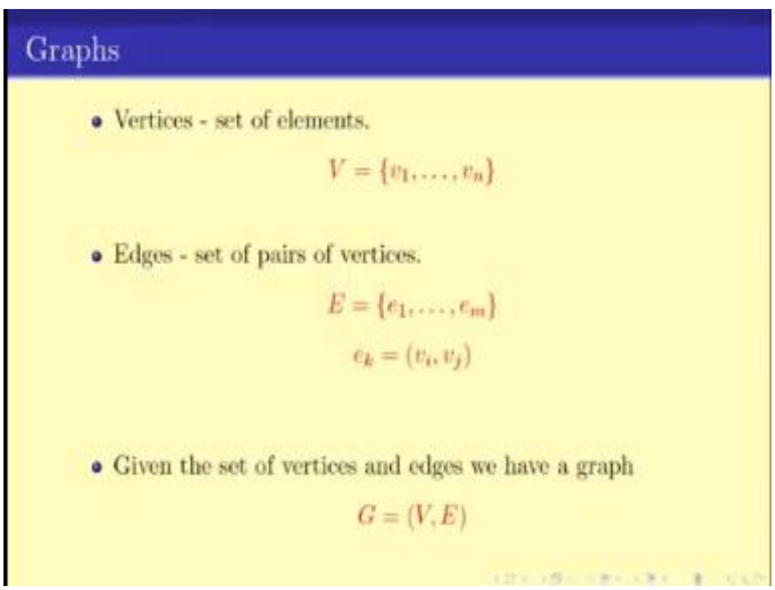
Discrete Mathematics
Prof. Sourav Chakraborty
Department of Mathematics
Indian Institute of Technology – Madras

Lecture - 28
Properties of Graphs

Welcome back, so in the last few videos, we have seen how to use graphs for designing or modeling problems and how that can be useful for solving many of the problems. In this video, we will be looking at graphs as a subject itself. Now graph theory is a big subject in itself it would usually take one whole course to study graph theory at least even for the elementary purposes. So in one video, I cannot do good justice on graph theory.

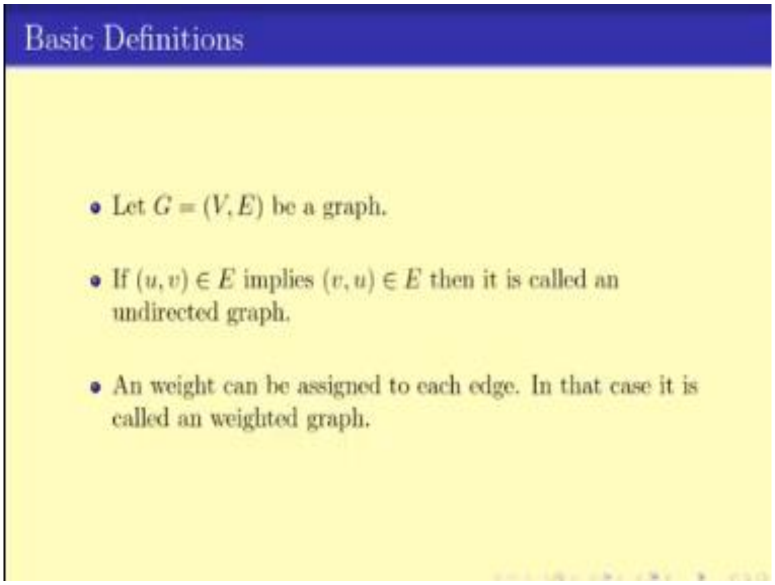
But I would like to go over some of the properties of graphs that can be useful for understanding how to attach graph problems, also we have already seen as some of the examples of how to use graphs for modeling problems. We will be continuing to see more such examples in this video and in the next video so to quickly recap what is graph.

(Refer Slide Time: 01:08)



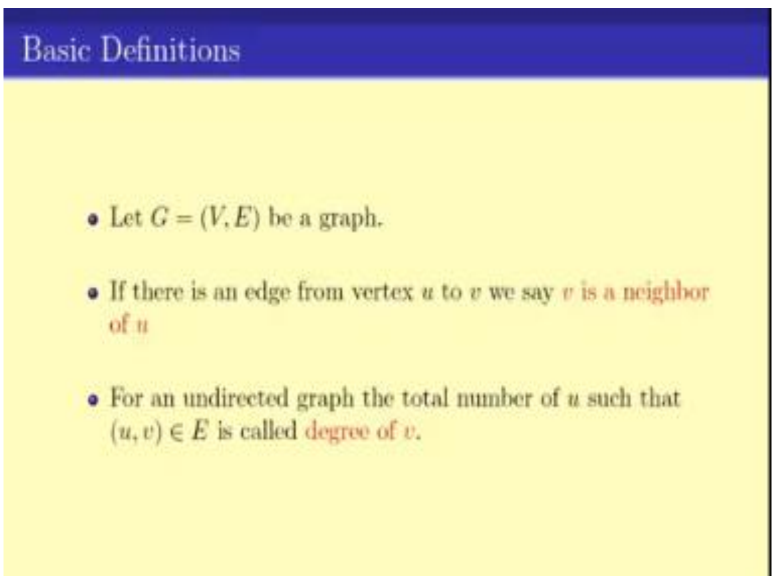
So it is a set of vertices and a set of edges. Edges are basically pairs of vertices and the graph is given by V, E where V is the set of vertices and E is a set of edges.

(Refer Slide Time: 01:29)



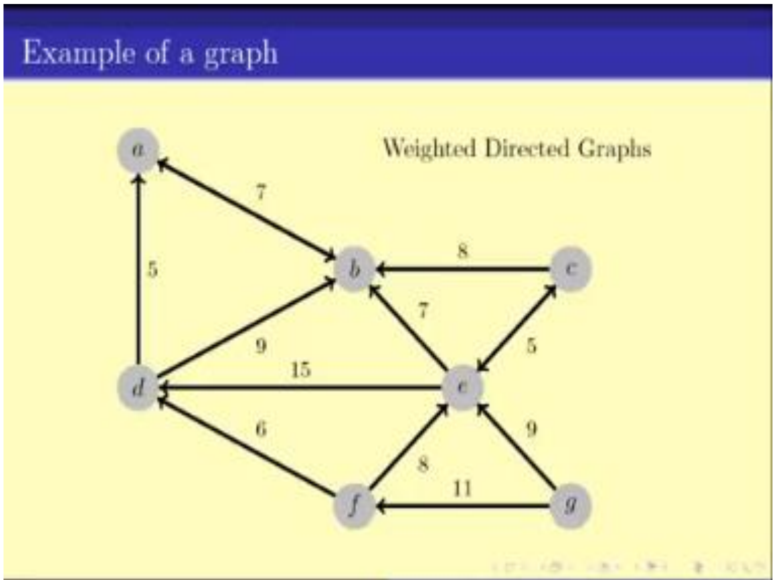
Now this is some of the definition I have always shown whenever I talked about graphs, namely if the edges that represent some kind of a binary relation is symmetric that means u, v is in the edge if and only if v, u is in the edge, then the graph is called undirected and sometimes for our purposes, we might add a weight to the edges.

(Refer Slide Time: 01:59)



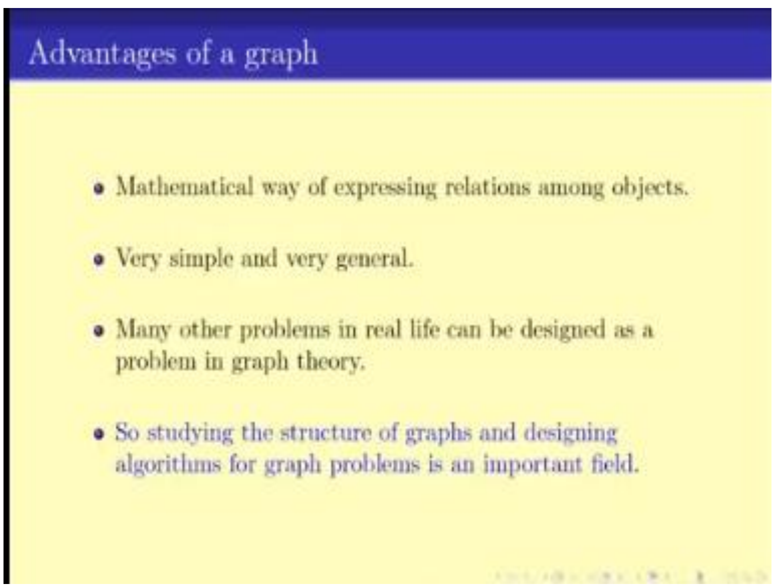
Also if there is an edge from u to v, we say v is a neighbor of u and the number of neighbors of v is known as the degree of v.

(Refer Slide Time: 02:22)



So pictorially, we have this set of vertices and this edges to give the undirected graph. We can have weights on the edges to have weighted undirected graphs and we can have directions on this edges to get weighted directed graphs. We have seen the use of undirected graph as well as directed graph. We have not seen the use of weighted graphs yet, but as you can imagine weighted graphs can also be useful for modeling various problems.

