Discrete Mathematics
Prof. Sourav Chakraborty
Department of Mathematics
Indian institute of Technology – Madras

Lecture - 03
Propositional and Predicate Logic

Welcome everybody to the third video lecture in discrete mathematics. In this video lecture, I will be talking about propositional and predicate logic. This will help us set up the mathematical foundation using which we will be writing mathematically sound proofs.

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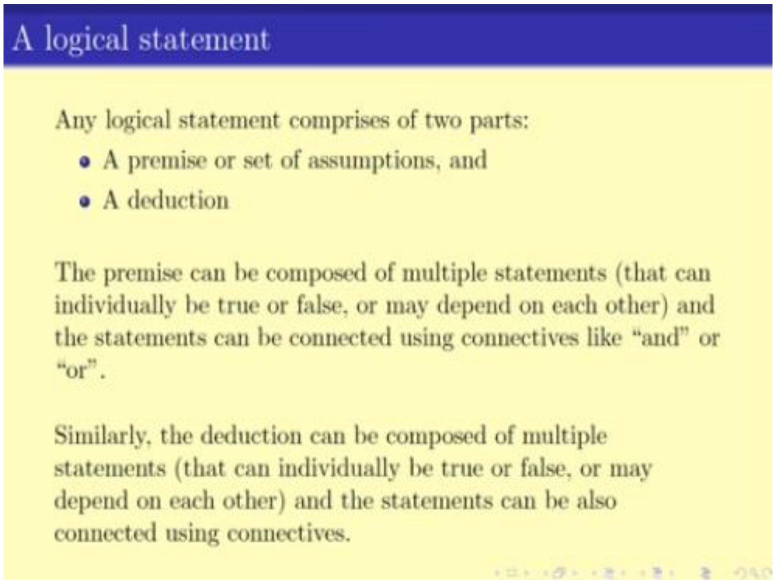

To start with, let us consider any logical statement. You can think of a theorem, for example, it comprises of 2 parts. First part is a premise or set of assumptions and the second part is a deduction. The premise or the set of assumptions can be composed of multiple smaller statements each of which individually can be true or false or in other words, each of the smaller statement can either be satisfied or not.

And these smaller statements are connected using connectives like “and” or “or”. Similarly, the deduction can also be composed of multiple statements, connected using connectives.

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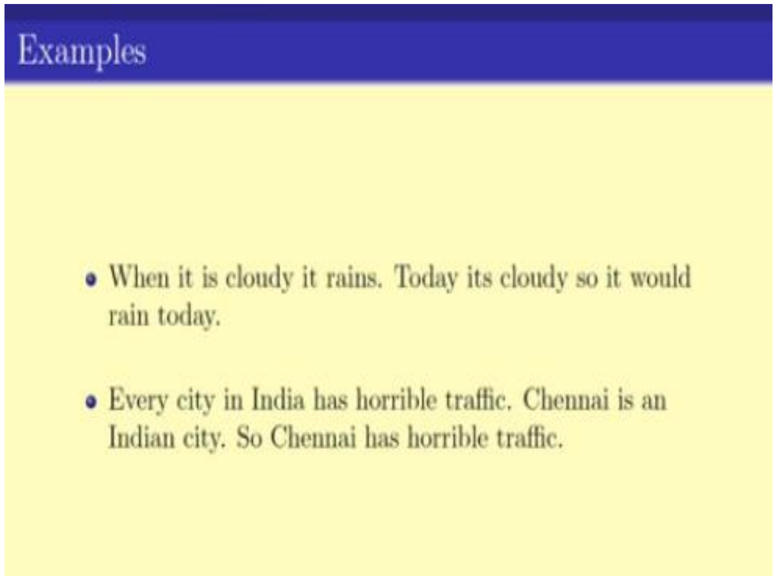

For example, let us consider this following statement. When it is cloudy, it rains. Today, it is cloudy, so it would rain. It is a statement a logical statement. It has some set of assumptions and a deduction. Now what are the set of assumptions? In this statement, this sentence when it is cloudy, it rains and the sentence, today its cloudy are both part of the assumption and what is the deduction? The deduction is it would rain today.

As you can see, the assumption comprises of two statements and ideally one should think of it as when it is cloudy, it rains and today it is cloud. That is the assumption and the deduction is it would rain today. So we see that the assumption is composed of 2 statements and connected by the connective “and”. Now when it is cloudy, it rains. This can be true of false. Similarly, today its cloudy can be true or false.

So the each of the individual statements can either be true or false. Similarly, consider another example, every city in India has horrible traffic, Chennai is an Indian city. So Chennai has horrible traffic. Now what is the deduction here, deduction is Chennai has horrible traffic what are the assumptions? or what is the assumption? The assumption is every city in India has horrible traffic and Chennai is an Indian city.

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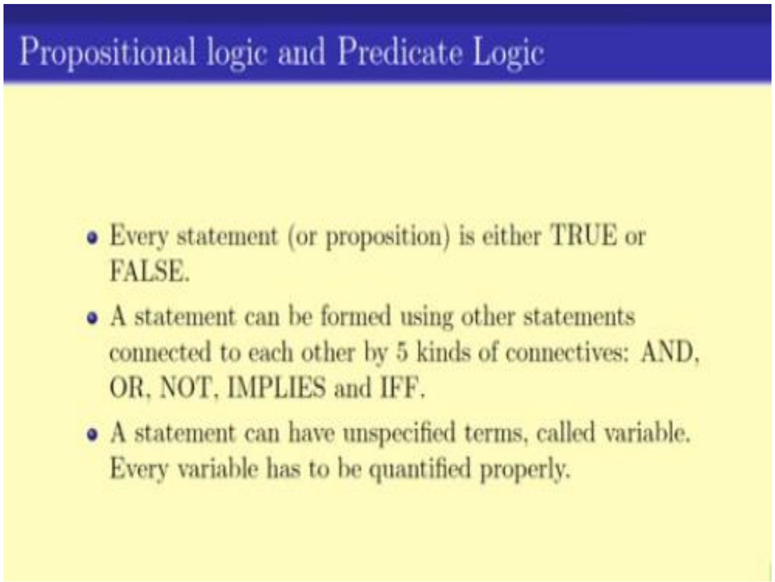

So, formally speaking, in propositional logic and predicate logic, we say that every statement or proposition is either true or false. These statements can be composed of other smaller statements and they can be connected using 5 different kind of connectives and, or not implies and if and only if. A statement can have unspecified term called variables, each variable has to be quantified properly.

