Discrete Mathematics
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Lecture - 27
Ramsey Problem (Part 2)

Welcome back. So we have been looking at the Ramsey Problem and how to solve the Ramsey Problem using graph theory. So to recap, so what does Ramsey Problem say.

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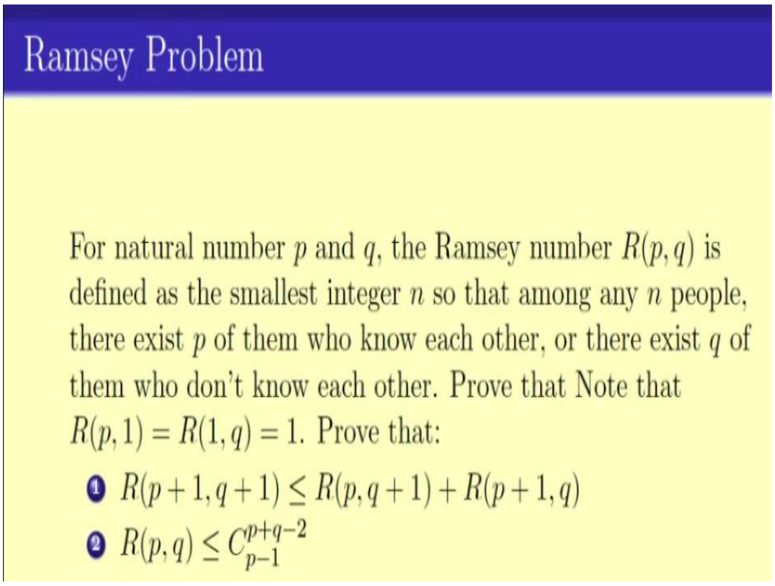

So given two natural numbers P and Q we define Ramsey number R (P, Q) as the smallest integer N so that among N people there exist either P of them who know each other or Q of them who do not know each other. The idea is to prove that R (P+1, Q+1) < R (P, Q+1) +R(P+1, Q) and using this recurrence to prove that R(P,Q) ≤ P+ Q -2 choose (P-1). Now this second one R(P, Q) ≤ P+ Q -2 choose (P-1) is something that we have already done a couple of weeks ago when we did induction on multiple variables.

So in the video, we would like to just prove this first recurrence (P+1, Q+1) < R (P,Q+1) +R(P+1,Q). Now to solve this first recurrence, we would of course first try to understand this problem using graphs here. So let us see how do we model it in graph here we did it in last video. Let us quickly recap. So to recap some of earlier graph which is the set of vertices, a set of edges and this is what the graph is given as a set of vertices and set of edges.

Now if the relation between vertices is symmetric that is U, V is (()) (01:58). we call it an undirected graph. We can also have weight assigned to edges and if there is an edge from U to V then we say that V is a neighbor of U and in an undirected graph the degree of V is a total number of neighbors of V.

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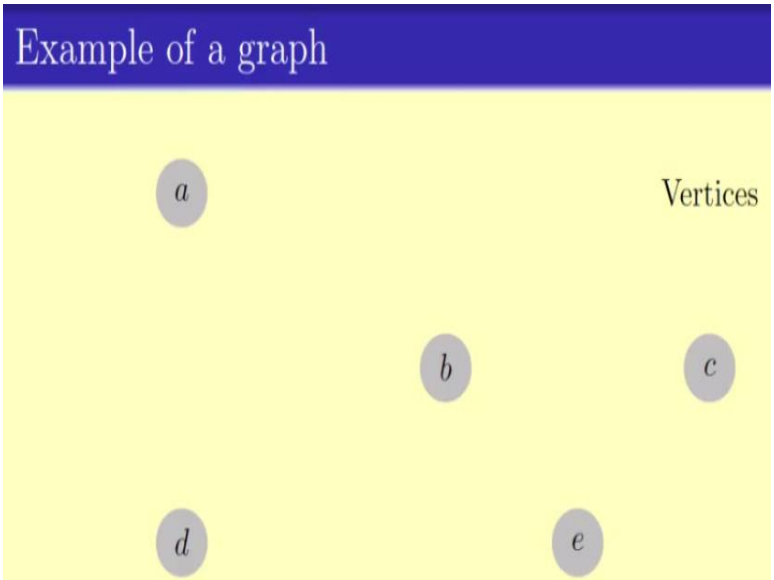

So pictorially this is the set of vertices the edges are drawn using lines joining the vertices. They can be weight on the edges and there can be direction on the edges to represent the asymmetric version of the edges. In the last video, we also looked at some other properties or some other definitions in graph theory.