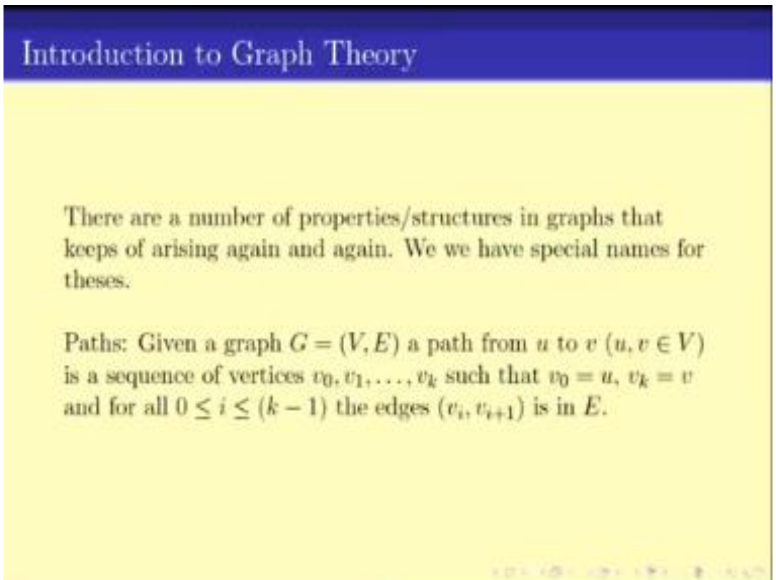
(Refer Slide Time: 02:59)



Now the main advantage of graphs is that they are very simple and yet pretty general and they can be used to design or model lots and lots of real life problems and hence studying graph structures is an important field by itself. In this video, I will be going through some of the properties of graphs and try to convince you that those properties are correct. I will not be giving any formal proofs in any of those cases.

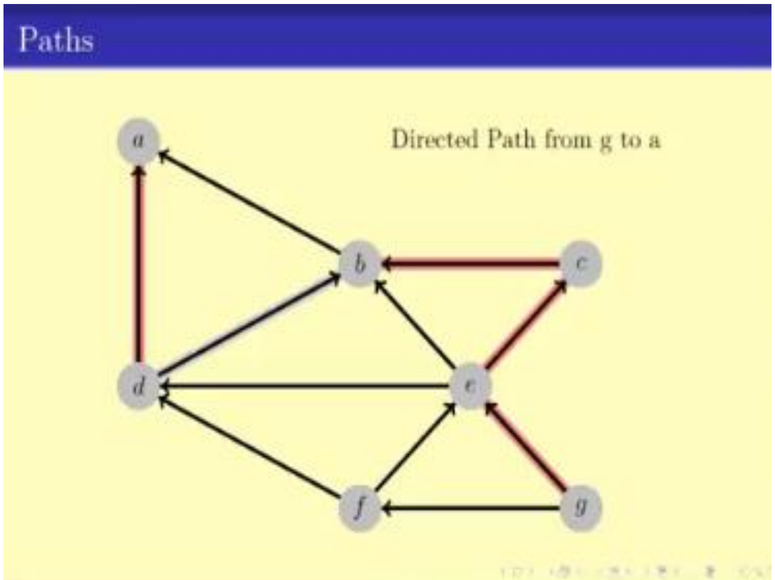
I would really recommend you guys to take up full course on graph theory, which will be much more useful for understanding the graph theory aspects. In the next video and the next other couple of videos, we will be showing about how to model other problems using, not only just graph theory, but other property other mathematical objects and you will slowly try to, you will slowly start to realize the importance of the subject of graphs.

(Refer Slide Time: 04:22)



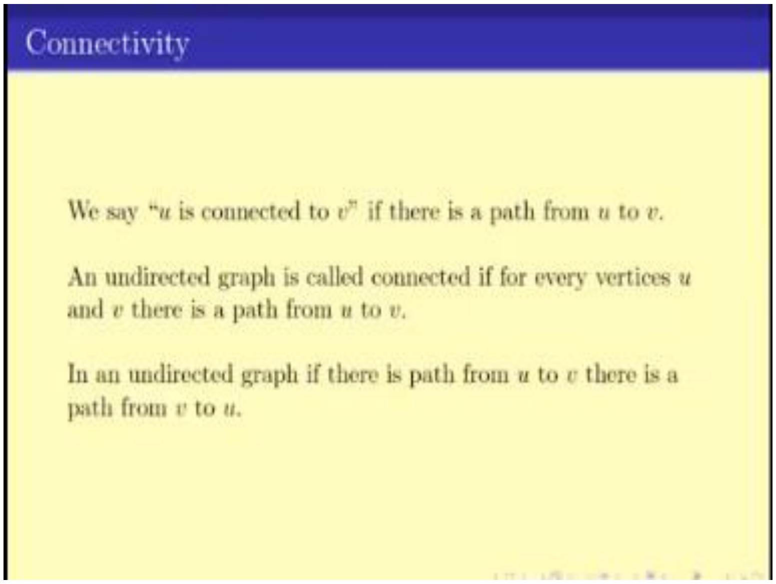
So there are number of properties and structures in the graphs that arise again and again and we need to study them by themselves so the idea is that theory should be there, so that whenever convert the problem to graphs, we can use properties from this theory to answer the problems right. So to start with we looked at the concept of parts so a path from u to v is basically a sequence of edges that start from a vertex, a sequence of vertices start from u and end at v, so that any consecutive vertex has an edge in between them.

(Refer Slide Time: 05:08)



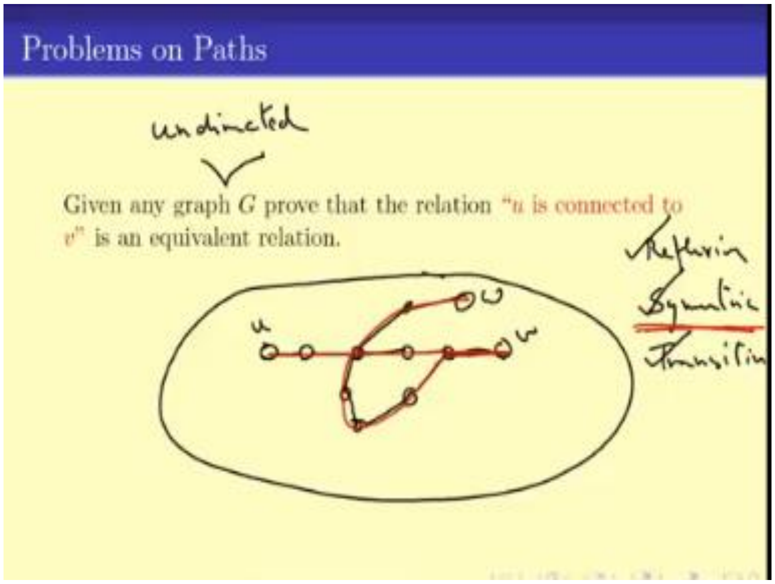
So in other words if I have this is the graph and I go on to draw a path from g to a, gfda is a path, so is gecbda is a path. Now if the graph is directed, then we have to talk about a directed path so if this is the direction of the edges, then this gecbda is unfortunately not a directed path because the edge between b and a is in the wrong direction, so this is not a directed path. But something like gecba is a directed path from g to a.

(Refer Slide Time: 05:58)



Now if there is a path from u to v, we say u is connected to v. In undirected graph if for any to vertex, u and v, it is connected then we say the graph is connected and as you can see since in an undirected graph, there is no direction, if there is a path from u to v, there is a path from v to u also.

Refer Slide Time: 06:41)



Now here is the first problem that we are asked to think about namely this relation v is connected to v, is an equivalent relation. Now let us try to prove this particular case. So if this is the graph G. First of all, to prove there is an equivalent relation, we have to follow three things namely symmetric, reflexive, and transitive. Now first of all is reflexive, why? Because u is related to u, u is connected to u.

Because you start from u, you are always for u, there is a path from going from u to u. This is that was it to that u is connected to u. So I have to prove reflexive, symmetric and transitive. Now reflexive is easy, what about symmetric? Now if there is path from u to v, by the way this one I have pointed out last time also, here we mean that the graph is undirected. The graph is undirected so there is the path from u to v, right.

