Discrete Mathematics
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Lecture - 26
Ramsey Problem

Welcome back, so we continue with our study of problems or solving problems using graph theory. In this video lecture, we will be looking at the Ramsey Problem.

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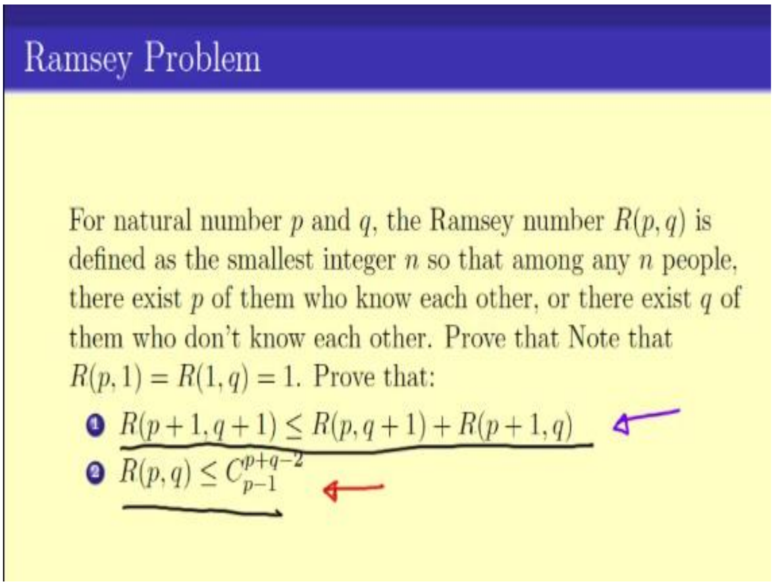

So this problem says that for any natural number p and q, if I define R p q to be the smallest integer n so that any n people, for any n people there exist p of them who know each other, or q or them who do not know each other. If this is the definition of R p q, then prove that this following. R p plus 1 q plus 1 is less than or equal to R p q plus 1 plus R p plus 1 q and the second part, which is R p, q is less than p plus q minus 2 choose p minus 1.

Now as you might remember, the second part is something that we already done in our earlier video while we dealt with induction on multiple variables. But in this video, we will be trying to solve this particular problem or proving this recurrence that R p plus 1 q plus 1 is less than R p q plus 1 plus R p plus 1 q. Now to be clear here what is R p q, let me quickly overwrite again.

So if a R p q is something like hundred, is basically will say its mean that give me any hundred people, I will find either p of the people, p people who will know each other or q people who do not know each other and R p q is the minimum such number. So hundred is may be a possibility or (()) (02:12) something right. So the recurrent is talking about the R p plus 1, q plus 1 is less than R p q plus 1 plus R p plus 1 q. Now to attend this problem, we will start with by trying to write in terms of graph theory.

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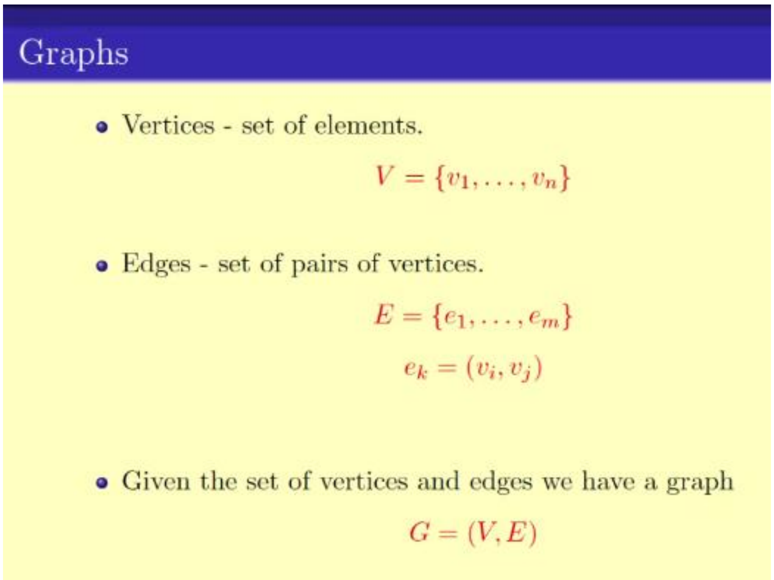

So we have already seen a bit of graph theory, so let us recap what have seen. So our graph, we have vertices, a set of elements, edges, which are basically set of pairs of vertices and the graph is represented as the vertex and the set of edges.

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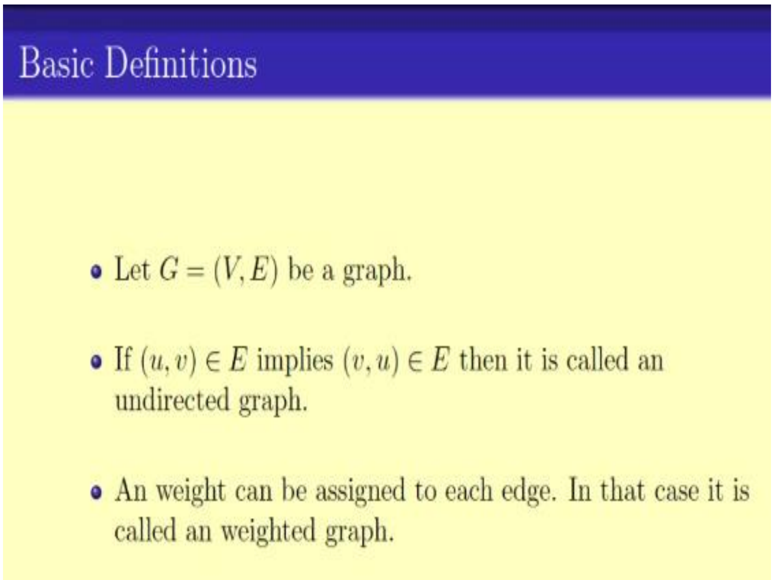

Now there are some basic definition that one can have particularly if the binary relation is symmetric that means u v is an edge implies v u is an edge then we call the graph undirected. We can also have weights assigned to the edges for if we need b.

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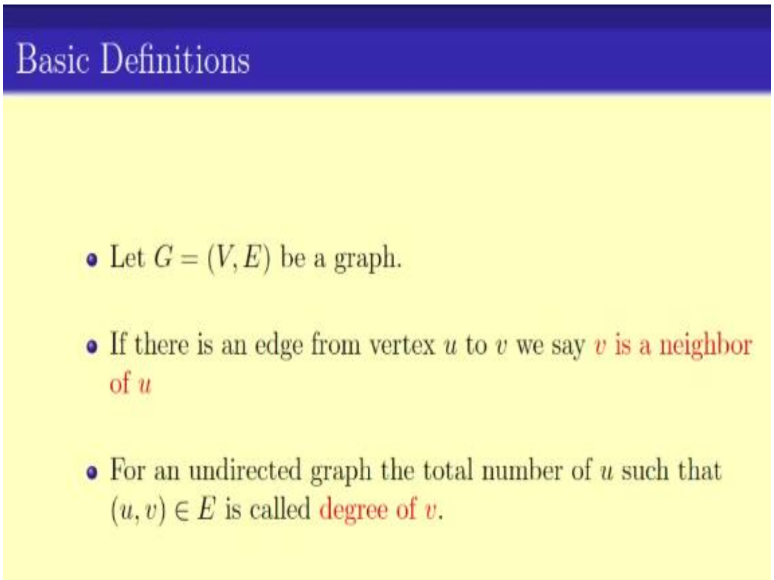

Also if I have an edge from u to v, we say v is an, neighbour of u and an undirected graph, the number of neighbours that v have is called the degree of v.