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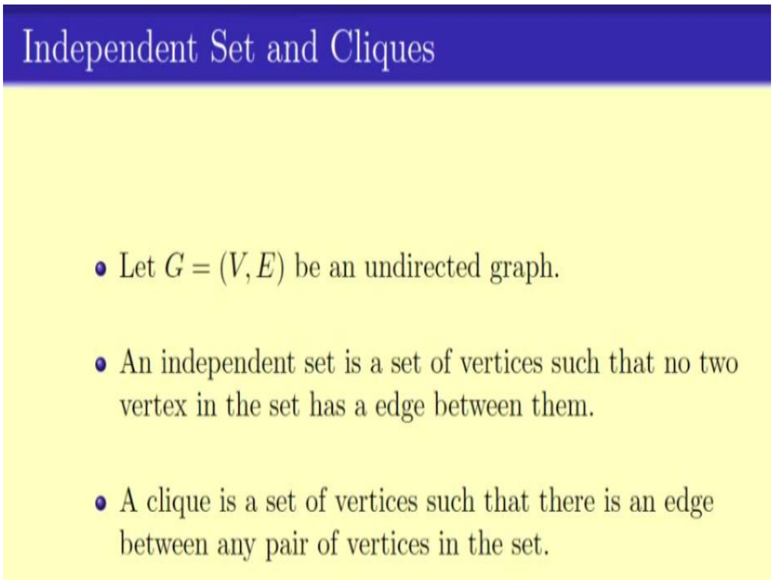

In particular, we looked at the independent set and Clique. So what is an Independent Set? The independent set is a set of vertices such that no two vertex in this set has an edge between them and the Clique is just the opposite of it namely a Clique is a set of vertices such that between any pair of this set there is an edge.

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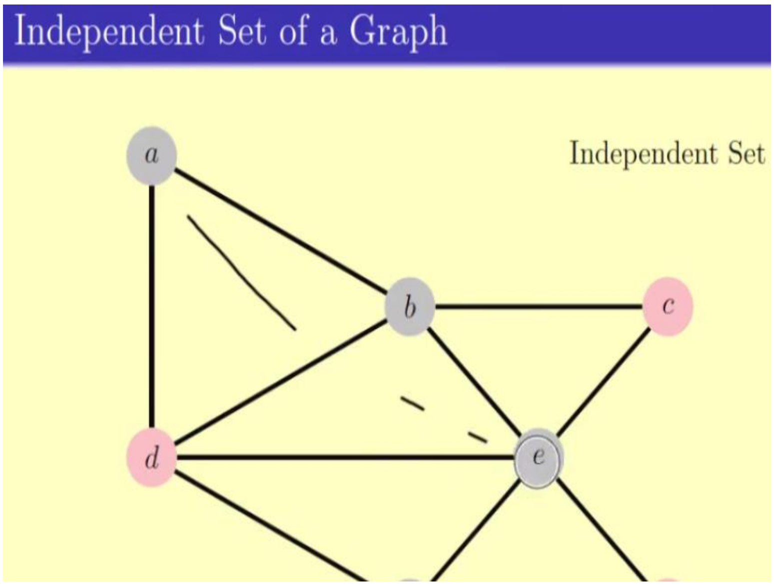

So, for example, if this is the graph that we have A and E is an independent set there is no edge between them. Similarly, D, C and G is also an independent set because there again no edge between B and C, C and G, G and E. Now as a Clique we can have A, D, D is a Clique because there is edge between any two of them. Similarly, we can have A, D, B, E as a Clique.