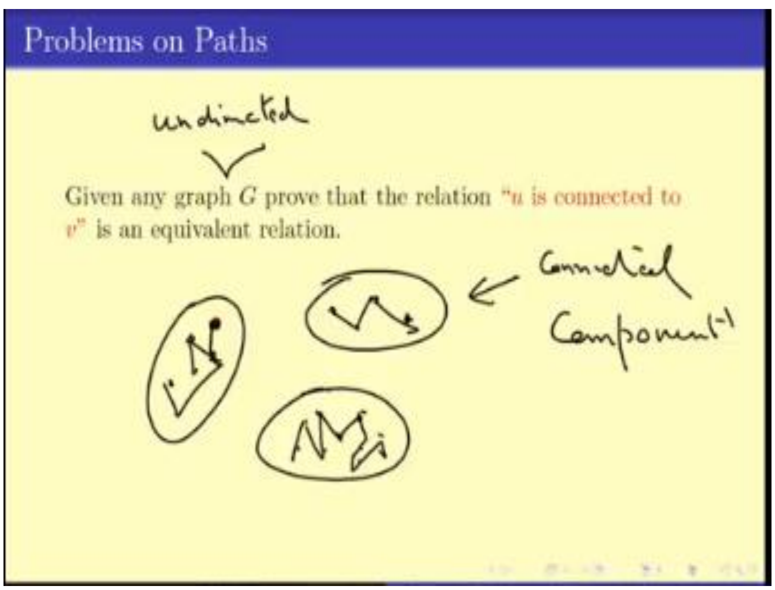
So if this one is as we have noted v1, v0, v2, v3 till vk. Now note that, we can also have a path for v to u by then renaming this one as v0, the next one as v1, till this one as vk. So u is connected to v, that means v is connected to u. So it is kind of able to see that they are symmetric. So symmetry case also. Now what about the transitive case? So the transitive case, the idea is that if u is connected to v and v is connected to w, so this may that v is connected to w through this path and this is w.

Status post there is a path from u to v, there is a path from v to w, it may be that some of the edges are u, some of the vertices are used, now the question is that is there a path from u to w and the answer is that, yes it is. For example, I can therefore then traverse from go from u to w, by going to u to v, and then replacing by the path and going back to w. So this may be the path from u to v, and then a path from v to w, then there is a path from u to w also.

Hence in the case of undirected graph, it is an equivalent relation. Now why do we use undirectedness, now undirectedness was specially used to the symmetric case. If it is not undirected, then the relationship is not symmetric, okay. So that means if the graph is undirected, then the relation namely u is connected to v is an equivalent relation and we can have an equivalent relation.

As you have pointed out then earlier, then the whole set of set on which the equivalent relationship is told, namely the set of vertices here, we can split then up into equivalent partitions, right.

(Refer Slide Time: 10:28)

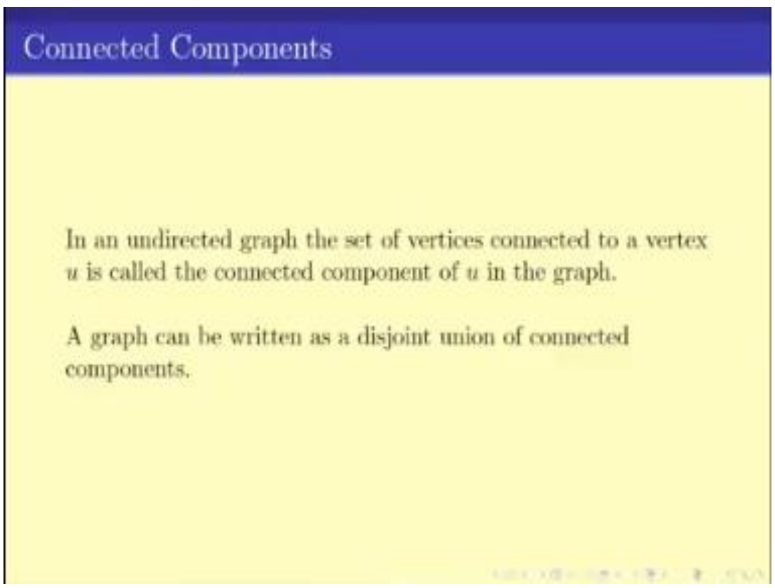


So namely we can therefore split up the graph G into chance like this where any two vertex here is related, any two vertices, all the vertices are related, and a vertex from this point to this point is not related. So we have the equivalent classes. These are the equivalent classes and what it means is that these are connected graphs somehow means you can go from any vertex to any vertex, but you cannot go from one connected component to the other.

These are called the collected component. So thus if the graph is undirected then the whole graph can be split up into connected components where individual components are connected and across components there is no way one can go from the other one to the other. Note that this is not this having uses the fact that the relationship is equivalent and in fact if the relationship is not equivalent and we could not have got it.

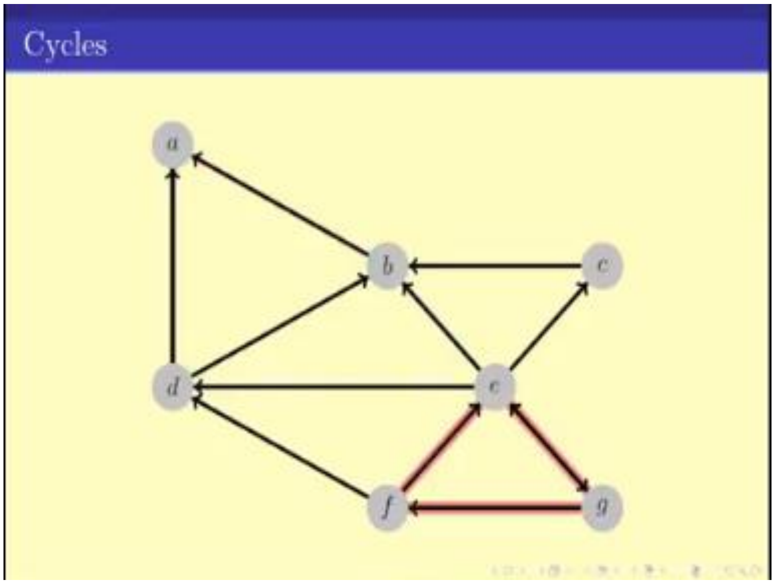
And hence if the graph G was not undirected but was directed, then we could not have got this particular property.

(Refer Slide Time: 12:17)



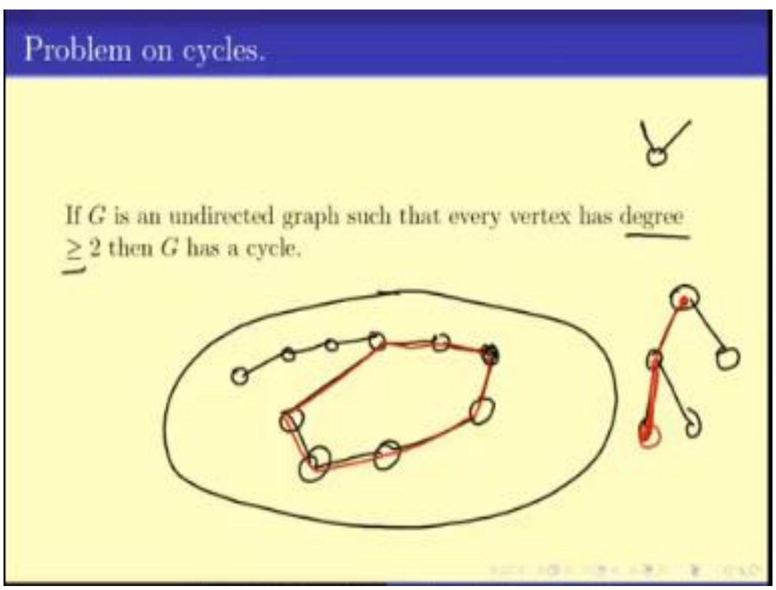
So we have this so an undirected graph. We can split up into connected components and the graph is the disjoint union of this connected components. This is about the case of undirected graph.

(Refer Slide Time: 12:37)



Now, moving on, there is a concept of cycle, which is basically a path that starts and ends at the same place. So for example, this is a cycle gfdbe cycle. So is this one dabdc is a cycle and we can also have directed cycles. For example, this is not a directed cycle, but this is a directed cycle, right. The cycles are very useful concepts in graph theory.

(Refer Slide Time: 13:36)



And here in one of the nice problem that we have. If G is an undirected graph such that every vertex has degree greater than two, greater than or equal to 2, then G has a cycle what does it mean by every vertex have degree greater than or equal to two means, every vertex have two neighbors. So let us see how to prove the statement. So if this is the graph what we can do is that we can start from a vertex and keep on walking on the edges, right.

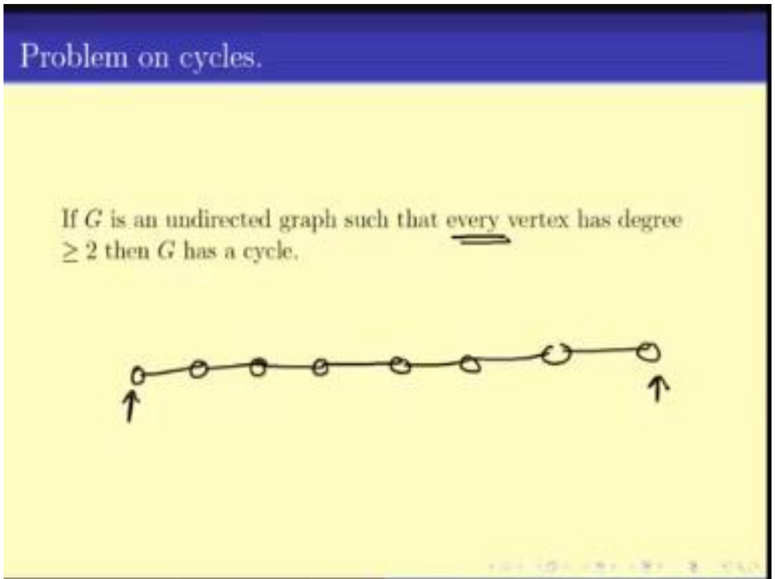
Now if I keep on walking on the edges because every vertex has degree greater two, that is whatever vertex, I whatever edge I enter from, I will try not to go out from the same edge. So enter a vertex on a edge and leave a vertex from a different edge because of the there would be two case, every vertex has two edges going in, two edges going in and out of it. So what I do is, start from vertex and keep on walking internal vertex from an edge and leave the vertex through the other edge.