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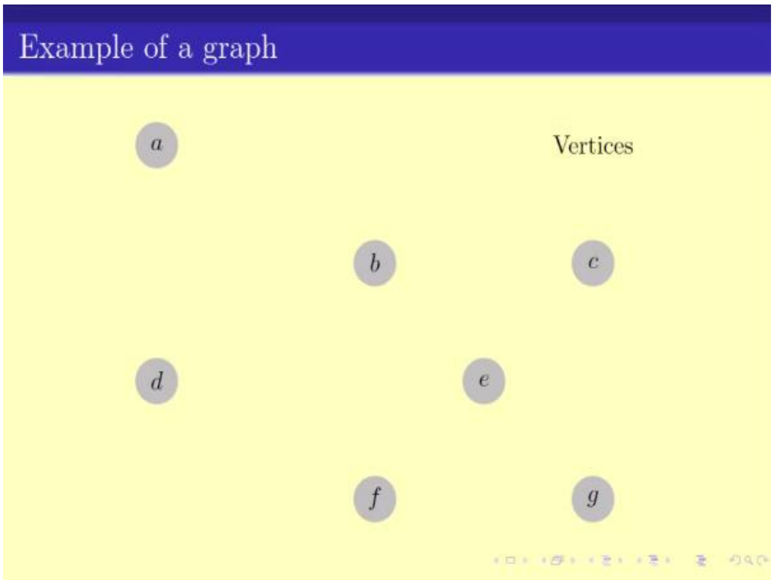

So pictorially, we draw it like this, the vertices of a, b, c, g, e, f, d. The edges are drawn by drawing lines between the two vertices that define the edge.

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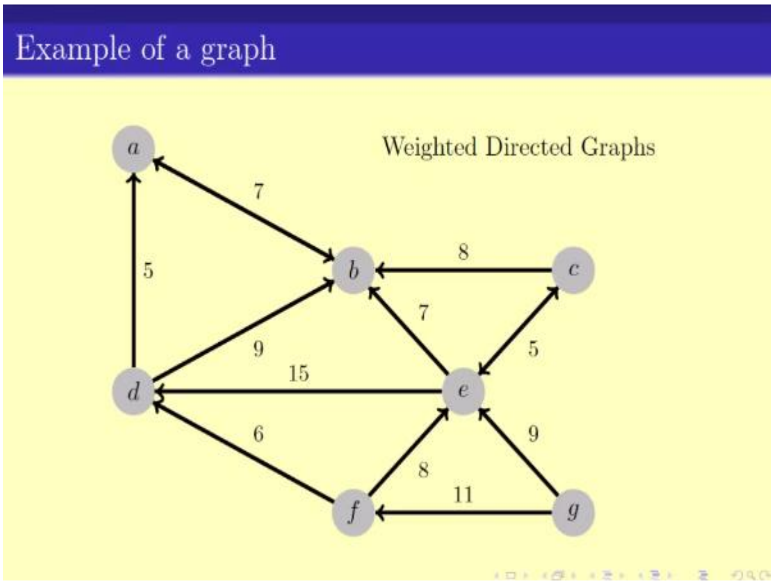

So here the edge is ab, bd, bc so on and there is no edge between a and c, that is a and c are related. We can also of course have weight form the edges and sometimes the edges can be directed, which basically means that the binary relation is not symmetric and we define or we draw it by the arrows. So here this a between d and a, but there is no a between a and d, A 2. Well, there is an edge from a to b and there is an edge from b to a.

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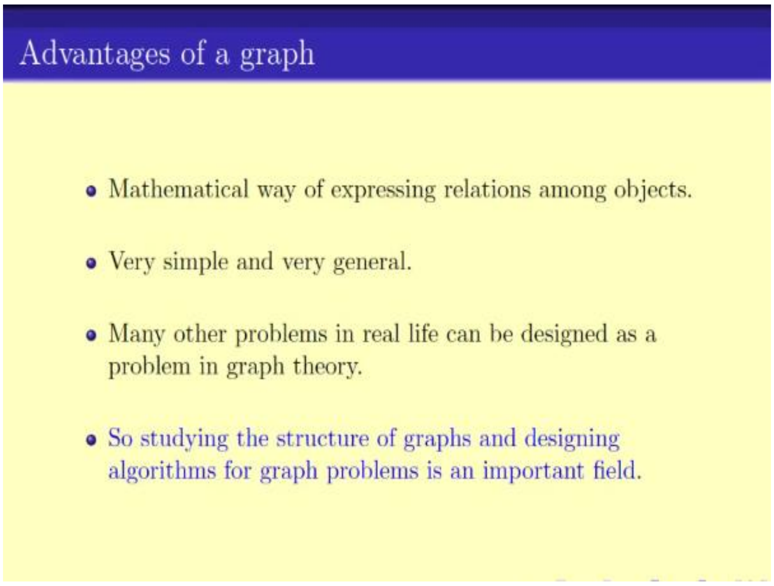

Now we have looked at some properties of graphs already in our earlier videos. The best part of the graphs are that they are very simple and general and hence many problems in graph theory, in real life can be modelled as problems in graph theory. And so starting this structure of graphs and designing algorithms for graphs is an important field to study.

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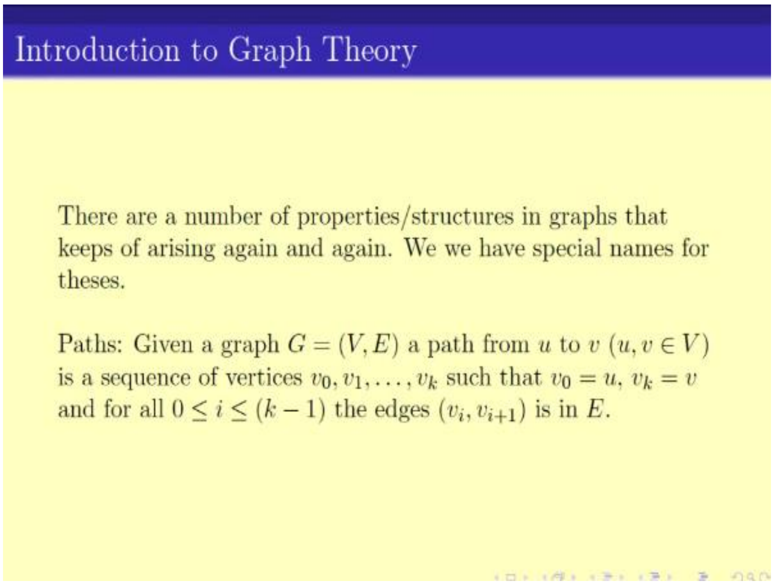

So there are various properties that we have already looked at, let me quickly go over once again. So one of them on the path, a path is basically a sequence of vertices such that between any two adjacent vertices there is an edge.

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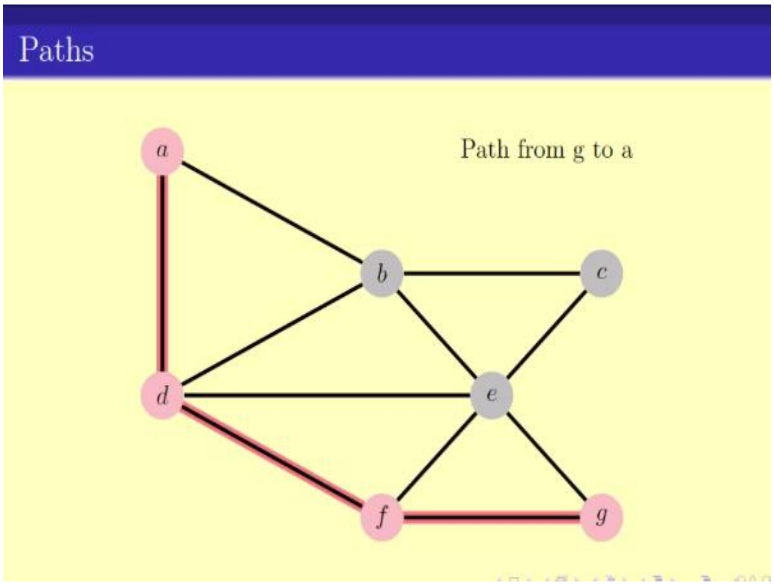

So it is like, if I have this graph and a part from g to a will typically something like this g f d a.

