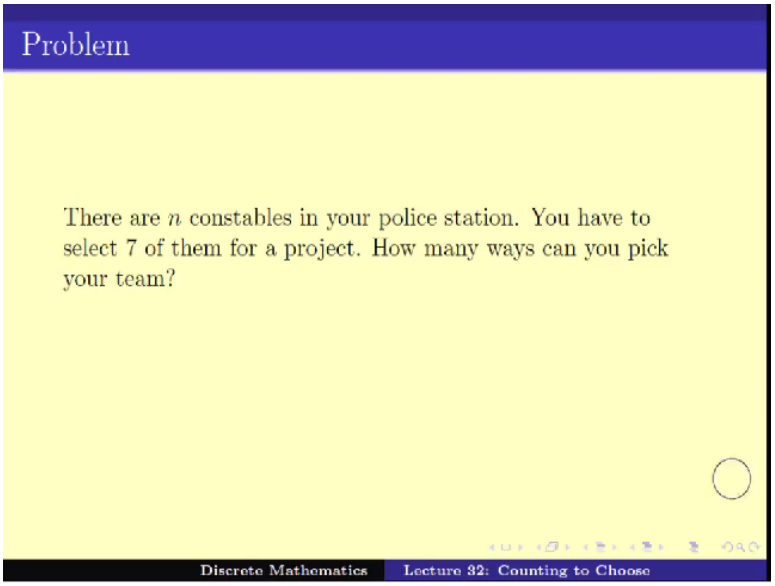

So we will look at this problem there are n constable in a police station. You have to pick 7 of them for a project. How many ways can you pick your team? Now, different from this and the earlier one was that here the ordering does not matter. I do not care about who is the leader of the team and all. There are 7 people in the team. How many ways can you pick them up? Now, okay may be what we do is default let us first write down the thing that we have done which is the seven days of the week.

And let us first try to assign the various constables for this seven days. So, let us imagine that I have a –So I have assigned Constable 1 here, Constable 2 here, Constable 4 here, Constable 3, Constable 6 here, Constable 9 here and Constable 10 here. So, this is one way of doing it. Now, this is one way of in the earlier problem assigning the seven constables to seven nights. For them in that problem even this one was a different way of assigning constables.

Where I have all done is an x square this constable for Monday and Tuesday. Note, that in the earlier problem they too were different but in this problem when I do not care about ordering I do not care about who is doing which day. I just want seven people these two equivalents. So, now because here I pick the same set of it. So, in set, question is that if I give you a set of seven constables as long as this set has the same set of constables they are equivalent.

Question is that how many such arrangement is equivalent to say this particular set? It is the same number of ways I can permute this seven constables. So, this is equivalent to 7 factorial number of other picking up. So, in the whole set which if you remember was n minus sorry it was n factorial by 7 factorial. It was n factorial by n minus 7 factorial. This was the total number of ways in which I could pick seven constables for the seven nights by ordering matter but now each of those arrangements is equivalent to seven factorial number of ways.

So, by the equivalent partition rule to get the number of groups that I can get which is in this case as I have to look at the number of equivalent classes is that I have to divide this number by 7 seven factorial and hence I get this number. So, you might have seen this particular thing earlier it is called n to 7 both number of ways I choose 7 people for group of n people. But the idea is basically this that you first see how many ways you can pick people in an ordered manner.

And now each of the ordered selection is equivalent to 7 factorial number of them and I have to divide by 7 factorial to get the number of equivalent classes.