I cannot keep on going about it infinitely right. So the graph is finite. So at some point of time, I must end up coming back to one of the vertices I have already visited. And once I have that I get that a cycle, this is a cycle. So let us try to prove it again by looking at the graph, which does not have a cycle. For example, this graph does not have a cycle. Now here, the thing is that if I start walking from here, I could have been here, then go here, then go here.

But now I have to replace that back to my path, on the same path, which is unfortunately, but I do not want to go. So it is very important that we whatever edge we enter from, we do not go out from the same path in this cycle. By doing so, we can ensure that that we always if I end up revisiting re-visiting a vertex second time, we actually get a cycle, okay. But this is a very simple quick observation.

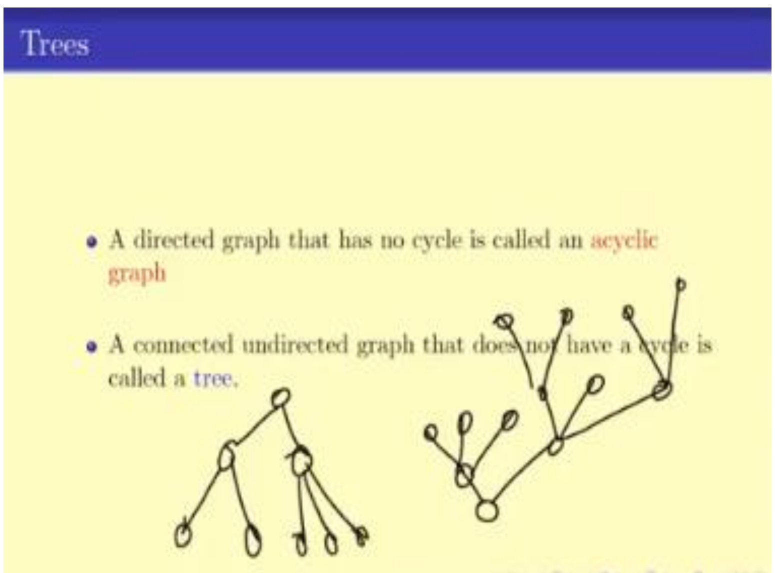
And there is several powerful observations, as we will see very shortly, what does this particular observation mean? That if a graph for a graph, if every vertex have degree greater than or equal to two, then the graph has a cycle.

(Refer Slide Time: 16.29)



Knowing that we can have a graph like this, in which case as you can see, every vertex except for the first and the last has degree equal to 2 and yet it does not have a cycle. So we always require that every vertex has degree available to may be equal to that, right.

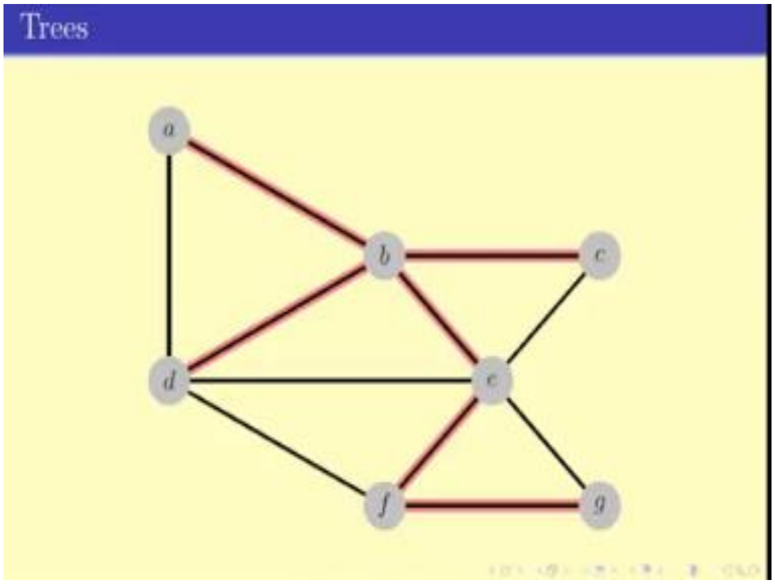
(Refer Slide Time: 17:06)



Now, we have the concept of trees. So what is a tree? First of all a graph that does not have a cycle an undirected graph that does not have a cycle is called acyclic graph and if that acyclic graph is connected then we call it a tree. So why is it a tree called? So it is basically because if you draw a graph of this form, it is usually like this. So this is the typical representation of our graph that does not have a cycle.

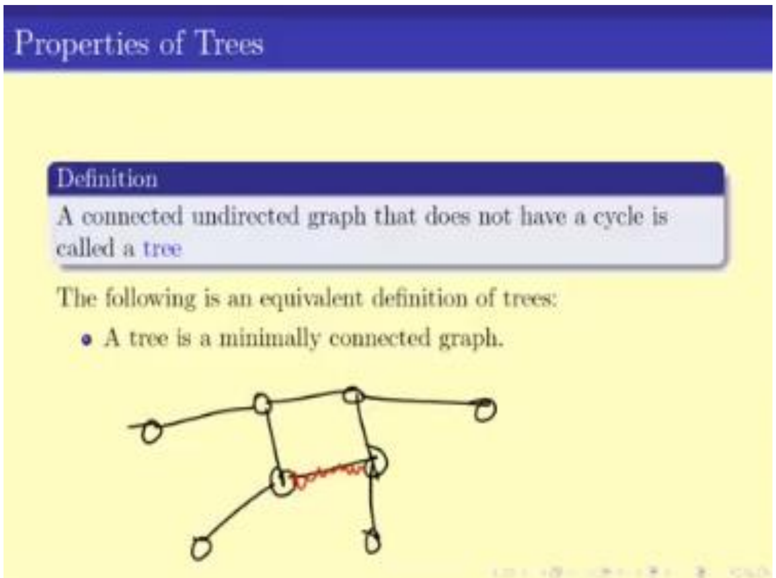
So as you can see it looks like a tree and therefore, it is called a tree, sometimes it is drawn in the opposite direction, so it is drawn like this.

(Refer Slide Time: 18:16)



So this is a graph, for example then we can have a this is a tree. A tree is a connected graph, the red edges gives you a connected graph and there is no cycle. So it is a tree similarly this is also a tree, right. There can be various kind of trees.

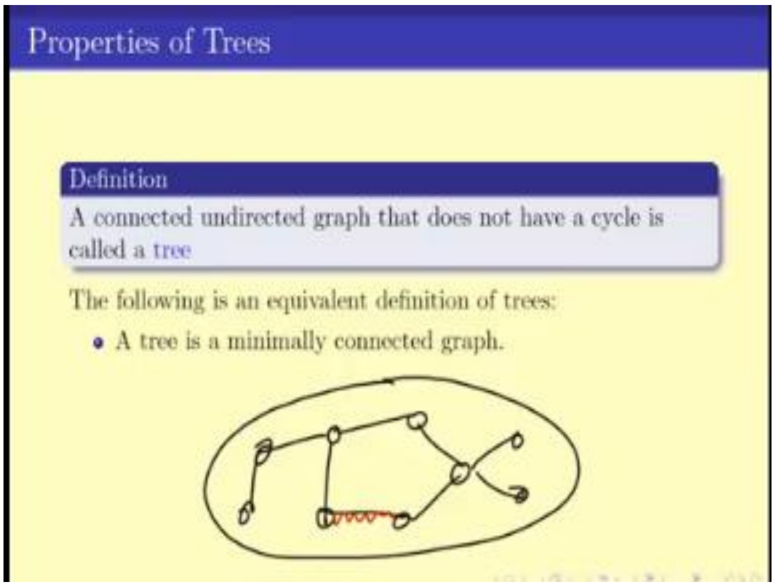
(Refer Slide Time: 18:44)



So, the definition of the tree, if a connected undirected graph that has that does not have a cycle is called a tree. Now here is our equivalent definition of a tree namely a tree is minimally connected graph. Now what does it mean by minimally connected? So graph is minimal connected if I can remove, for example if this is a graph and I say that this is not a minimally connected graph, why? Because I can remove this edge and still the graph can be connected.

So, graph is minimally connected if I cannot remove any edge and yet it is rather if I you move any edge, the graph becomes disconnected a graph is minimally connected. Now the thing is that if a graph has a, if a graph has a cycle then the graph cannot be minimally connected.