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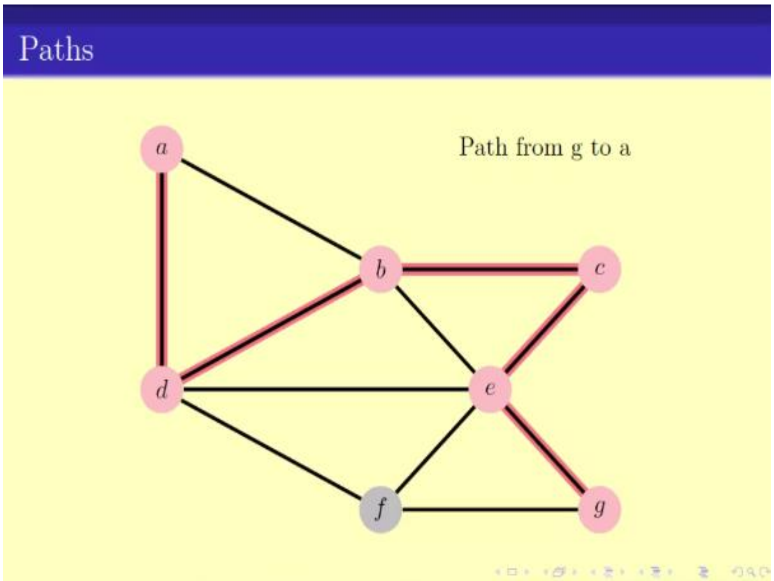

They have a multiple path between g to a, for example, g e c b d a. Also this is in the undirected case. In the case of directed case, but they might be direction on the edges. A particular path which is the path in the undirected case may not be of valid path. For example, this path is not a valid path as from b to d does not exist. It is going to the opposite direction.

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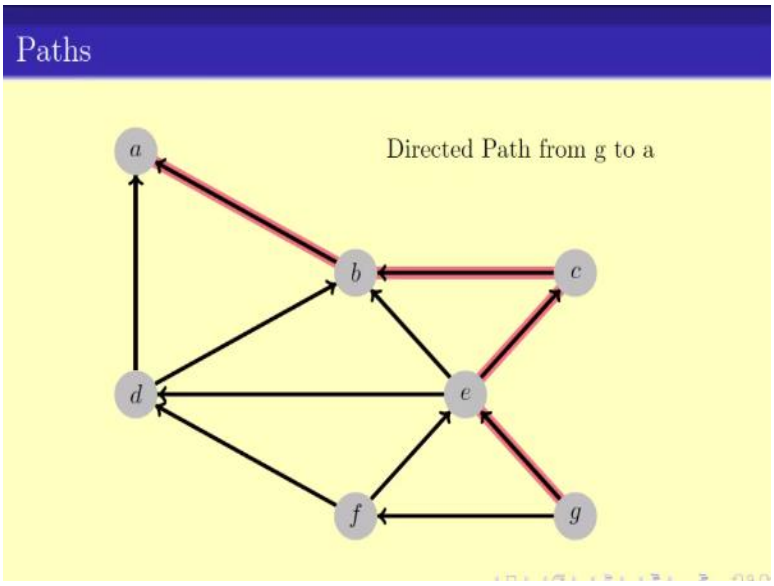

But then there can be other paths for example, this path is a directed path from g to a, g e c b a, the directed path. So you have already seen how to use this concept of paths for various problems.

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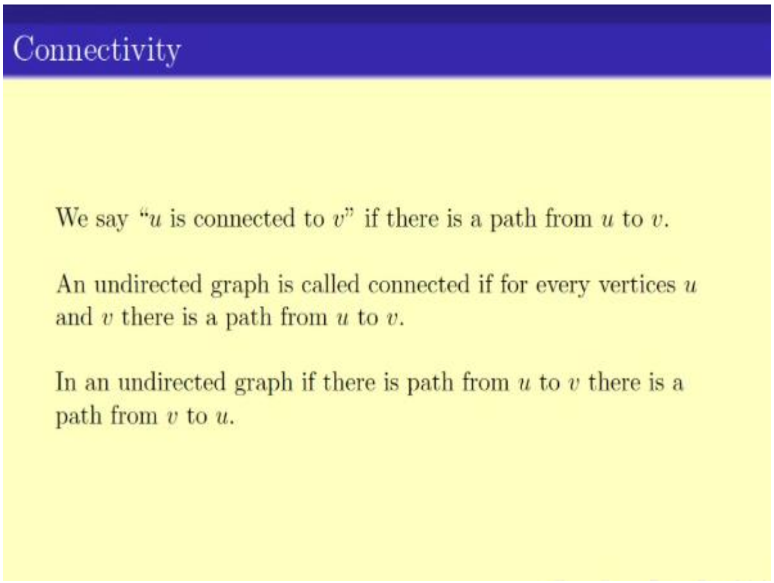
graph.



So you are going to say that u is connected to v if there is a path from u to v. In an undirected graph it is not hard to see that if there is a path from u to v, there is a part from v to u and we call that a graph is connected it's any two pairs of vertices is connected, in case the path from any one vertices to any other vertices. So these are the usual initial definition of graphs.

In this video we will be going through some of the properties of graphs and giving you some of the problems to think about will be coming back in a couple of videos to solve them.

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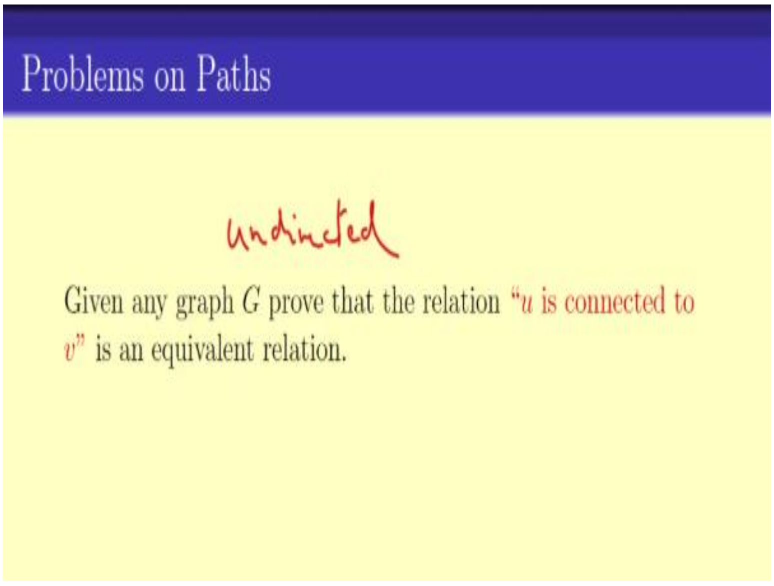

So one of the first problem that was that I prescribed last time was that prove that this relation, binary relation that u is connected to v is an equivalent relation in an undirected graph.

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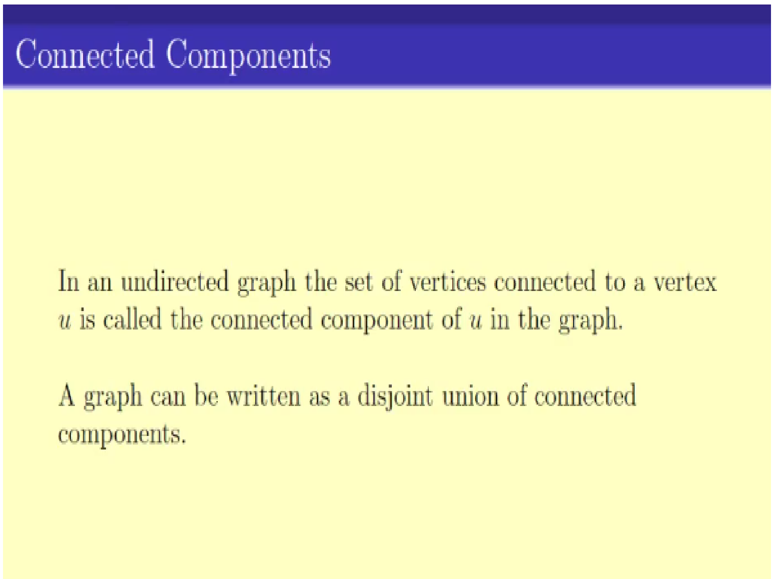

So, this - in this thing was g is an undirected and once you have this thing, if you can prove this statement then by the properties of the equivalent relations we will get that the graph can be split as a disjoint union of equivalent classes which are the basically the connected components. So namely, as of components which is connected to each other, so all the vertices are connected to each other and the graph can be written as a disjoint union of connected components.