Let us go back to the examples that we were seeing just few some time ago. So in the first example, as you can see there are no variables. It says that when it is cloudy, it rains and today, its cloudy implies it would rain today. Thus we had 3 different statements and they are connected by the connectives and implies. On the other hand, in the second example, there is this steeped every city. Now I have not told what city or which city and so on.

So this statement, every city in India as horrible traffic is a statement. So I have this city either variable and this term every city is kind of used to quantify the variables. So we have every city in India has horrible traffic in the statement having of variables which is quantified using this the quantified every city and we have the whole sentence as every city in India has horrible traffic and Chennai is an Indian city implies Chennai has horrible traffic.

You should understand that when we write something in English, we need not use the exactly same words as the connectives. For example, and, or, implies, if and only if (IFF) and not. We might use something else some other synonym. So to understand an English sentence, we have to understand what is the inner meaning of that so in that 'and' be replaced by a full stop or something like that as you can see in this (()) (06:32).

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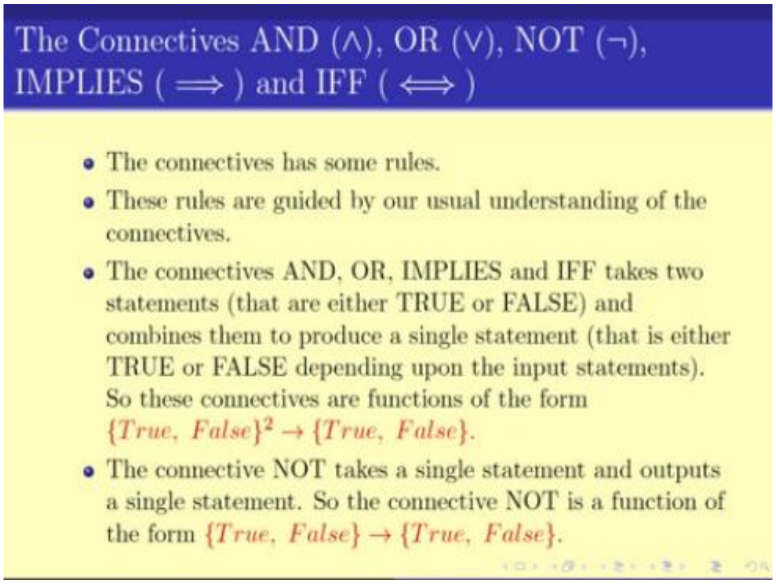

Now, the connectives of and, or, not, implies, IFF satisfy some rules and these rules are guided by our usual understanding of connectives. By the way, the connectives 'and' in mathematical term is represented using this particular notation or represent within the inverted V, not in this example with this notation, implies with this implication mark and IFF with the double implication mark in either direction.

Now these connective AND, OR, IMPLIES and IFF takes 2 statements, each of which are either true or false combines them to produce a single statement that is either true or false depending on the input set. So in other words, these connectives are a function from {true, false} whole square to {true, false}. Connective NOT takes a single function and produces a new function, so the NOT is a function from this set {true, false} to the function {true, false}.

Let us see some of the rules that are used for AND, OR, IMPLIES and IFF.

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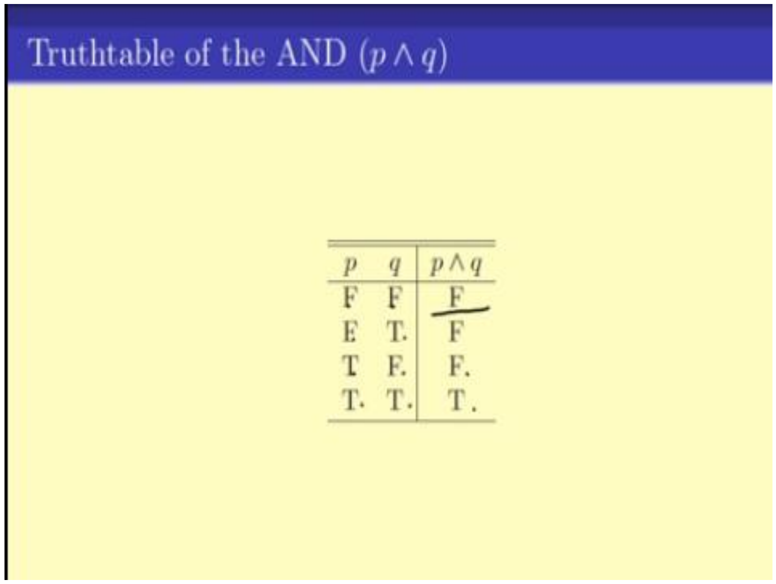

So here is the truth table, you can go over the truth table and you understand what does it mean by each of the statement. Now this is the way, we should predict. So if P and Q are the 2 statements and we are saying statement like P and Q. So if P is false as well as Q is false then P and Q as a statement is false. This is kind of natural like when we say that this happens and this happens.

And if any one of them does not happen, it means that the whole thing as a whole also does not happen. So false and false gives false. Similarly, false and true gives true, true and false gives false and the last one is true and true gives true that means only when both P and Q are both satisfied or they are both true statements only then give combined function of P and Q is also true.