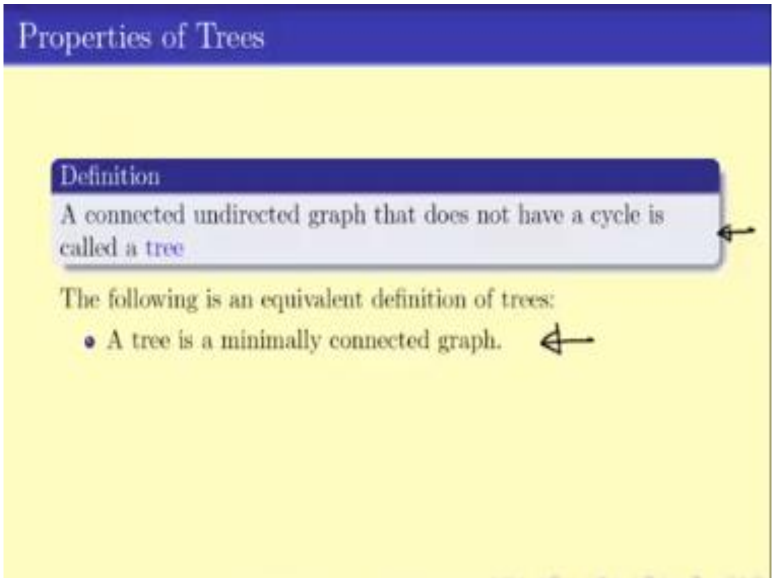
(Refer Slide Time: 19.50)



So if this is a graph and there is a cycle sitting here and I do not care how things are they otherwise but in this cycle I can take any of the edge and remove that and yet the graph will remain connected. So this is not a minimally connected graph because I could remove an edge and still the graph could remain connected right. So as you can see, I just now proved to you that if a graph has a cycle, then the graph is not minimally connected.

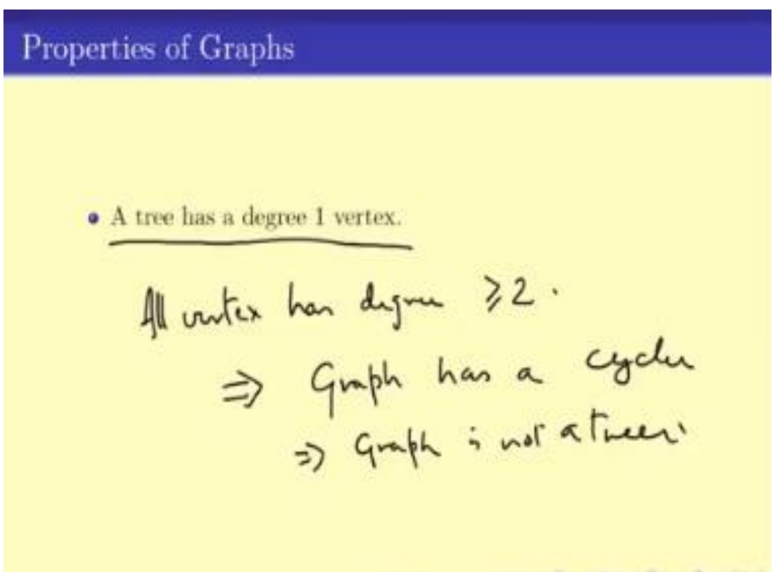
Now how do I prove the opposite direction? If the graph is not minimally connected, it must have a cycle, okay I leave this to you guys to think about. But basic idea is that basically the most important thing is that a tree has two or three different definitions all of them are equivalent.

(Refer Slide Time: 21:02)



This is one of them that this is a connected graph that does not have a cycle and second one is this one a tree is the minimally connected graph, meaning given a graph, a graph is minimally connected if by removing any edge you can make it disconnected. It is not that case then it is called minimally connected.

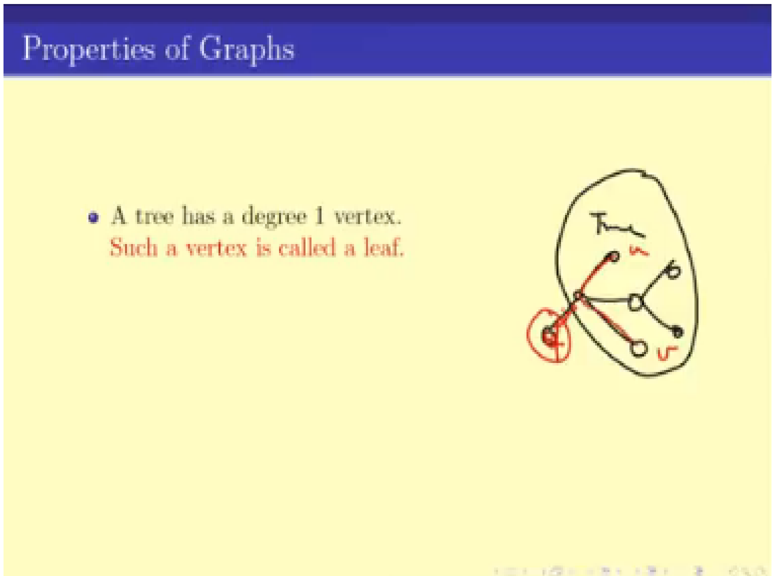
(Refer Slide Time: 21:35)



Now here is one very important property of graphs, sorry a tree. A tree has a degree 1 vertex. Now can someone see how to prove this statement? We prove it by contradiction. So if this is not the case that means what? that means all vertex has degree greater than or equal to 2 and if that is the case, then we just now proved some couple of slides ago that it means that the graph G, the graph has a cycle, which means that graph is not a tree by the definition.

So a tree must have a degree one vertex at least one degree 1 vertex. If it does not have any degree one vertex, then it must have a cycle and in that case, we contradict the definition of a tree.

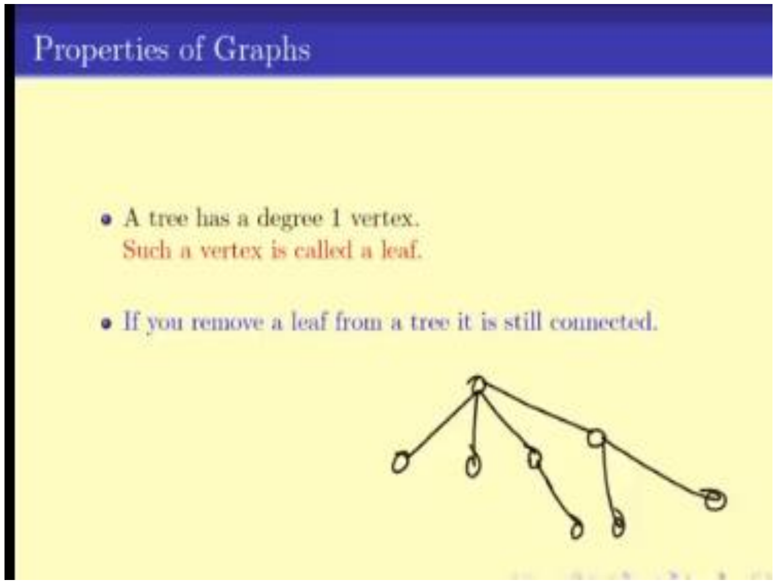
(Refer Slide Time: 23:02)



So once we have such a vertex, we call that vertex a leaf. Now does this leaf look like. Let us say, I have the graph G, this is a tree right, this is a tree and I know that there is a vertex like this, a vertex of degree 1, Now what does it mean? That means this tree has lots of other things like this. Now whatever if I remove this vertex, will the graph remain connected? If I do this vertex as well as this edge.

The answer is yes, it will remain connected why because any path from say u to v, cannot use this vertex because this must be going by a path without using this vertex. So if I remove this a, no pair of vertex is disturbed. The only pair of vertex that gets disturbed is the paths from this particular vertex to any other, but given the fact that I have destroyed this particular vertex, so the rest of the graph is still connected.

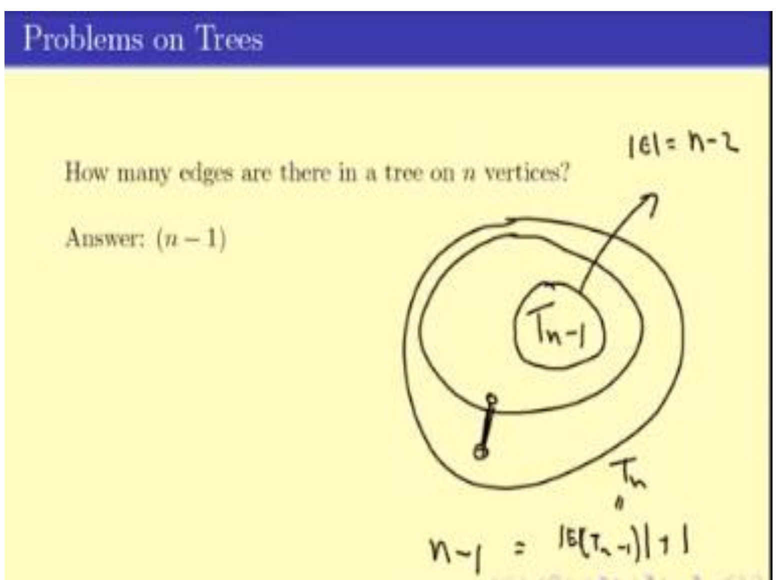
(Refer Slide Time: 24.28)



So I get the following that if you remove the leaf from a tree, the tree still remains connected. So here note that I am not just removing the edge from the leaf to the rest of the tree, but the leaf itself. So I get a graph on a smaller number of vertices right. So for example if this is a tree and if I remove this edge and this vertex, then I get this graph is the graph on one less number of vertices, but still it is a tree and still it is connected right.