Now if we have an edge between A and E then in that case A, B, D and E would have been a Clique also because between any pair of between A, D, B and E there is an edge. So we have an understanding of what is an independent set is and what is Clique is. Now using these two notions we can now visualize this problem. To visualize it, we have to understand what does it mean (()) (04:52) created graph.

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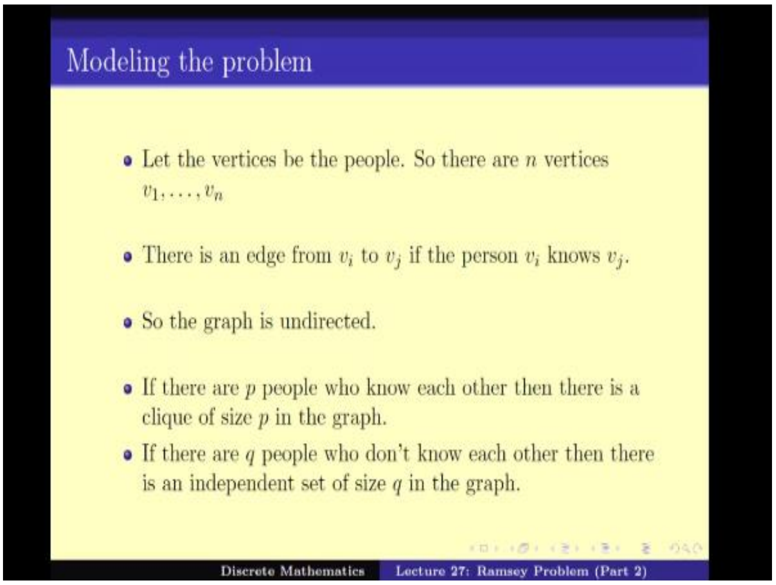

So namely let the people be N vertices V (1) to V (N). So the first thing is to try to understand what does the R (P, Q) implies. We draw an edge between the vertex V (I) and V (J) if the person V (I) know person V (J). Now since person V(I) knows person V(J) implies person V(J) knows V(I), so the graph is an undirected graph because the relationship is symmetric. And what are we looking at.

We want to find something like P people who each other. So P people who know each other meaning there are P vertices such that between any two vertices, there is an edge. So that means there is a Clique of size P and similarly if I am looking for Q people who do not know each other or in other words there are Q vertices such that between any two pairs in pair of vertices in this set there is no edge. So we get Q independent set

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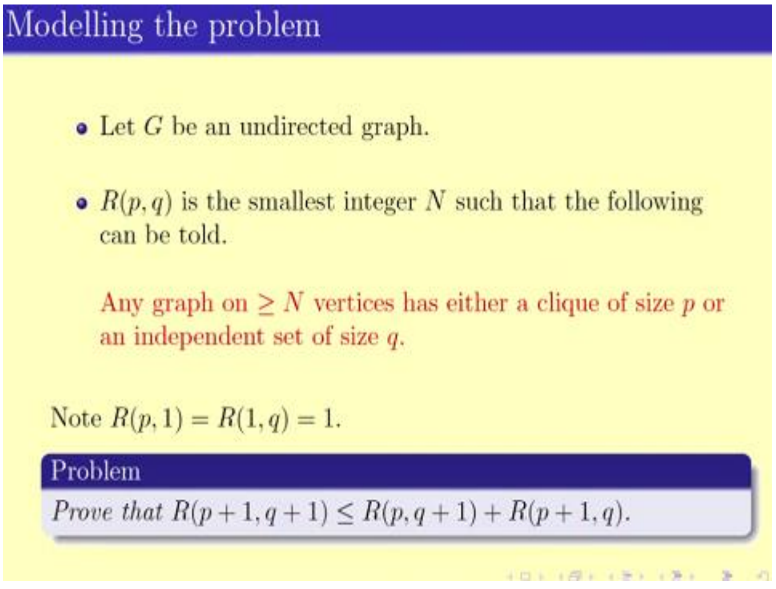

So if this is how we look at it the definition of R (P, Q) becomes R (P, Q) is the smallest integer N such that the following can be told. Any graph on at least N vertices has either a Clique of size P or an independent set of size Q. So this is how we would like to define R(P,Q) and what we have to prove so note that R (P,1) = R(1, P) =1 because if you have given 1 vertex just there exist 1 independent set which is our P1.