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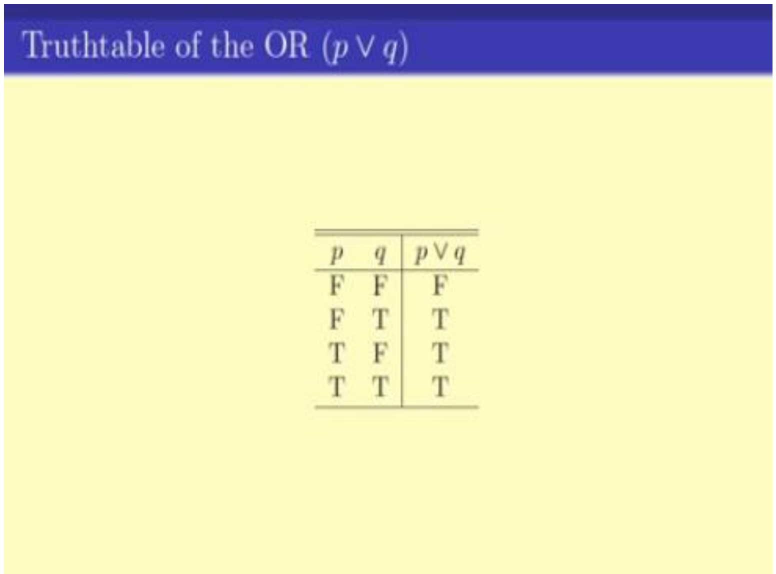

Similarly, for or for an example, say if I say that I will either give you an A grade or I will give you a B grade. But this says that as long as I do either of those 2 objects, the whole sentence is true. So in other words, as long as, one of P and Q is true, I get true where as if both of them are false then the function evaluates to false.

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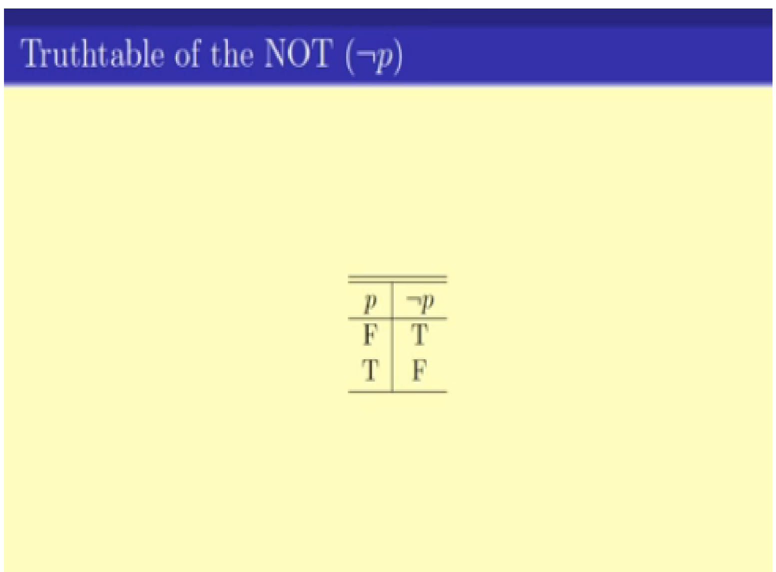

From the connective NOT, it just flips the function, it just adds a negation to the statement and just a false statement becomes a true statement and a true statement becomes false statement. Now let us consider this interesting function of IMPLIES.

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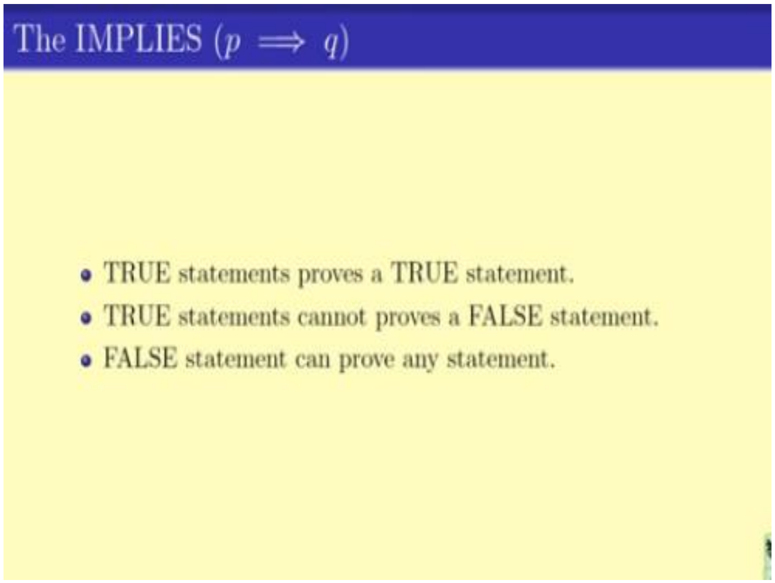

When can we say P implies Q is a right statement. So when the mathematicians and the logicians were setting up the mathematical foundations. They were divided as to how to define this particular function implies. So one thing that they decided is that a true statement always proves the true statement. Hence if P is true and Q is true then the function P implies Q is also a true statement. In same time, P cannot imply the false statement that is the other way of saying.

So if P is true and Q is false then the whole statement has to be evaluated to false. Here is one slightly more complicated thing that a false statement can prove any statement. So in other words, if P is a false statement from that, I can imply anything which means false implies true; It is also true as well as false implies false is also false.

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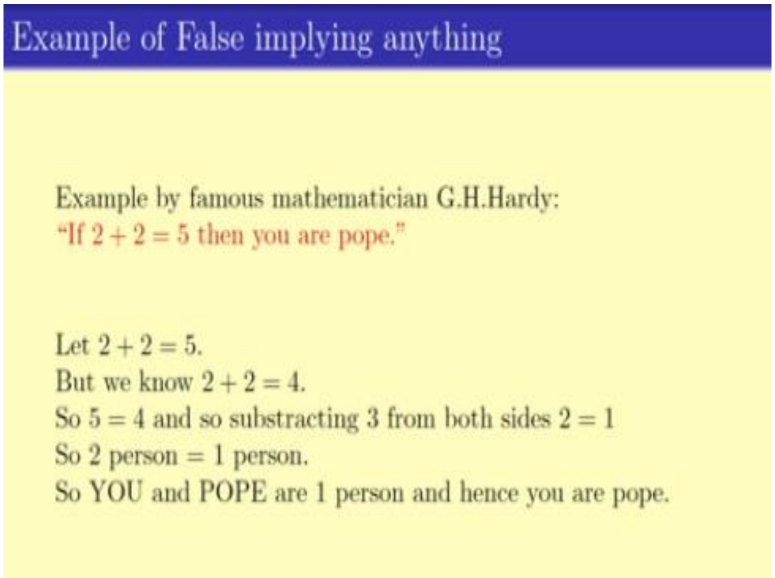

There is a lot of debate on this and one example of one beautiful story that goes along is that when the mathematician G.H. Hardy, was asked to comment on this statement or was asked to comment on this particular rules for implies. He was asked to prove that can you prove this following statement. If 2 plus 2 equals to 5, then you are Pope. Now clearly 2 plus 2 equal to 5 is false and you are Pope is also a false statement.