This is, if you important note of observation that if you remove of leaf from a tree then the rest of the graph is still connected.

(Refer Slide Time: 25:42)



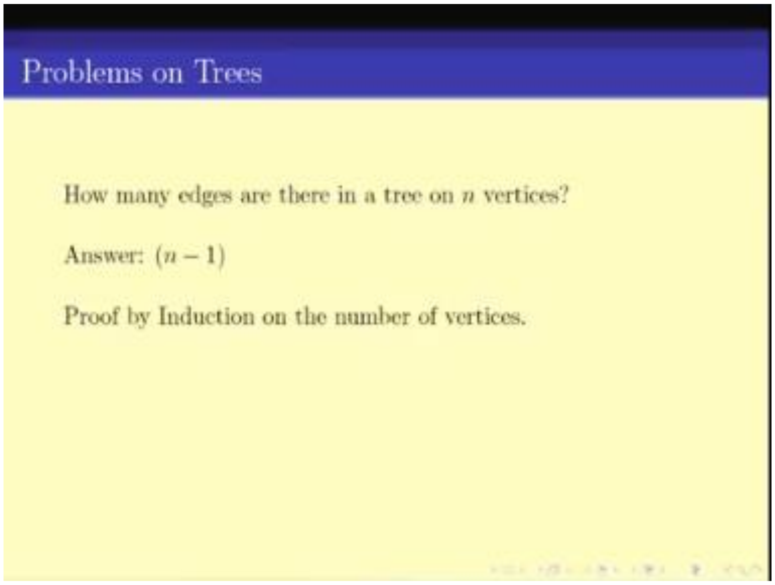
Now can you answer this question, how many edges are there in a graph on n vertices? Now to answer this one, let me say that okay let me first give you the answer. The answer is n minus 1 and how do prove it. Now the proving can be done by induction. So here was a tree and here was a leaf. A tree has other leaf, right. So this is the graph on smaller, so this is a tree on Tn minus 1 and this whole thing is a tree on n vertices.

So how many edges are there in this tree on n minus 1 notes, number of edges is on this step is by the induction hypothesis is n minus 2 and how many edges are there in Tn. This is number of edges in Tn minus 1 right, number of edges in Tn minus 1 plus this edge plus 1, which is Tn, n minus 2 plus n minus plus 1 is n minus 1. So basically by using induction one can prove induction all the other properties that we have used.

So what other properties we have used. We have used the property that a tree has a leaf, okay of degree 1 vertex, and we have also used that if you remove the leaf from tree, you get a tree on a smaller number of vertices right and then we apply induction hypothesis on this particular case to get the answer that the number of edges in a tree on n vertices is n minus 1. Now I request you guys to go and solve this particular induction completely for yourself.

In this course, it will be hard for me to judge whether you have understood the induction techniques.

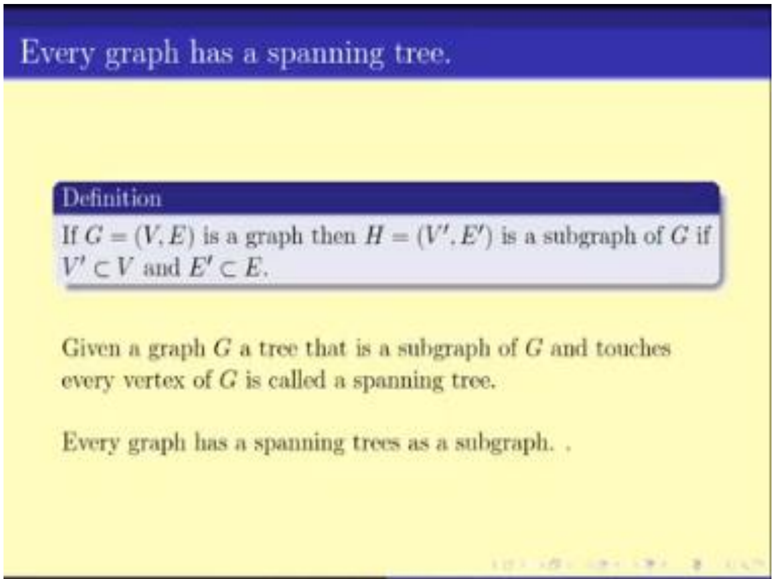
(Refer Slide Time: 28:19)



How far, you have understood the induction techniques, particularly because the quizzes and assignment are all multiple choice questions, which are not the most perfect way of judging the understanding of the topics like induction hypothesis, but this is a good example how many edges are there in a tree on n vertices, a good example where you have to do induction on graphs. It is a very important concept.

We have done induction graphs earlier also, particularly in the case of the tournament problem and so on, but this is something very useful and very crucial for your understanding of graph theory and induction, right.

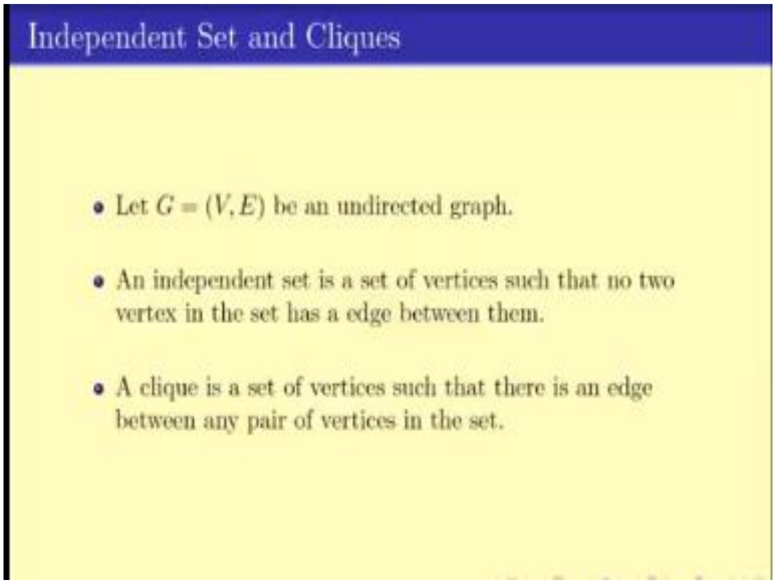
(Refer Slide Time: 29:17)



Okay, now moving on other than trees, we also okay there was a concept spanning tree. Now given a graph if I take if I remove some of the edges or some of the vertices then whatever we get, we call it sub graph. Given a graph G, a tree that is a sub graph of G and touches every vertex is called a spanning tree. Now the question is that every graphs has a spanning tree as a sub graph. Now this follows from the fact that if this graph is not already at tree.

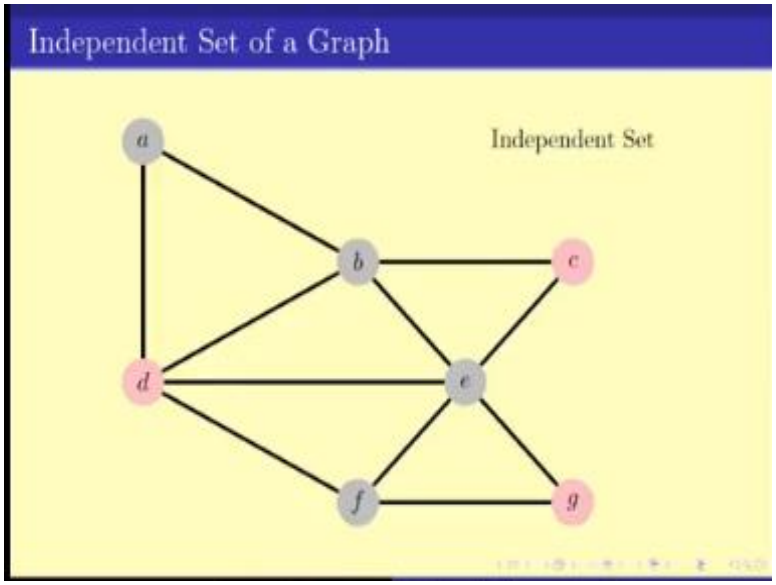
First of all, if the graph is already a tree and the graph is connected, then the graph itself is a spanning tree. If the grass is not a tree and is connected so that means it is not minimally connected in that case, there must be a cycle or an edge that I can remove. So remove that edge, I get a smaller graph which is connected and we recursive on that again. So using this recursive idea, we do get the fact that every graph has a spanning tree as a sub graph.