And there is also 1 Clique with R (1,Q). So this notes is easy to see and we have to prove that R (P +1, Q +1) ≤ R (P, Q+1) and R (P+1, Q).

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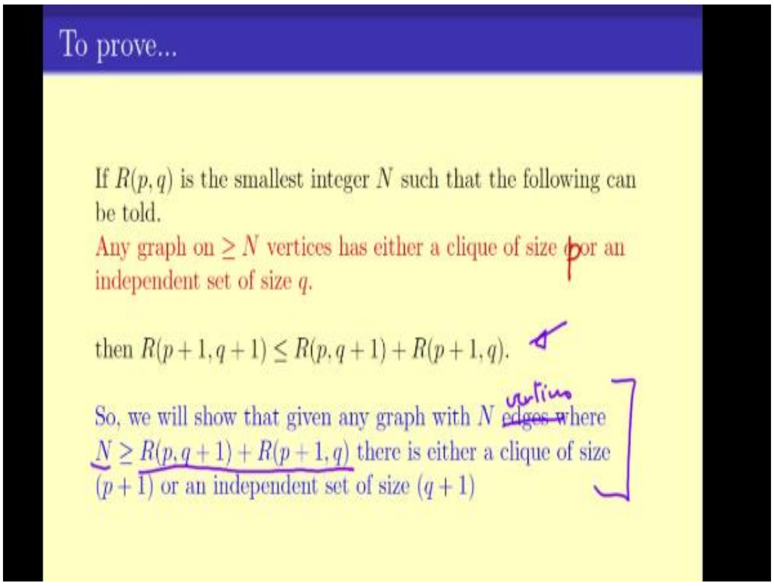

Now how we prove it? The hint is we have to give a direct proof of it. Now let us try to understand what we have to prove first. So this is the technique that we have. R (P, Q) is the smallest integer N such that the following statement can be told. Any graph on at least N vertex either has a Clique of size P or independent set of size Q. Then we want to prove that this relation holds.

So what we will show is that given at any graph of N vertices where N is ≥ R (P, Q +1) and R (P+1, Q) then there is either a Clique of size (P+1) or an independent set of size (Q+1). Now why is this sufficient? So if we prove this one, this statement what we will prove it? We are proving that the smallest N the R (P, Q +1) + R (P+1,Q) which is exactly what we have to prove. So we prove is that if I give you a set of any graph with more than N vertices.

But N is bigger than R(P,Q+1) + R(P+1,Q) then there is either a Clique of size (P+1) or a independent set of size Q. So this is what required to prove.

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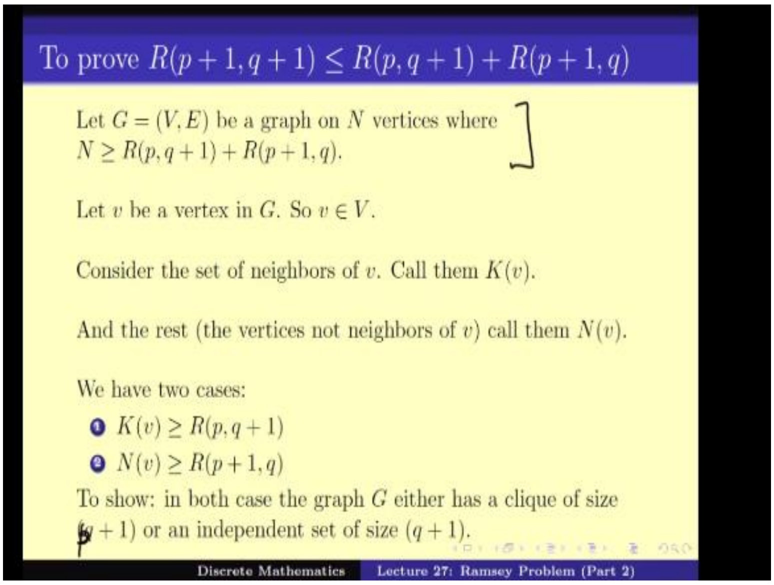