Now can the false statement imply any false statement. For example, here and G. H. Hardy did it, give a proof and here is a very cute proof of that, it says that 2 plus 2 equal to 5. Now everybody knows 2 plus 2 equal to 4 that means 5 equal to 4. If you subtract 3 form both sides, you get 2 equals to 1. So 2 person equals to 1 person. Therefore, you and Pope are one person and hence you are Pope. It is a very cute proof or very cute example that shows that if you assume something is false, in this case 2 plus 2 equal to 5, then you can prove any ridiculous statement or not, for example, you are Pope.

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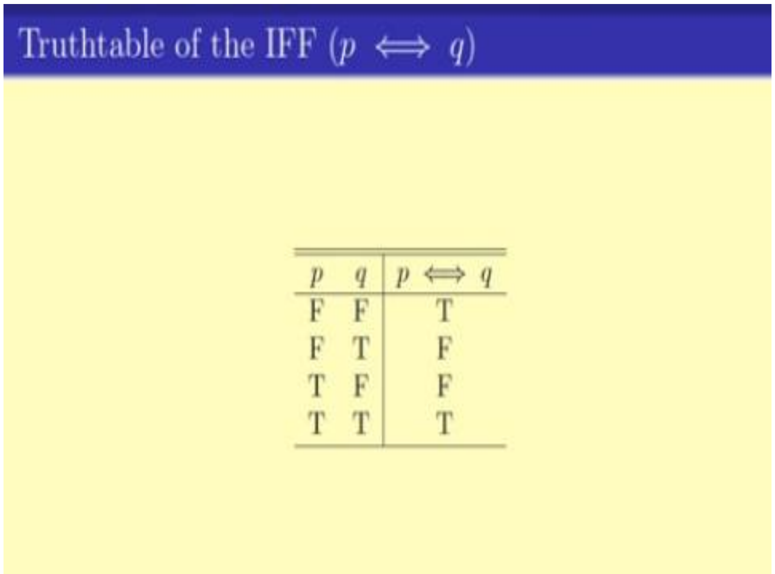

p	q	$p \iff q$
F	F	T
F	T	F
T	F	F
T	T	T

So in other words, for the case of implies, it is accepted that a false statement implies any statement. Thus, the truth table of the implies function has false implies false is true, false implies true is true, true implies false is false and true implies true is true. Similarly, we have the other thing of IFF and I leave you guys to convince yourselves that this particular truth table also makes sense.

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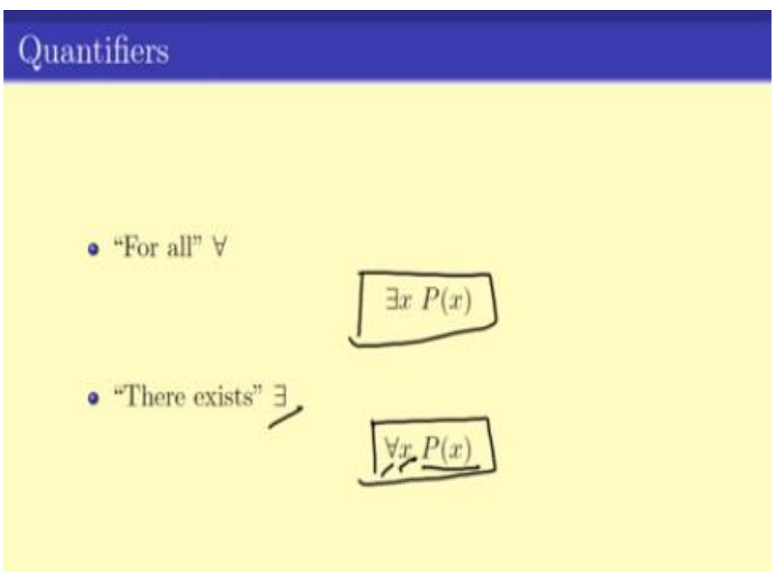

Now there are 2 quantifiers also, that are there, as I told you, every variable has to be quantified and these quantifiers are "For all" and "There exists" and you write statement like this. So this says that "There exists" X such that P of X is true. This is how you should read the statement. Similarly, this one says that for all X this statement P of X is true. If you recall the example we had used, there was this example of every city in India has horrible traffic. So in that respect X is the city every means for all. So for all city and statement was the city has horrible traffic. So it should read as for all city in India has horrible traffic.

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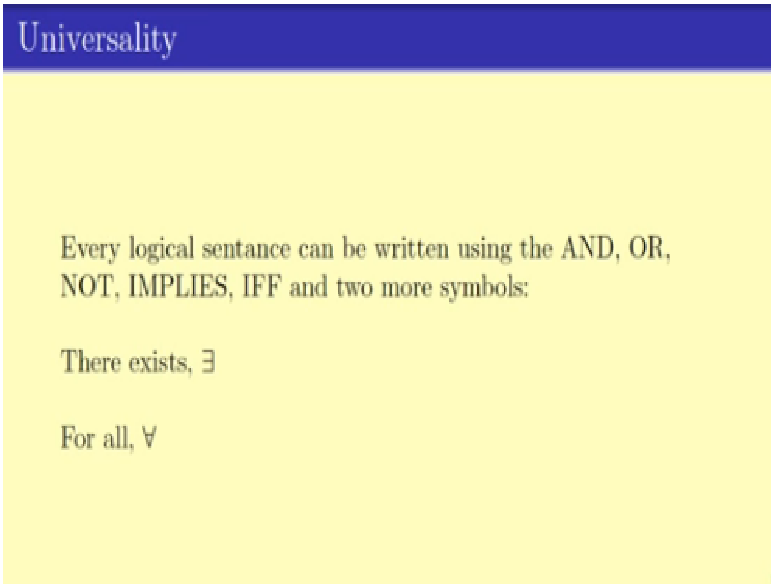

Now, it is a nice important fact that using this 5 connectives and the 2 quantifiers, "There exists" and "For all". One can write any logical statement. So these are called kind of universal set of notations using which any logical statement can be solved or written.