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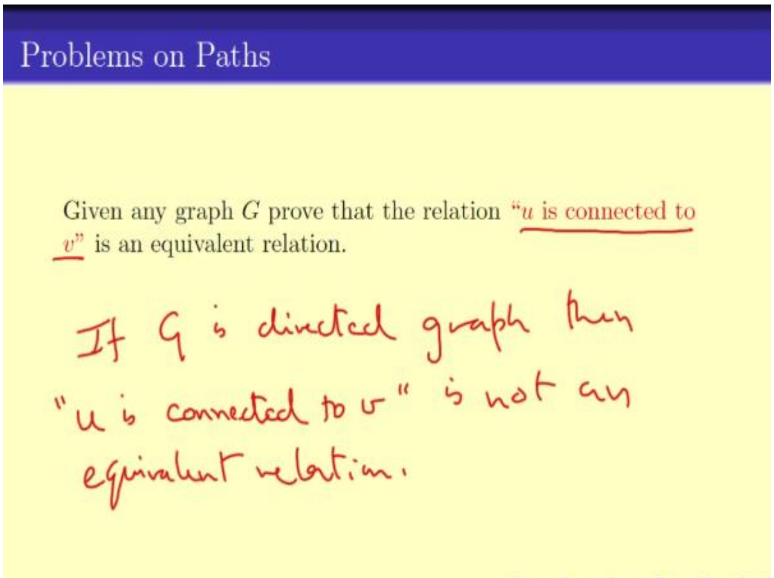

Let me also ask the question that, in this problem prove that u is connected to v is not an equivalent relation if g is directed. So g is directed graph then the relation that u is connected to v, now what you have mean by connected to v, it means that there exists an edge, directed edge from u to v is not an equivalent relation.

Now this is important because that would mean that in the case of directed graphs we need not necessarily able to write it as a disjoint union of connected components. We will come back to this particular connected components case later on in this course. Now there is another concept of cycles.

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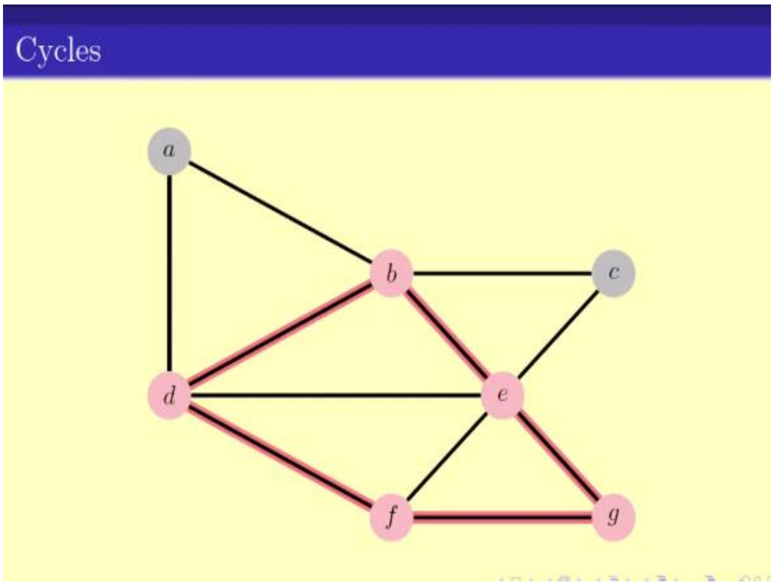

So cycles are basically paths that end up with wherever they started. So this cycle in the undirected graph. So g f d b e g.

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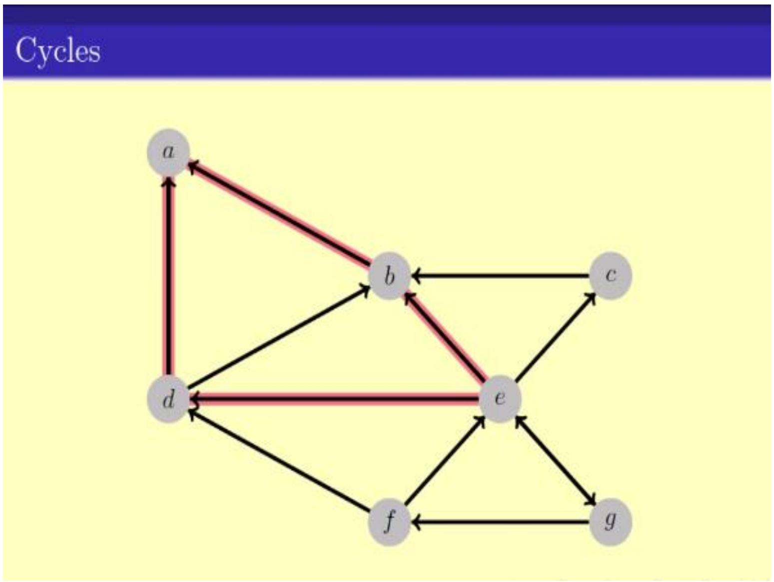

There can be of course many cycles for example, this is also another cycle a b e d. And also in the directed graph, we can also talk about directed cycle, so this is not a directed cycle because this is not going any direction. For example, d a b e is not a valid way of going around it because the path from or the edge from a to b is not there, similarly, b a d is also not a valid way of going around it because the path - edge from a to 0 is not there.

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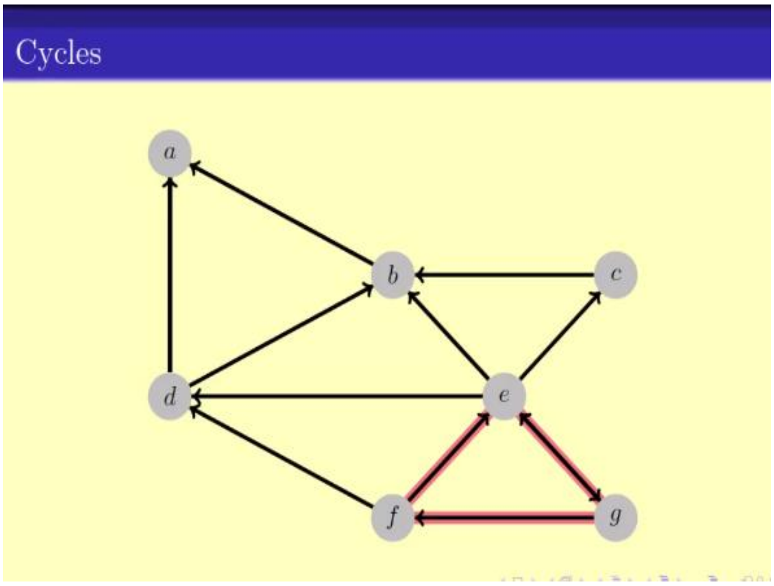

But for example, this one f e g is a cycle because one can go from f e g f and so on.