So to prove this let start with a graph on N vertices where N> what we just mentioned R (P,Q+1) + R(P+1,Q). Now take any vertex V in the vertex set and consider the neighbors of V. So how does a graph look like? So here you have the graph G, here is our vertex V. Now there are some neighbors of V. So these are the 3 vertices that have some A:B and then there are some vertices here that they do not have any A:B.

So we have some vertices here and some vertices here. Vertices that are neighbors of V and vertices that are non neighbour's of V. So we call them vertices of K (V). We call K because it actually known we know this set of people. Remember this graph is just a Mathematical presentation of the problem and we will call the other one as N(V). So the set of neighbors of V is called K(V).

And with set of neighbours vertices that are not neighbors of V are called N(V). Now given the fact that this N ≥ R (P,Q+1) + R (P+1, Q). We have two cases. First of all, either K (V) ≥ R(P,Q+1) or N(V) ≥ P+1,Q). So why are these two the only two cases? So to prove that these two are the only two cases what we have to see is that if neither of the case holds when something wrong is happening.

So what is the first case are in hold the first case that are in hold-K(V) is strictly < R(P,Q+1) which means this is ≤ this-1. Similarly N(V) is ≤ R(P+1,Q) -1. Now what is the size of the vertices set? So this was of course N to be started with and this is if you will call it is the neighbors of V + the vertices which are not neighbors of V + the vertex itself which is 1 which is ≤ this + this which is R (P,Q+1) + R(P+1,Q) - 1, - 1 is minus 2.

And there is this + 1 so I am getting -1. So if neither of this case, case 1 and case 2 hold then number of vertices < R(P,Q+1) + R(P+1,Q) -1 or in other words this is strictly <. So N is strictly < R (P,Q+1) + R(P+1,Q).which is unfortunately contradicting this assumption that N≥. So if case 1 and case 2 does not hold we get a contradiction and hence it cannot be that neither case 1 nor case 2 holds.

So either case 1 hold or case 2 hold or maybe both hold, but of them cannot hold is not an option. Thus, we have get this 2 cases namely K(V) ≥ R(P,Q+1) and N(V) ≥ R(P+1,Q) and what we have to prove. If we can prove that in either case either we have a Clique of size (P+1) or independent set of size (Q+1) then what we have prove that we started with a graph G with N ≥ R(P,Q+1) + R(P+1,Q).

And we have proved that the graph either has a Clique of size (P+1) or an independent set of size (Q+1) and this is exactly what we want to do to do proof in the first place. So we will go by proving it case by case.

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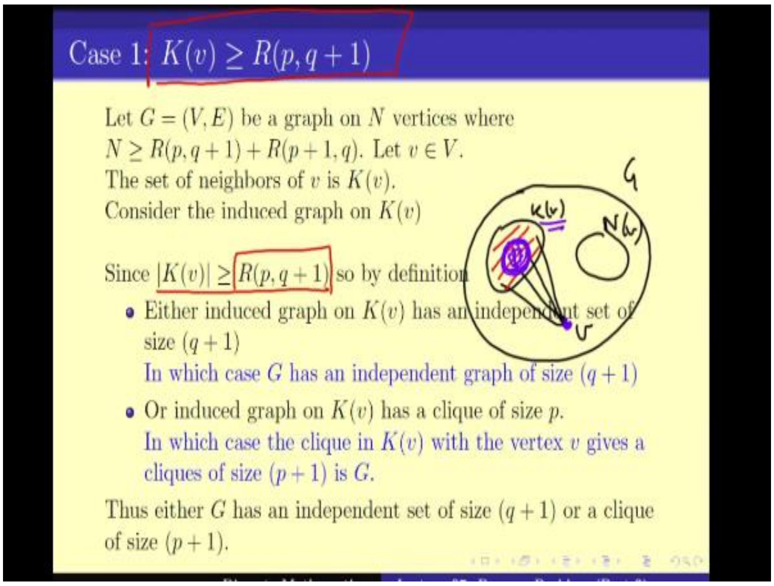