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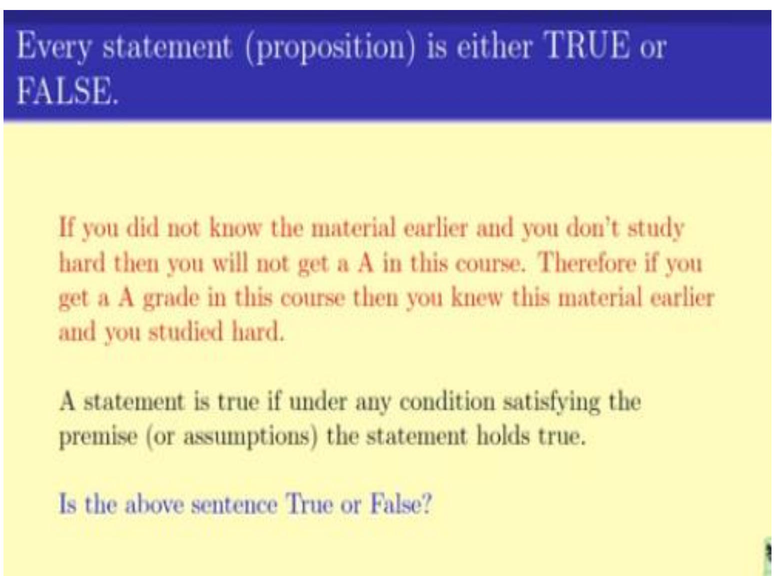

Let us see some examples, let us take a normal logical statement to a mathematical logical statement. For example, consider this particular para, this says that if you did not know the material earlier and you do not study hard then you get then you will not get a A in this course. Therefore, if you get an A in this course then you knew this material earlier or you studied hard. Now I would like to check the logical validity of this statement.

First of all, we should understand that this statement at the whole is either true or false and this statement is true or the statement is valid, if for any condition satisfying the premise or assumptions this statement holds true. So in this particular case, you have to understand what are the premises, what are the deductions and what are the connectives. So, if you want to ask is the above statement true or false? Let us try to convert into a mathematical statement.

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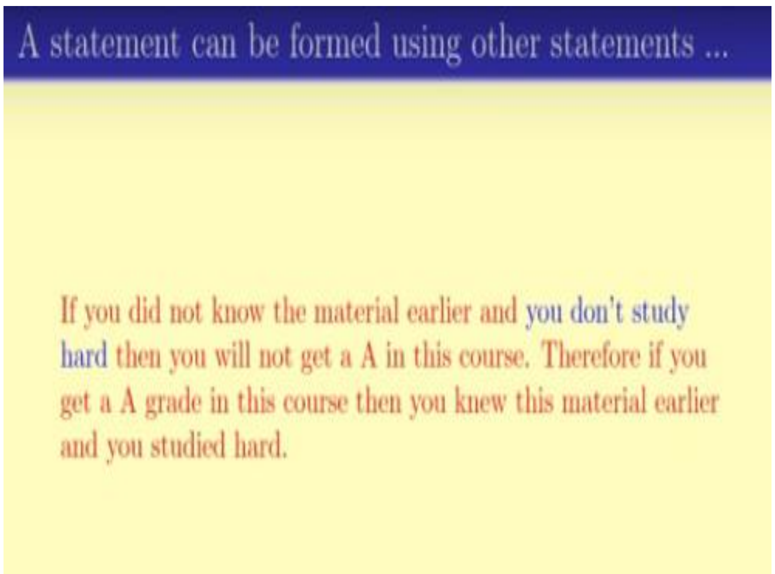

To start it, as you have seen, a statement can be formed using other statements. For example, here you did not know the material earlier either smaller statement. This statement can either be true or false. Either you knew this material earlier or you did not know the material earlier. In any case this, if the statement that can be true or false. Similarly, you do not study hard, the statement that can be true or false.

There are other statements here. I let you guys to look around inside the para and identify the set of statements.

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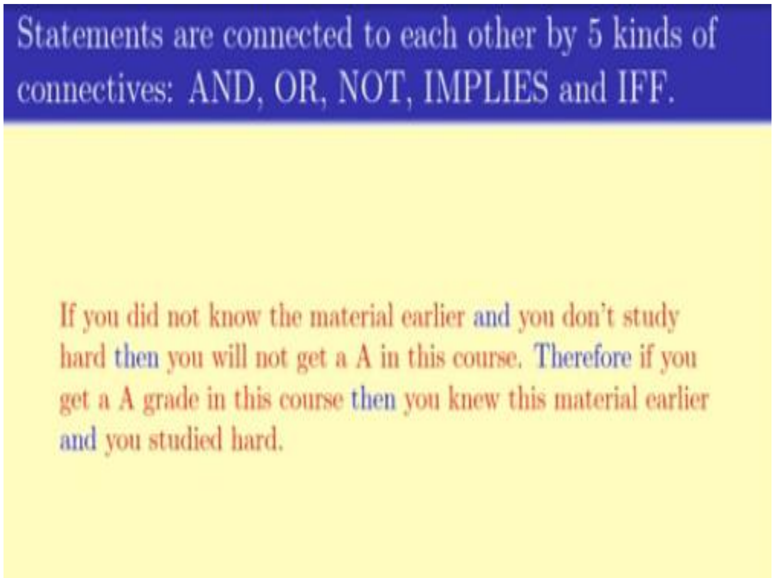

Now these statement are connected using connectives like AND, OR, NOT, IMPLIES and IFF. As I told you earlier, not necessarily this exact set of words of AND, OR, NOT, IMPLIES, IFF will be used, one can use other synonyms of it. For example, in this case, and, then, therefore have been used. So then and therefore are actually synonyms of implies. Now that we have understood what are the connectives and possibly what are the statements. Let us try to convert this statement or this para, this English para into a mathematical logic statement.