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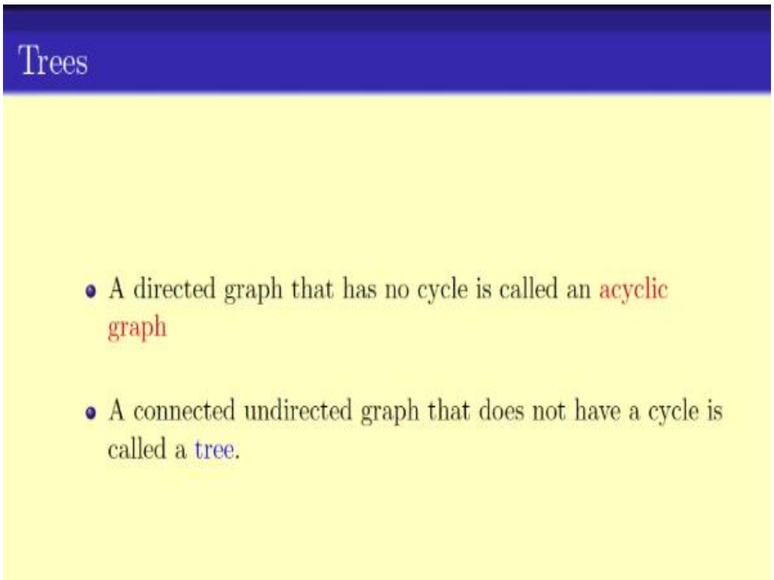

Now the cycle is a very important concept in graphs. If a graph does not have a cycle, we call it an acyclic graphs and if the graph is connected, many it is an undirected graph then we can go from any vertices to any other vertex but it does not have a cycle then we call it a tree. Trees are one of the most simple to understand graphs and hence one of the most powerful concept that is there also.

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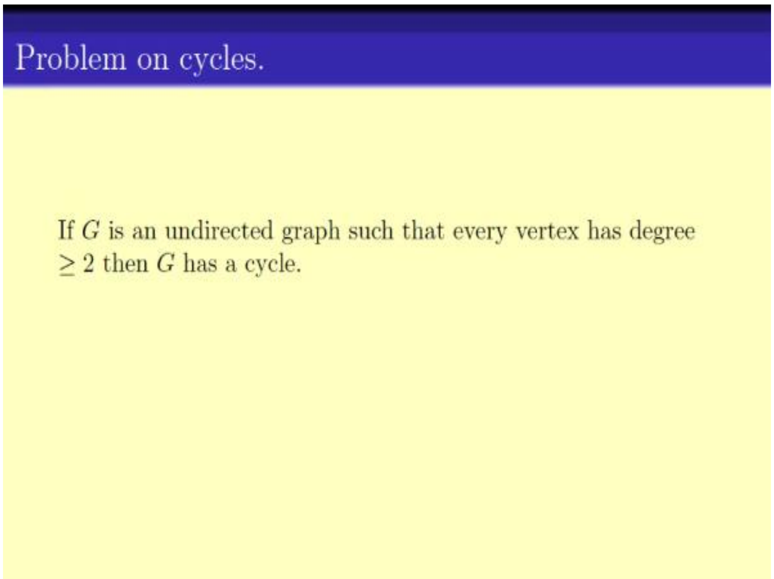

So I will give you a couple of problems on this thing, on this cycle. So prove that if g is an undirected graph such that every vertex have degree greater than or equal to 2. Mean every vertex have two neighbours at least then G has a cycle. So if the problem for you to think, I will be solving these problems after couple of videos.

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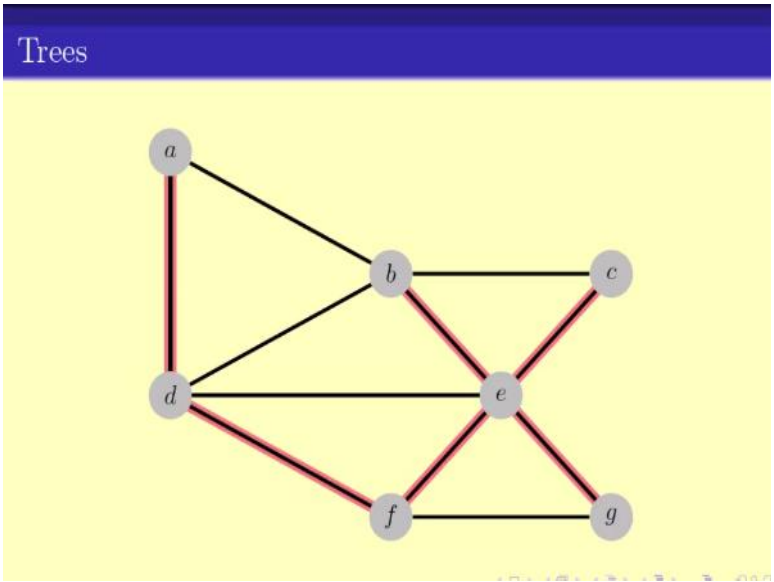

So what are trees, say let us look at this example. So this is a tree. The red edges on a tree. There can be mainly if you look, note that the range is does not form cycle, there is no cycle if you just use the red edges.

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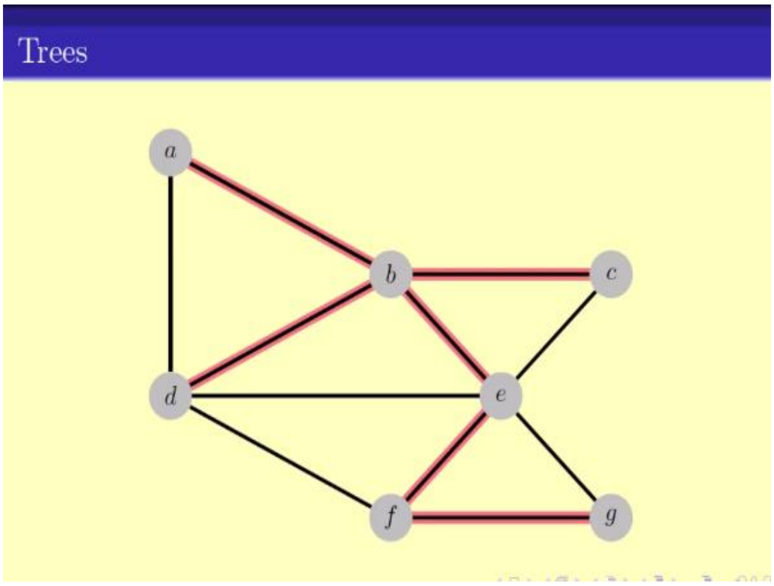

They can be other cycles or other trees also. For example, this is the tree and so on.

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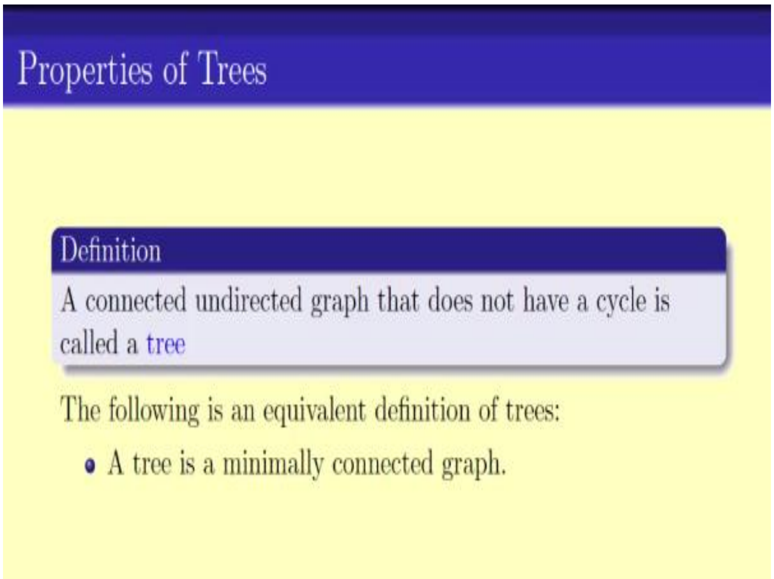

So the definition of a tree is as I told you earlier connected undirected graph that does not have a cycle is a tree. Now prove that the following is the equivalent definition of tree namely a tree is the minimally connected graphs. What do I mean by that, meaning given a graph G, is that so the graph is minimally connected. If you remove any edge then the graph becomes disconnected okay.