So to start with let us start with the first case. Namely K (V) > R(P,Q+1). Now let us look at what is going on before I start with the explanation. So here is the graph, here was the set of K (V), here is the set of N (V) near the vertex. So every vertex in K (V) is attached to this vertex V. Now this K (V) > R (P,Q+1) what does it mean? If I look at just this graph just the graph on K (V).

Then we have this two options this one either K(V) has an independent set of (Q-1) that's what the definition of R (P,Q+1) means. K (V) > R (P, Q+1) either there is an independent set of size (Q+1) sitting inside K(V). In that case that independent set is also an independent set of size (Q+1) set.

If this is an independent set here there will no two vertex here has an edge between them then the whole graph G I still do not have any edge between them and hence I get the same set of vertices either independent set of size (Q+1). Now the definition that KV > R (P, Q+1) and the definition of R (P, Q+1) either it has an independent set of size (Q+1) or it has Clique of size P.

Now what happens if it has a Clique of size P. Let us think of this blue thing area as a Clique of size P that is between any two vertex here there is an edge. Now if this is the size P say consider this set and this vertex V then this blue set classes vertex we get a set of size (P+1) and it is a Clique because first of all any two vertex in this Clique has a edge and this all the vertices is a set of K (V) which are neighbors of V.

So between V and any other vertex there is a Clique of size there is K, so it is an edge which means that this vertex along with this vertex V gives me a Clique of size (P+1). So either G has an independent set of size (Q+1) or a Clique of size (P+1). Now in case 2, we have a similar argument except that now we will look at N (V) set.

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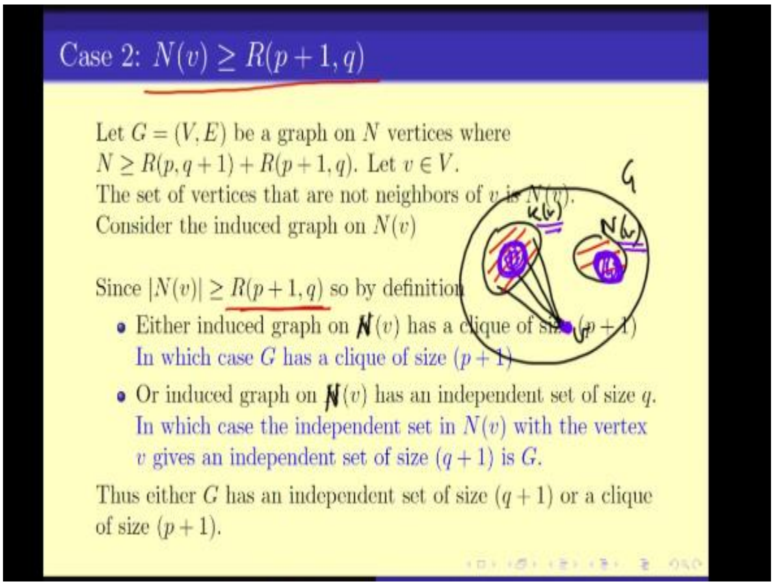

So the case 2 the N (V) > R (P+1, Q). At this time again look at the induced graph on N (V). Now since N (V) is bigger than R (P+1, Q), so by definition 2 things holds. Either this N (V) has a Clique of size (P+1) so either is already help Clique of size (P+1) it is a Clique here. In that case this whole thing is still a Clique of size (P+1) in the original graph. And if not the other thing is that the N (V) has a independent set of size Q.