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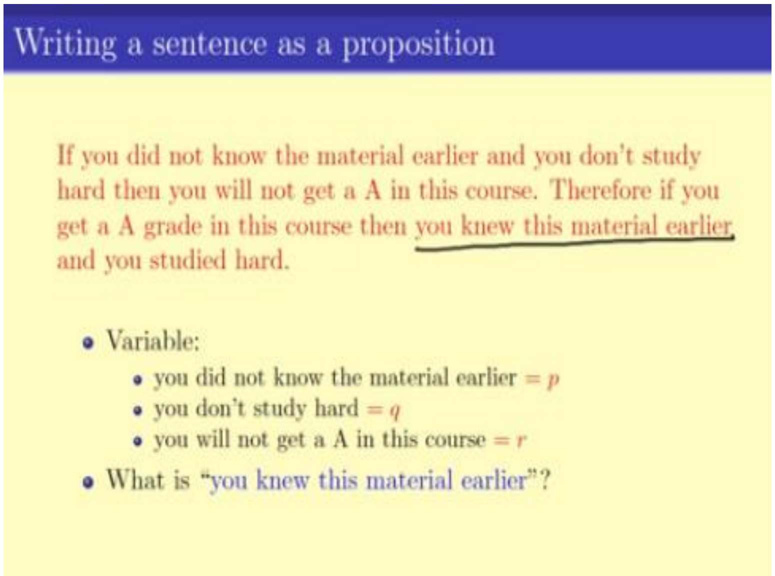

To start with, let us start, take the first statement that is there and that is you did not know the material early. This is a statement I do not know how to break the statement into smaller statements and hence I call this one a variable, let us call this P. What is the next statement that are there? It is – “you do not study hard”. Let us call this variable Q. For the third one is – “you will not get an A in this course”. Let us call this one a new variable R.

Now what is the next one, here is another statement, “you knew this material earlier”. Now what is this statement, so what is the statement? you knew this material earlier, now what is this statement, so that is the statement “you knew this material earlier”. Note that if P indicates the variable - is the variable indicating, you did not know the material earlier. Then you knew this material earlier is nothing but NOT of P.

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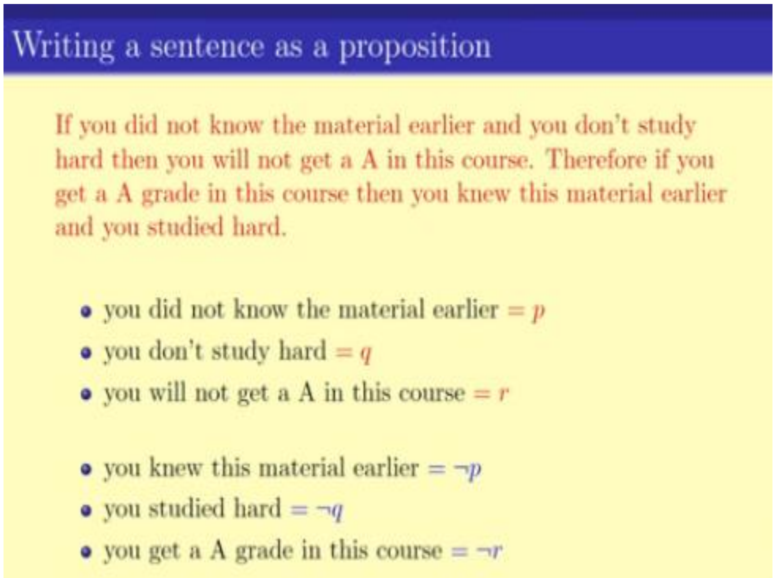

Thus, after we have put variables for these 3 statements, we can have, “you knew this material earlier” as not P. Similarly, “you studied hard” is nothing but not Q and “you get an A grade in this course” is nothing but not R. Now using this set of variables, what can we? How can we represent this statement? Let us see.

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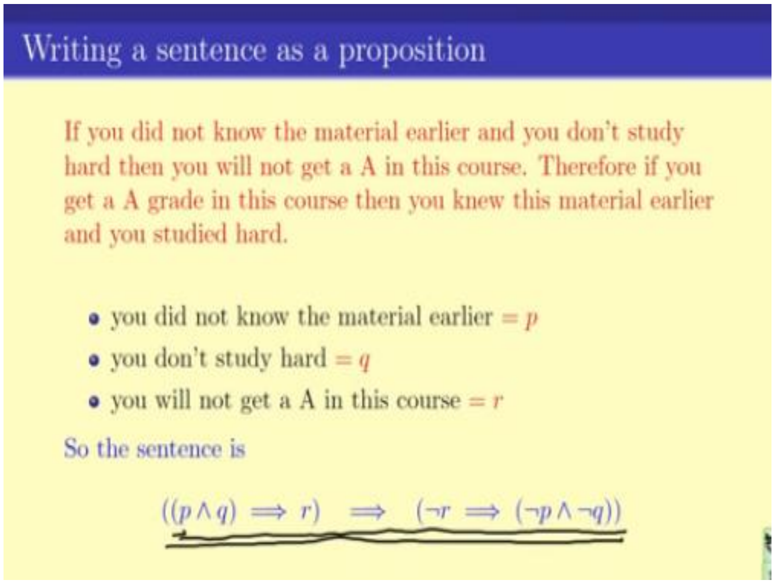

You can check for yourself that we can write it in this following fashion and the reason we can write in this following fashion is that, we can read this sentence from left to right and you will see that this para coming up completely. For example, since P is "you did not know the material earlier" and so on. I will just replace the P, Q and R with the English sentence. You did not know the material earlier and you do not study hard implies you will not get an A in this course.

Thus or therefore, you get an A in this course implies Q knew the material earlier and you studied hard. So this English sentence has been converted into a mathematical logic statement.

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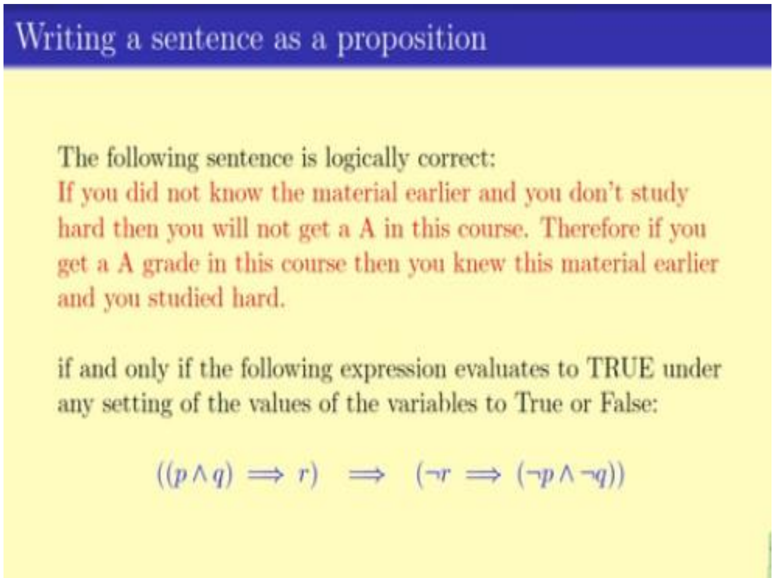

Thus, the sentence is logically correct, if in whatever ways I put P, Q and R as true, false, true or false, this following equation or statement must evaluate to true.