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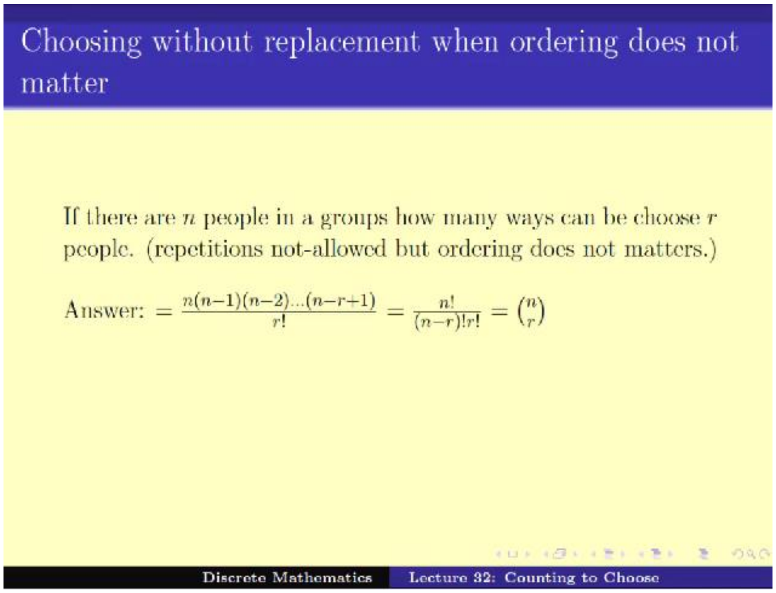

Again, in a general setting same kind of ideas if I have n people in a group and how many ways can I choose r people repetition is not allowed and ordering does not matter it is whatever the answer that I had last time by r factorial (()) (44:27) for the constable case which is n factorial by n minus r factorial and r factorial which is n choose r. So, this is number of ways in which we can choose when repetitions are not-allowed but ordering does not matter.

Now, what happens if I ask this following question?

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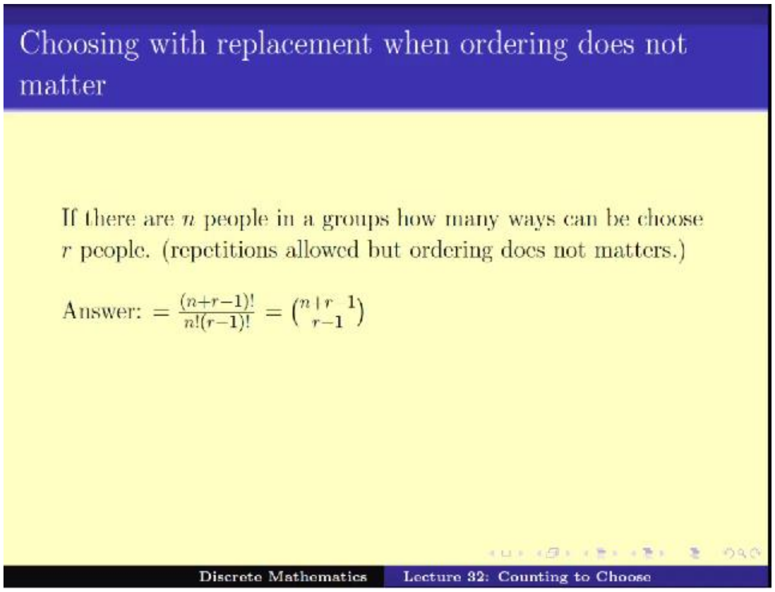

There are n constables in your police station. You have to give the names of 7 constables. You can give a constable name more than once. How many ways can you give this 7 constables? So, in other words what I am asking is if we go for the general problem here and there are n people in a group how many ways can you choose r people when the same person can be chosen a few number of times repetitions are allowed but ordering does not matter.

I do not care about the ordering inside each group. Now, how do you solve this particular problem? So, solve this problem let us first start with the position like we did. But earlier we were writing the position as p 1 to p r unfortunately we cannot such seats because the positions are not the same in the sense that I do not care about the ordering. Since, I do not care about the ordering so I cannot write this P 1 P r is that let me just try to write as some round balls.

So, there are round balls and I have, how many of round balls I have? I have r round balls, and these are the postings that are there. Now, I have to kind of give the positions to people. So, if people r marked right? so I have the person one P1 or maybe I go back it say I will give the times that they are constables. So, they have the Constable 1, Constable 2 till Constable n. I have n constables. Now, question is that how many times should constable 1 name be in the group?