Now it is where an independent sections, the blue thing the independent set what this means is that between any 2 vertex in this blue thing there is no edge, but then again consider this blue with this V and then we get this thing + V gives an independent set of size (V+1). This is because this set now is completely continuos at N (V) with by definition of the edges on which there is no vertex to see.

So in that case also either there is an independent set of times Q+1 or Clique of size P+1.

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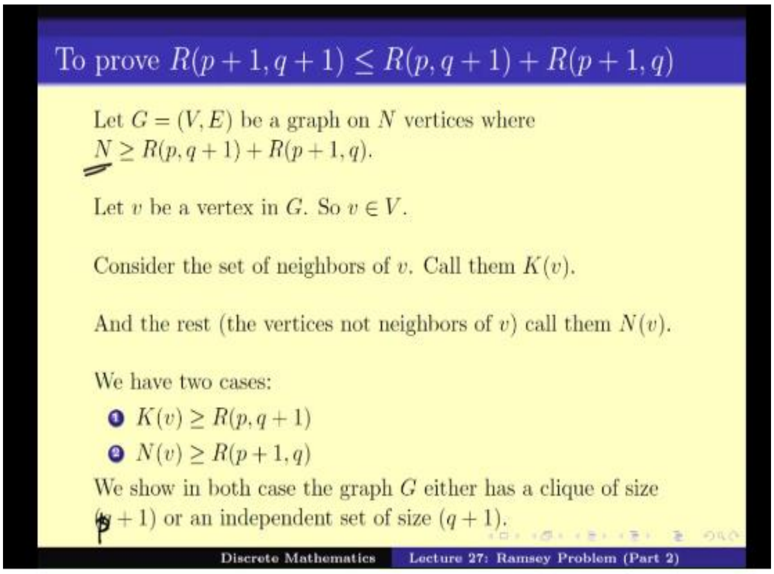

Thus we started with this graph G which has number of edges was ≥ R (P, Q+1) + R (P+1, Q). We looked at picked up any vertex N (V) looked at the neighbors and the non neighbors. We had 2 cases and we prove that either case either there is a Clique of size (P+1) or independent set of size (Q+1) and hence we have enabled to prove the Ramsey theory.

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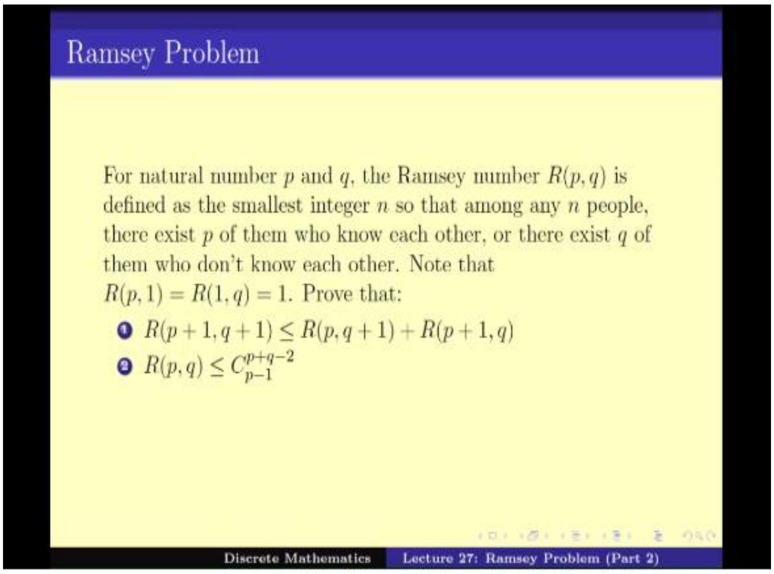

I mean, we ended up proving dis reconciliation the fact that this thing follows from the reconciliation we did it a couple of weeks earlier. So again we are seeing how graph theory can be used to model a problem, visualize it correctly and we solve it used graph theory to solve the problem. In the next video, we will be looking at graph theory; we will start looking at graph theory at itself and try to see how previous properties of-graph theory can be proved. So we started looking at some properties of graphs. Thank you.