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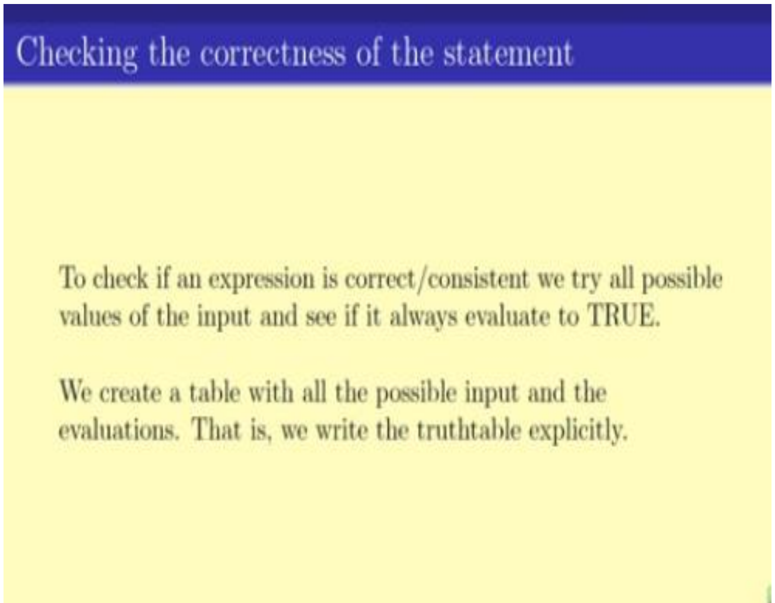

So to check if an expression is correct or consistent, we will try all possible values of the input and see if it is - if it always evaluates to true, one way of doing it is to write down the truth table of the equations exactly.

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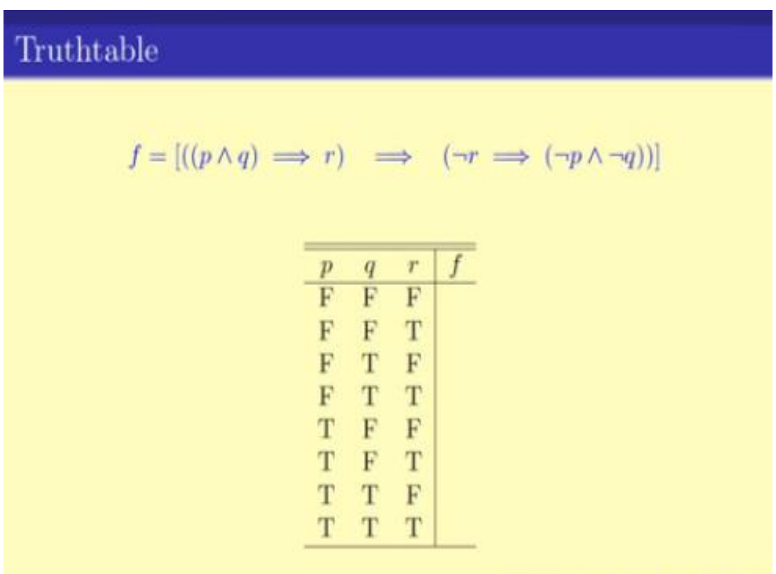

That means, for all the possible choices of P, Q and R set to true and false, so there are 8 of them does the function F which is this function on top evaluate to true and the way to do it is to do the whole truth table. As you can see that the evaluating the truth table is a slightly tedious job and we break it down to smaller parts namely, let us say F is S implies T, that means this is S and this is T.

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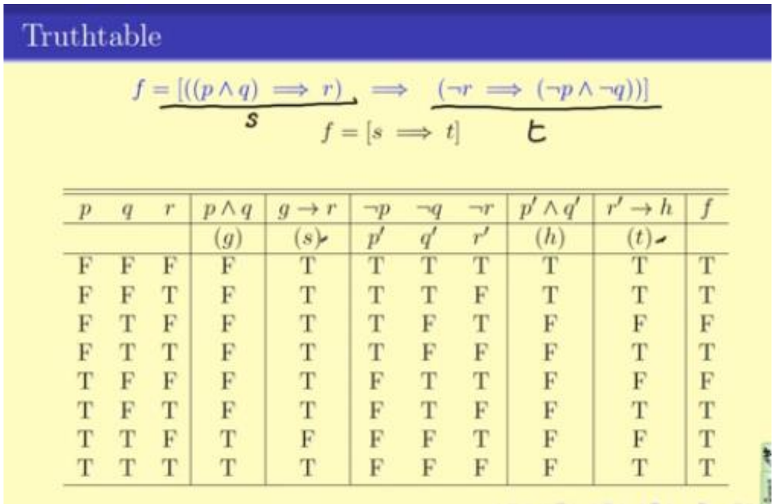

p	q	r	$p \wedge q$ (g)	$g \to r$ (s)	$\neg p$ p'	$\neg q$ q'	$\neg r$ r'	$p' \wedge q'$ (h)	$r' \to h$ (t)	f
F	F	F	F	T	T	T	T	T	T	T
F	F	T	F	T	T	T	F	T	T	T
F	T	F	F	T	T	F	T	F	F	F
F	T	T	F	T	T	F	F	F	T	T
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T	T	T	T	T	F	F	F	F	T	T

Now to evaluate S, I have to write - I have to have P and Q implies R. So I first has to evaluate what is P and Q and then I have to go for P and Q implier. So here is the first few steps, say P and Q is following the truth table of AND is - so that means, the both of them are false, it is false if both of - If one of them is false is also false and only when both P and Q are true, it evaluates true.