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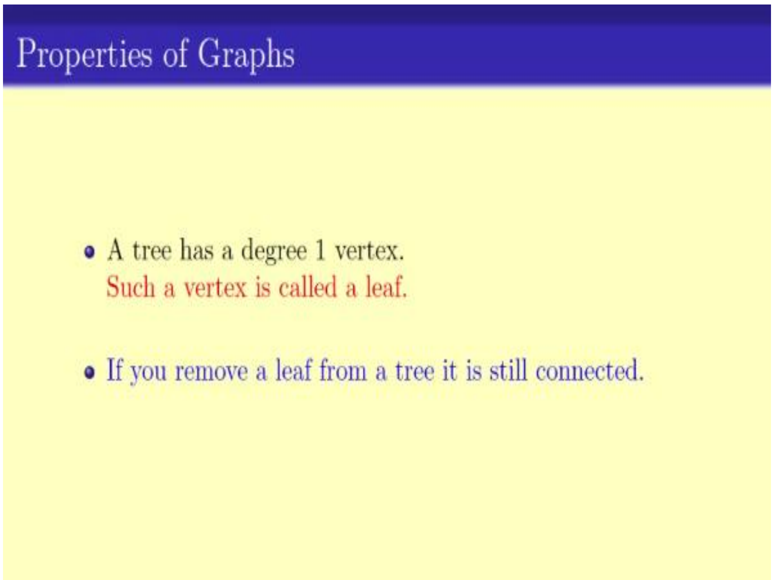

So this is one of the problems that I prescribe or ask you guys to think about. Another problem is that prove that a tree has a degree one vertex, so such a vertex is called a leaf. So in other word, prove that tree had the leaf and using this whatever we have proved in now, prove that if you remove a leaf from a tree then the tree is still connected okay. So remove a leaf, and then the tree is still connected.

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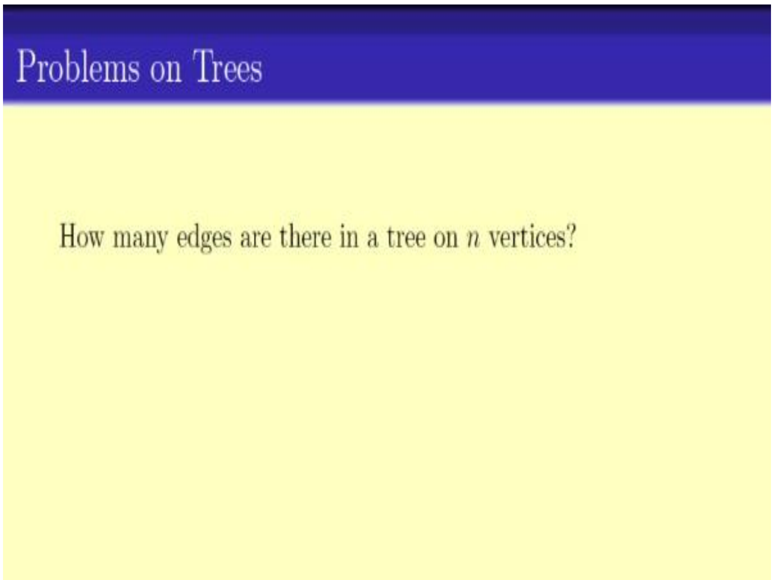

And using all of these things, can you tell me, can you answer this question, namely how many edges are there in a tree on n vertices, so if have a tree on n vertices how many edges can there be? can you, is there a fixed number or can be anything. So these are the problems on trees that will come back after two videos to solve but before that I wanted to tell you the problems so that you can think of the problem by yourselves.

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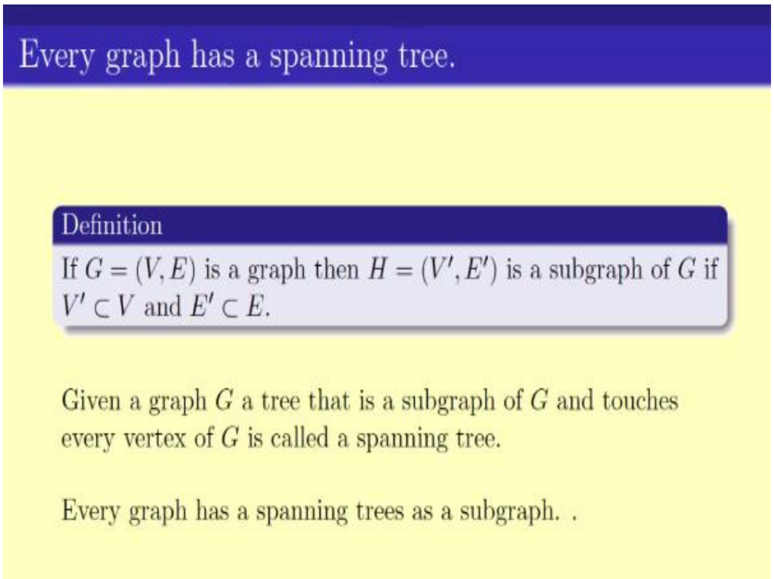

Now there is one more definition, If I give you a graph, and you remove some set of ages and some set of vertices then we get something called as subgraph of G. So the graph can have a number of subgraphs. A subgraph G or subgraph G is called as spanning tree, if it is a tree and if it touches all the vertices. So the question is that prove that every graph has our spanning tree as a subgraph. So in other words, even a graph G, I can remove some of the edges such that the graph is still connected and it is the tree okay.

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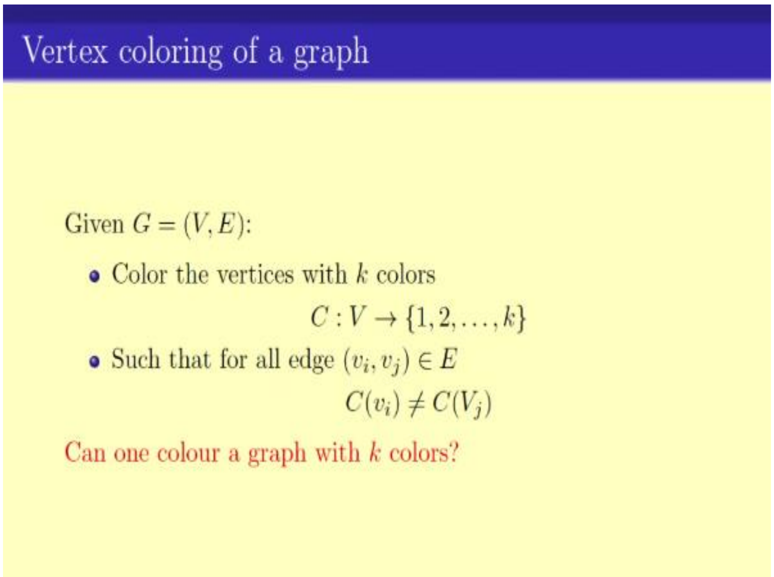

So let us move on to a new concept of graph. So we call, this is a vertex colouring of graphs. So given a tree a colouring basically says that can we colour the vertices using some colours so that means, there is a map from the set of vertices to colours that I number them, 1 to k. Such that if two vertice are adjacent to each other or neighbours of each other. So that mean if I have a vi vj in an edge, then the colour of vi vj are different. The question that are asked is that can one colour a graph using k colours.