Should it be once, twice or no times or r times (()) (47:26). So, in other words, it is like I have this points and if I draw a line I kind of say that okay if I give the constable 1, two positon then I will draw line here and I say that okay everything left of the line is for Constable 1. It basically implies that Constable 1 gets two positions in that r. I want the r positon. Similarly, in constable 2 I will draw a line here and whatever the number of point that is there is in Constable 1 and Constable 2 line in this case zero is the total number of position that Constable 2 gets.

Now, in other words here Constable 2 does not get any position. So, may be Constable 3, I will draw a line over here and that is saying the Constable 3 get 1 position. Maybe Constable 4 has line next to Constable 3 as Constable 4 does not a position. Here, Constable 5 does not get a position. Here Constable 6, get 3 positions. So, in other words and at the end of the day how many lines should I draw?

If I have drawn n minus 1 such vertical lines at the end I will say that okay whatever you left it goes to constable n. So, in another word this is so a particular arrangement of this round circles and straight line indicate that since there has n minus 1 bars or straight lines I have n intervals in between these straight lines as that indicate the number of times the ith constable name is in the group.

So, in other words the number of ways you can permute or number of ways you can write down the set of string for me with r circles and n minus 1 red lines is going to be the number of ways you can pick r constable from m constables where we draw it with repetition but ordering does not matter. Now if you do it you see that how many objects are there? There are the r circles and n minus 1 straight lines which in turns become r + n minus 1 objects are there.

So, in number of ways I can write a string with r circle and n minus 1 straight line is r + n minus 1 factorial, I am just pointing to all of them but since all the positions are identical in since all these circles are identical I should divide by r factorial and I should also divide by n minus 1 factorial because this bars are also identical. So, this way I have been able to count an indirect way how many ways I can select r people from n people where repetitions are allowed and ordering does not matter.

So, answer is in fact n + r minus 1 factorial by n factorial times r minus 1 factorial which do become n + r minus 1 choose r minus 1. So, this particular way of speaking in this particular problem the last part of it is reasonably complicated and tricky. We will come back it again next video in a different light. How to and we will see the same problem in the next video also.

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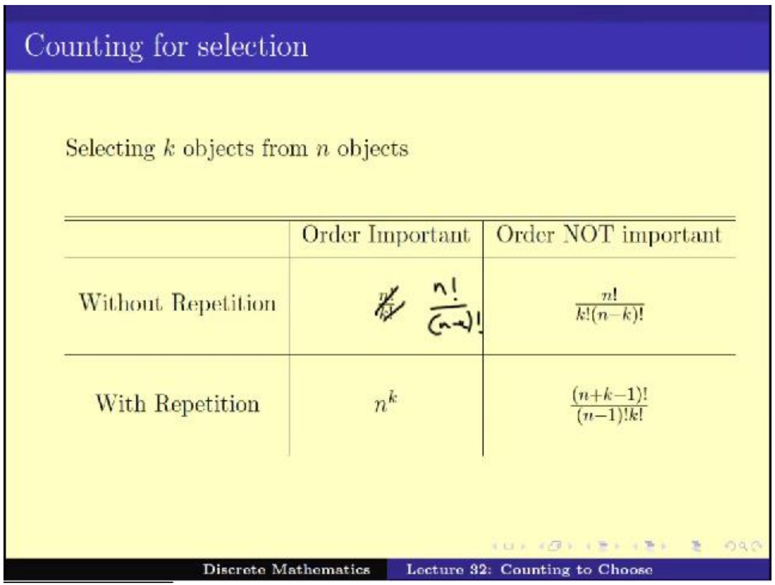

So, for now what we have seen? We have seen that if you have to select k objects from n objects the number of ways of doing it is in the case of which repetition is this n factorial - without repetition order important n factorial by k factorial. Sorry, I have made a mistake here it should be n factorial by n minus k factorial. This is n factorial by k times n minus k factorial and the last one which we just did now.

Yes, this one, n + k minus 1 factorial by n minus 1 factorial times k factor. In the next video, we will come back and see some more counting problems. Thank you.