Now if this is G, G implies R when is that true. Now, for example, say false implies false is true because as we have discussed false statement implies any statement. So similarly, you can do it for all them, then all implies true is true, false implies false is true and so on and so forth. True implies true is true, true implies false is false and so on. Similarly, we can do the NOT of P and NOT of Q and NOT of R we just using the truth table of NOT and then I can you evaluate what the values of T is by doing the same thing.

First of all, doing it as NOT of P and NOT of Q and then NOT of R implies NOT of P and NOT of Q. And once we have that we have this S and we have this T and if S implies T, let us see true implies true right, true implies true correct, true implies false and that is not true that is false. So in fact, when you write down this whole equation, if realize that the function F evaluates to false at 2 cases and true in the rest.

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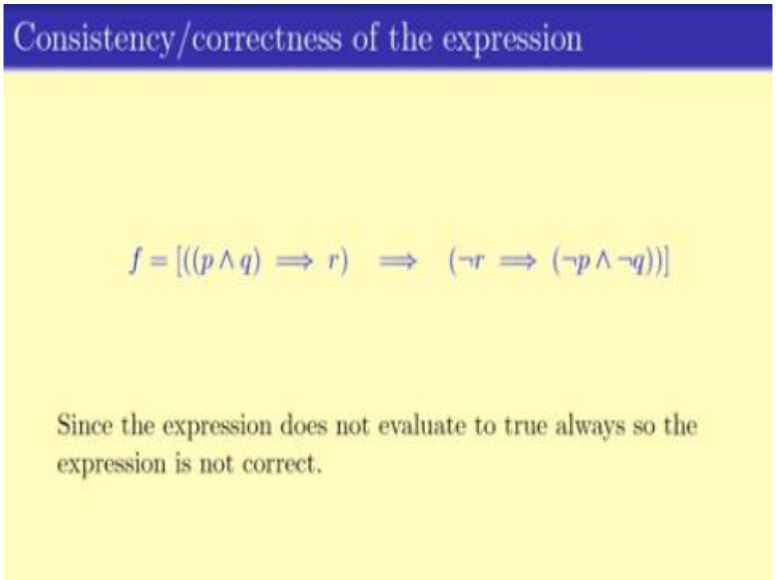

So since the expression does not evaluate to true always, so the expression is not correct. This possibly something that you could have understood by looking at this sentence as a English sentence earlier also but here is a mathematical way of proving that this English sentence is wrong.

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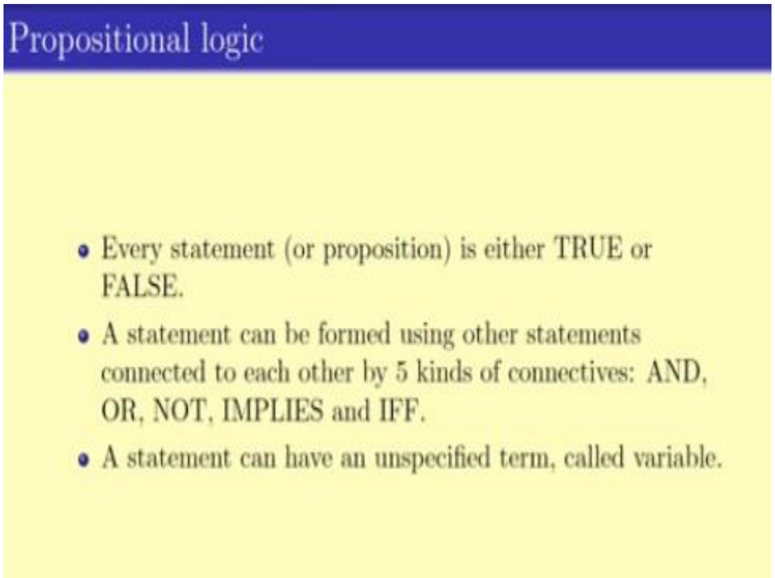

So to recollect what we are done till now, we have set up the basic for the propositional logic where every statement is either true or false. These statements are connected using 5 connectives that is AND, OR, NOT, IMPLIES and IFF and a statement can have some unspecified term but that has to be quantified using 1 of these 2 quantifiers “For all” or “There exists”.

We have also seen that any logical sentence particularly any logical English sentence or mathematical sentence can be converted to a logic statement and that can be used to check whether the logical sentence is consistent or not and the way of checking it is by ensuring that for any input of true and false to the smaller statements, does the whole statement evaluate to true, so does the whole statement evaluate to true at all the time.

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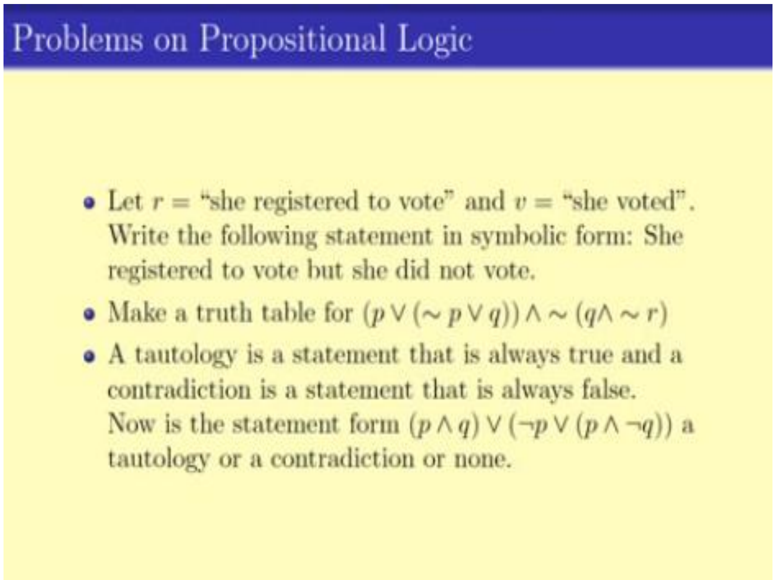

Here are a few problems on propositional logic. This brings us to the end of the video lecture, this was an introduction to propositional logic and predicate logic. Next week, we will see how this particular framework helps us to understand proof techniques. You should try to solve this problem yourselves. I would discuss this problem in the video lecture where I do a lot of problem solving. Thank you.