(Refer Slide Time: 30:43)



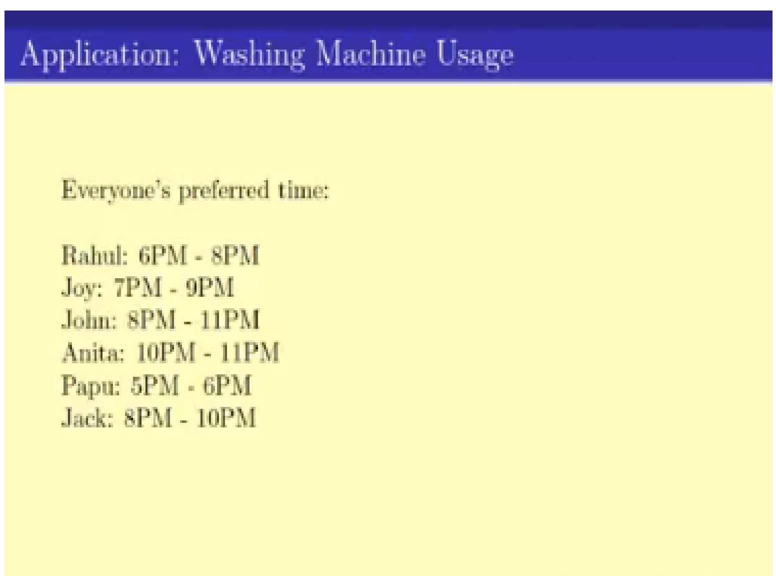
Now other than the trees and paths, we also looked at things like independents set and theory. Now these are two very important concepts again. An independent set, is a set of vertices such that no two vertex in this set has edged between them and the clique is a set of vertices, such that there is an edge between any pair of vertices in the set.

(Refer Slide Time: 31:16)



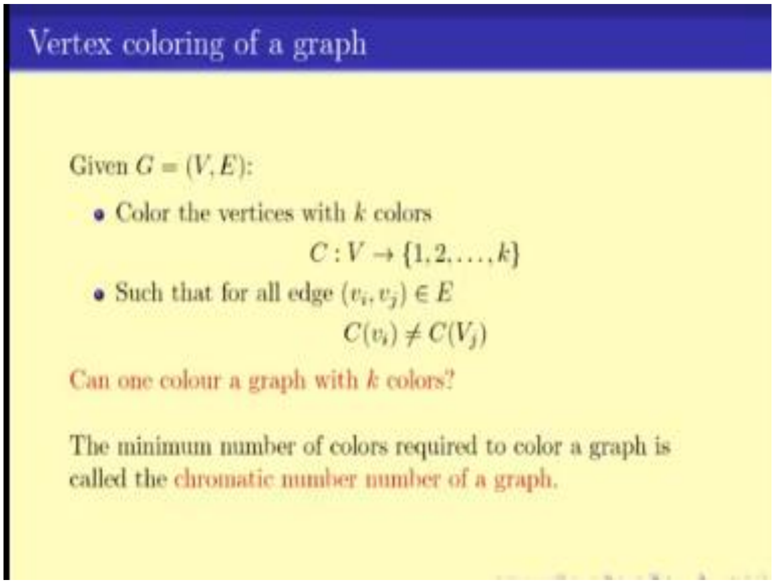
So for example if this is a graph, independent set is this one, say a and e because there is no edge between them or d, c, and g forms another independent set. On the other hand, say c, f and g is a clique of size 3, similarly, we can have a, b and d is a clique of size 3. Now finding the largest independent set or the largest clique in a graph is a pretty complicated or hand problem, a very challenging problem, very well studied problem also.

(Refer Slide Time: 32:09)



So we have seen the application of independent set I am going to skip this case.

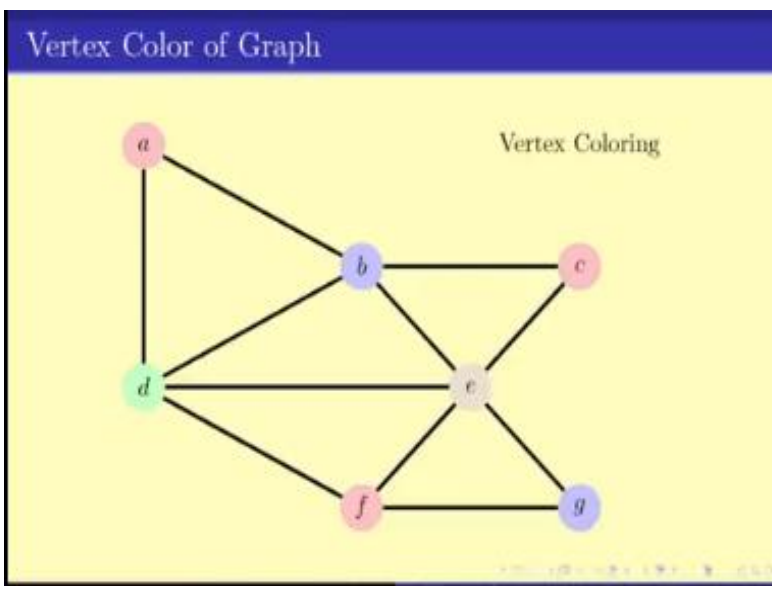
(Refer Slide Time: 32:19)



Let me go on to a different concept namely the coloring. So given a graph G a coloring is basically the set of vertices using colors namely they are the mapping from the vertex set to one to k, such that no two consecutive edge in a vertex has the same color. That means if there is an edge between vi and vj, then color of vi is not equal to color of vj. The typical question is that can one color a graph with k colors, right.

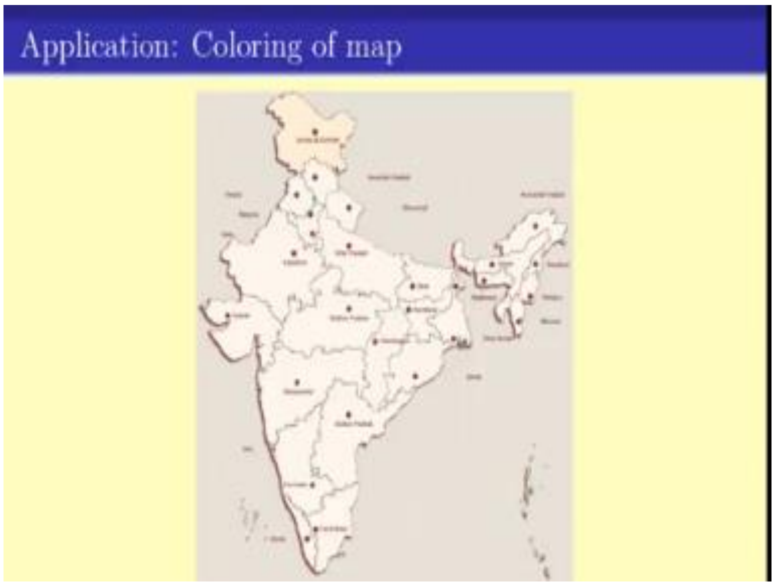
So in this case yeah the minimum number of colors required to color a graph is called the chromatic number of a graph.

(Refer Slide Time: 33:12)



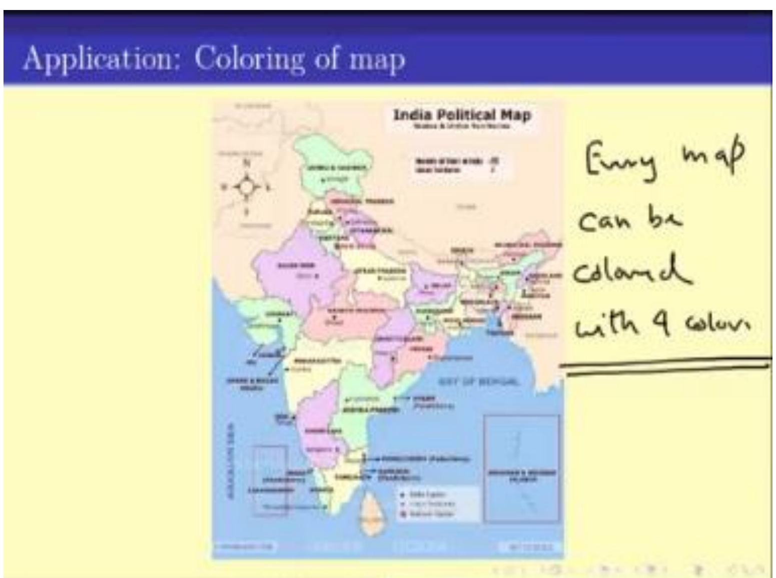
So for example here, if this is the gas if is colored red, b is colored blue, c can be colored red again, d because it is adjacent to both red and blue can be colored green, e because it is adjacent to all red blue and green, it has to be colored yellow and f can be colored red and g can be colored green and here we have a coloring with four colors now can you color this graph with less than that, namely can you color this graph with three colors. This is the kind of question that usually is asked.

(Refer Slide Time: 34:00)



And we have seen that the application of coloring can be seen in the coloring of maps.