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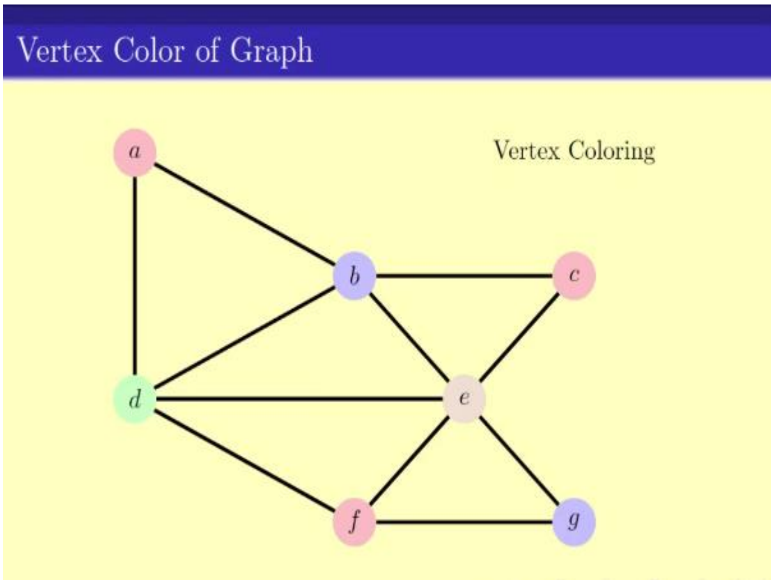

So here is an example of a graph and say If I colour this, the first vertex a as red, colour this one as blue, then I colour this one as grey. So in d, I cannot colour it as grey or blue, d as it is the neighbour of a red and the blue vertex.

This edged vertex e cannot be coloured with either green, blue or red, we have to colour it in a different colour because e is an neighbour of all of these three, f we can colour it with grey and g with colour with blue. So this gives as a colouring of the graphs using 1, 2, 3, 4 and four colours right. Note that no two adjacent vertex at the same colour.

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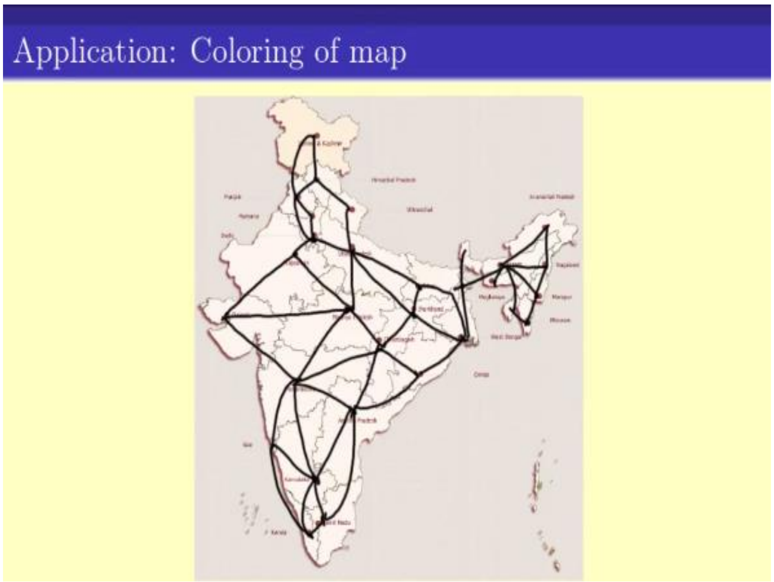

So a typical application of this problem is colouring of a map. So if you remember when we in our geography or atlas when we see the map of India, all the states have coloured differently and in that case we have to ensure that no two adjacent states has the same colour. For example, Rajasthan and Madhya Pradesh which are adjacent to each other should not have the same colour.

So we want to ensure that some how can I colour them so that no two adjacent states have the same colour. Now of course, we want to use minimum number of colours. We do not want to use a different colour for all these states. So the question is that how many colours were need to colour the whole map. So this is a very useful application of graph theory again. So the way to go about this is that you have a vertex for each state and you draw an age between the vertices, if the states are adjacent.

So for example, between Gujarat and Rajasthan, I can draw an edge because they are adjacent. Rajasthan and Madhya Pradesh, we can draw an edge to the adjacent. Madhya Pradesh and Maharashtra, can draw an edge to the adjacent. Gujarat and Maharashtra is also adjacent, Gujarat and Madhya Pradesh also is a neighbour adjacent. So I do not draw an edge between Rajasthan and Maharashtra.

So the idea is that I can colour Rajasthan and Maharashtra with the same colour and there is no problem with that. I will keep on doing it to get a complete graph of with all this various states and if you keep on doing this, you finally get a graph with vertices following the states and the edges represent whether the states are adjacent or not right. And so on you complete this thing.

So for like this, you can draw the whole graph for the whole India's map and by doing so you will be able to create a graph and question is that how many colours do we need to colour the graphs and this is the number of colours that require to colour the map of India. So this is a very nice application of graph's theory and we use the lot.

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Now moving on to some more, so this is a example of a colouring of a graph. As you can see that here is the colouring where no two vertices or the no two adjacent states have the same colour and they have used only four colours. Now the question is that is it that always we can use four colours to colour a map or do we need more colours. We will answer this question later on in this course.

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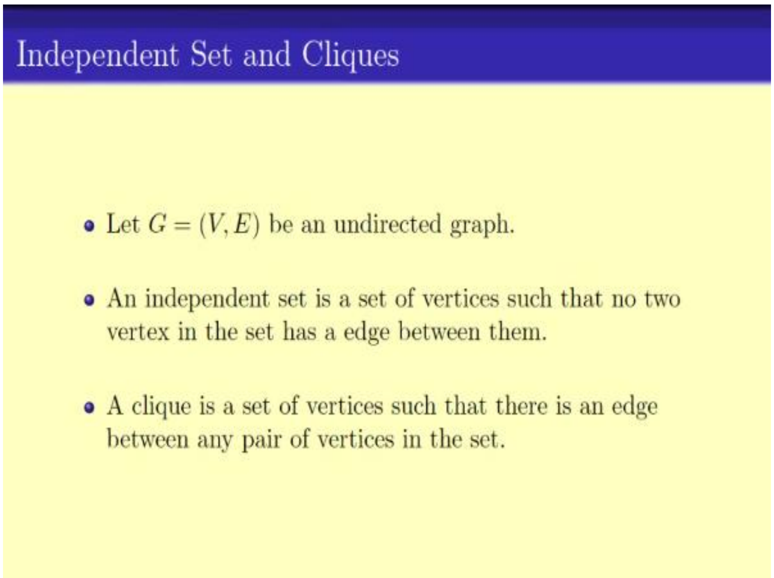

Another important concept for graph theory are this independent sets and cliques. So if you are given an undirected graph G, an independent set is a set of vertices such that no two vertex in the set have an edge between them. So they are independent. So no two of them are an edge between them. The clique on the other hand is just the opposite. It is a set of vertices such that there is an edge between every pair of vertex in the set.

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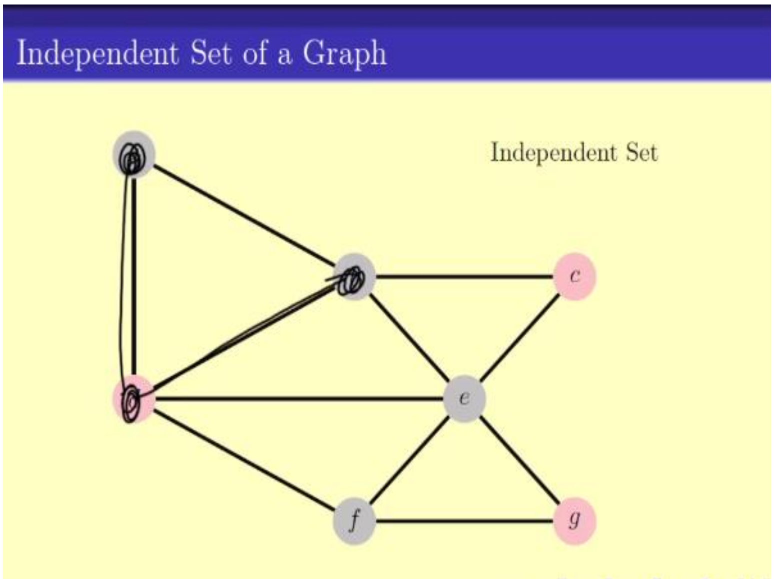

So for example, if this is the graph, this is an independent set a and e, because they know edge between a and e. Here this is the independent set d, c and g, there is no edge between d and c, there is no edge between c and d and there is no edge between d and g right. On the other hand, say a, d and b is the clique between because there is a edge between a and b, b and e and a and e.