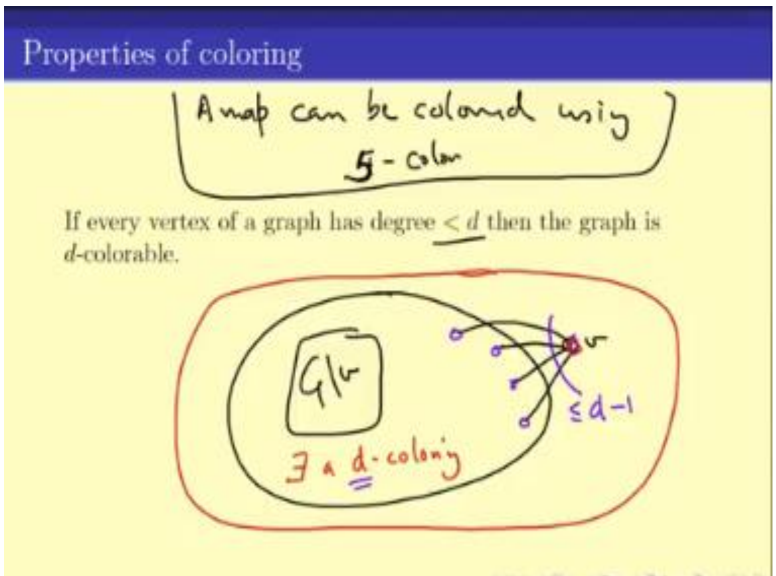
(Refer Slide Time: 34:09)



So for example here is coloring in the states of India using colors, so that no two adjacent states has the color. You can see it has uses of four colors. Now let me just tell that I cannot prove it that every map can be colored with four colors. So every map can be colored with four colors, this is called the four coloring theorem and a very beautiful theorem unfortunately it is beyond the scope of this course to tell you the proof of this theorem.

But I encourage you to go and take a look at it in some other place maybe online or something there you will find a four coloring theorem.

(Refer Slide Time: 35:20)



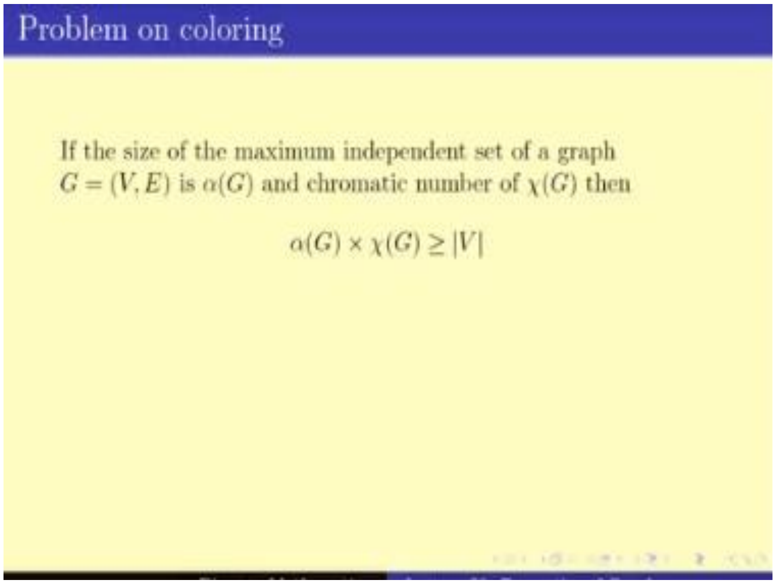
Now here is one very nice thing. So if every vertex of a graph has degree less than d, the graph is de-colorable, the graph can be colored using d colors. Now why is it so again the way to prove it by induction. Take a graph G where every vertex has degree less than d, take a vertex v and let us consider G minus v, right. Now this is G minus v, where G minus v is again a graph where the degree of every vertex is less than d.

Because if you remove a vertex, the degree cannot go up, so without loss of generality or other by induction hypothesis, I can assume that there exists a decoloring of this graph. There exists a decoloring. Now consider v, the question is that what you color v. Now v has how many neighbors. It has less than d, that mean at most d minus 1 neighbors so that means there must be d minus 1 colors that the neighbors have.

So if I have any of the d colors, one of the color must be missing from the neighbors. I can use it to color vertex v, right rank and hence I get a decoloring of this whole graph. So if all the vertex of a graph has less than d colors, then the graph decolored, right. So this is a very again a nice property that would come in handy many times and using this one actually you can prove that a map can be colored using four colors.

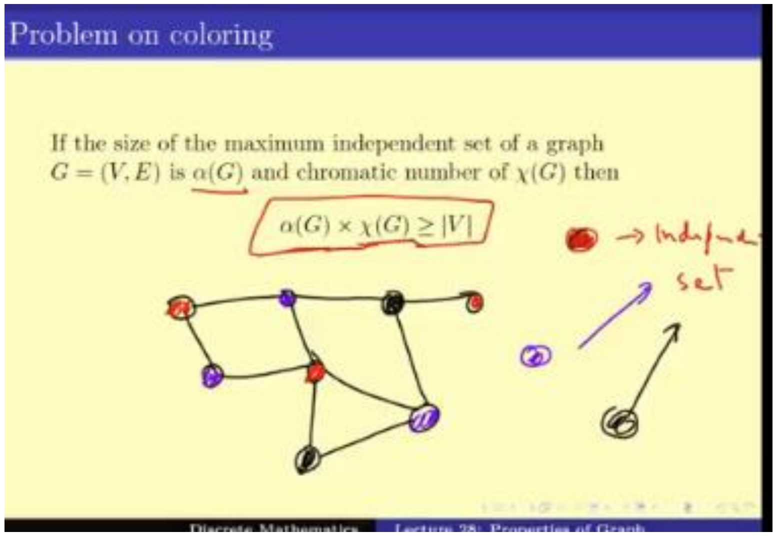
A map can be colored using four colors, sorry five color. Four colors are slightly hard, but five color is easy. So you can get this proof using this particular theorem that I have proved. I leave it to you guys to go and check the proof of this whole thing. Now one more problem on coloring.

(Refer Slide Time: 38:24)



So if alpha G is the set of independent set and this chi G is the chromatic number, the number of colors required to color the graph, this proves that the product of them is greater than the size of the graph on the size of v, why is it so? Let me quickly.

(Refer Slide Time: 38:55)



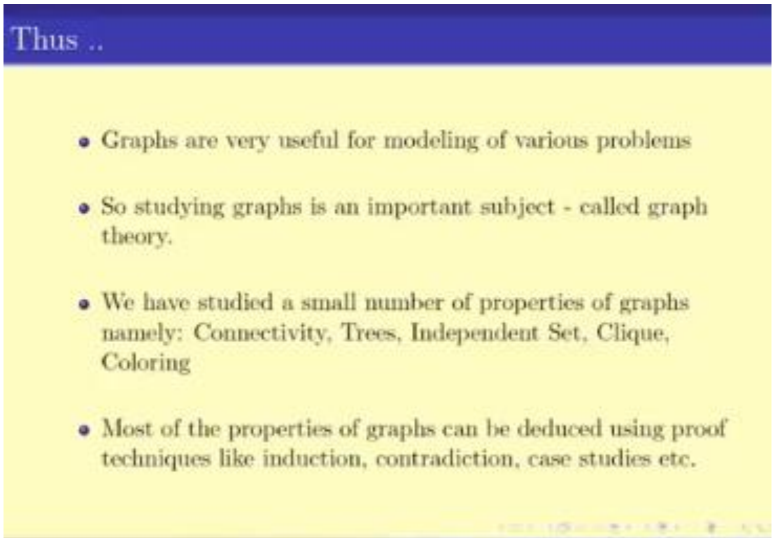
It is basically follows from a namely the definitions. So if I am coloring of this graph, so this one I can color red, this one I can color blue, this one also I can color blue, this one I can color red, this one I can color blue, this one I can color black. I think we have made a mistake here. This is not the best coloring. The best coloring to do is with this one colored blue, this be colored black, then this can be colored black, and this can be colored red.

Now note that if you look at the red colors the vertex colored red, then they do not have an edge between them. This is by definition, right, that no two red colored vertex can have an edge between them because no two vertex which are adjacent have the same color. So that means the red color is an independent set, similarly as you can see the blue colors will also give you another independent set and similarly black colors will also give you an independent set.

So and by definition of this is the maximum independent set that means the number of, the number of vertices colored red is less that alpha G, the number of vertices colored blue is also less than alpha G, number of vertices colored black is also less than alpha G. So the number of vertices colored is less than the number of colors times alpha G. So namely number of vertex therefore we have the size alpha G times chromatic number is greater than or equal to the number of vertices

So here is another problem or property of graphs, that can be solved not using induction, but just by plain definition on the plain definition right.

(Refer Slide Time: 41:45)



Okay so what we have till now, we understand that graphs are very useful for modeling various problems. The graphs are studied by themselves in a subject called graph theory. We have studied some of the properties of graphs and most of the properties can be deduced using the similar proof techniques namely, induction, contradiction, case studies, etc. Now this is clearly not the end of graph theory.

Graph theory is huge subject itself and we will not be able to spend too much time on graph theory. But I hope that you have understood the importance of graphs and how that can be useful for modeling real life problem. In the next few videos, I will be talking about modeling in general. Taking a problem and modeling it into some other mathematical things, particularly as looking at something called linear programming and graph theory and see how problems in real life can be modeled in linear programming and graph theory. Thank you.