So I have a clique and our independent set, a clique of sets three and an independent set of size three. Question that we usually asked is that how many, what is the largest independent set that is there in our graph or what is the largest clique that is there in a graph, we have very nice concepts also.

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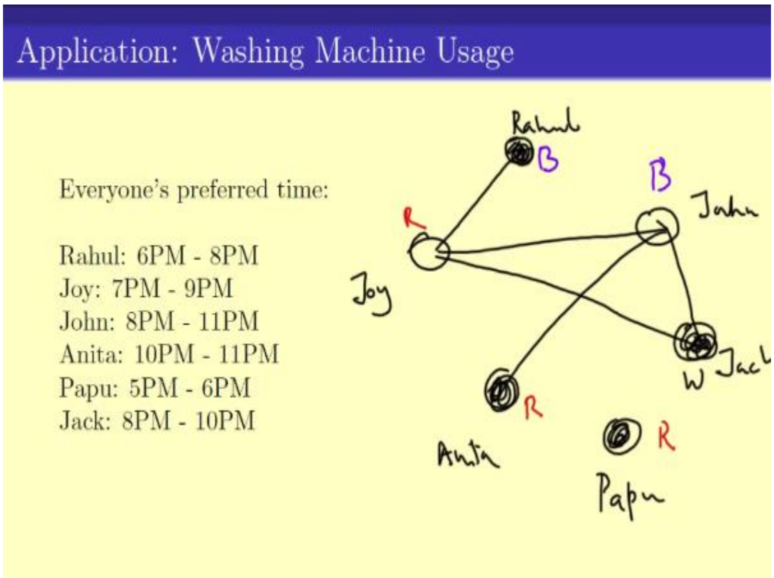

So here is a very nice application of independent set. So say in a hostel there are six people and they want to use the washing machine. Everyone has a particular preferred time. Saying 6 to 8, 7 to 9, 8 to 11 and so on so for John and. So now question is that how many washing machines do we need for that. So one of way of putting is that making defined edges in vertices.

Let the vertices represent the people, so this is say Rahul is a vertex represent the Rahul, there is a vertex representing joy, there is a vertex representing John and there is a vertex representing Anita, there is a vertex representing Papu and there is a vertex representing Jack. Now the edge is we draw an edge between Rahul and Joy, if there is a class of time. The idea is that if they have a clash then they cannot use the same washing machine at the same time.

So here for example, Rahul's preferred time is on 6 to 8 but Joy's from 7 to 9, therefore Rahul and Joy as an age. See, Joy, Rahul and John they do not have an edge because they do not have a clash right. Note that Rahul does not have a class with anyone else. But Joy has a clash with John. Joy does not have a clash with Anita. Joy does not have a clash with Papu but Joy has a clash with Jack. Whereas, John has a clash with Joy, John had a clash with Anita and John has a clash with – now John has a clash with anyone else.

Now Anita have a clash with John, Anita does not clash with anyone else. Papu for example, can clash with no one, Papu is completely independent and Jack has a clash with Joy and that is about it. So Jack has a clash with John of course. So now this is a graph representing this data. Now question is that does this graph so I want to, we can ask the various things on this thing.

How many washing machines are required and that would be basically how many number of colours is required to colour this graph because the number of colours that is required if for example, this one requires three colours they can red, I can define this one as well, red, red, blue, blue, and this one has to be white. So if I buy three washing machines I will satisfy everybody need. The other thing is that how many people can use the washing machine that day.

That depends upon, what is the largest independent set that I can get. For example, here I can have Rahul, Anita, Papu and Jack, all of them using it in the same time. So I have an independent set of (()) (28:53) so, at least four people can use the washing machine in the same day. So note that the representing this data in a graph and then trying to answer the number of colour (()) (29:12) or what is largest independent set and so on, gives a lot of information about our data.

So graph theory is used a lot for modelling our problems and thus by starting graph theory you will be able to answer quite a number of this real life problems also. Note that this is not just a toy example. We do have the application of this problem in things like say a location of jobs to servers and so on.

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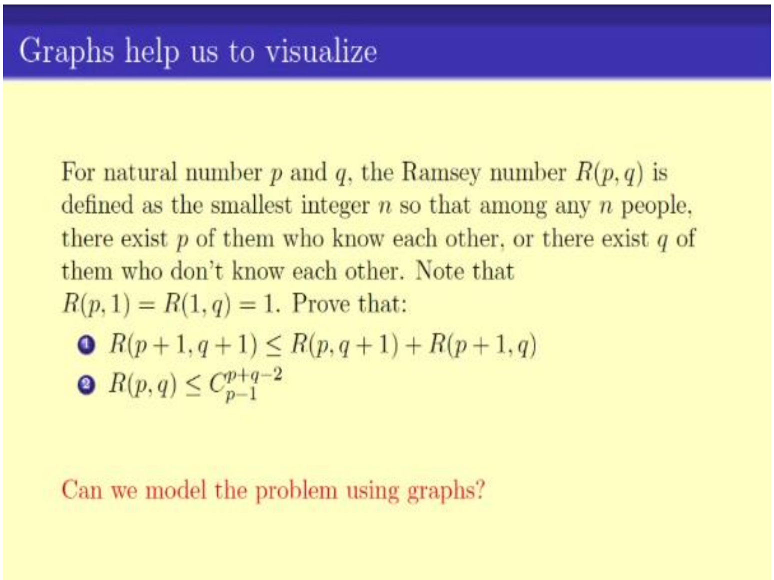

Now let us with all this nice knowledge about graphs clearly that we got, can we try to visualise this problem that we have to recall again p and q are two numbers Ramsey number to this R p q is the smallest number n, so that among n people, among any n people there exist either p of them, who know each other or there exists q of them who do not know each other. Then prove that this thing holds, this equation holds. Now forget what you have proved let us try to see the represent the data using the graphs.

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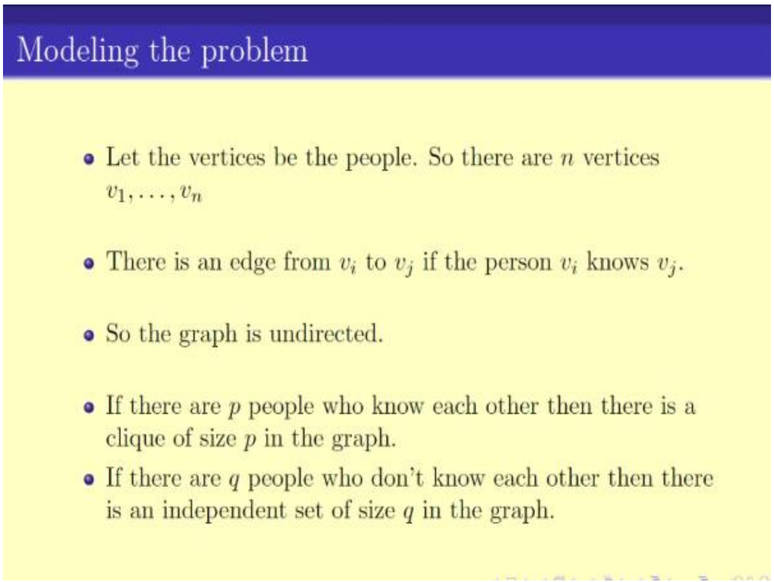

So first of all let the vertices be the people. So there are n people, v1 to vn. Now you join edge between vertex vi and vj, if this person vi knows vj. So this of course represent the currently being data above who knows whom and what are we asking. So by the graph is undirected because we have assumed that vi knows vj then vj knows vi and if we ask that if p of them know each other that means they should be p of them such that between any two of them there is an edge or in other words there is a clique of size p in this graph.

Similarly, if q of them do not know each other that means they will be q of the people or q of the vertices such that between any two of the vertex in this set, there is no edge. So independent set is of size q in this graph.

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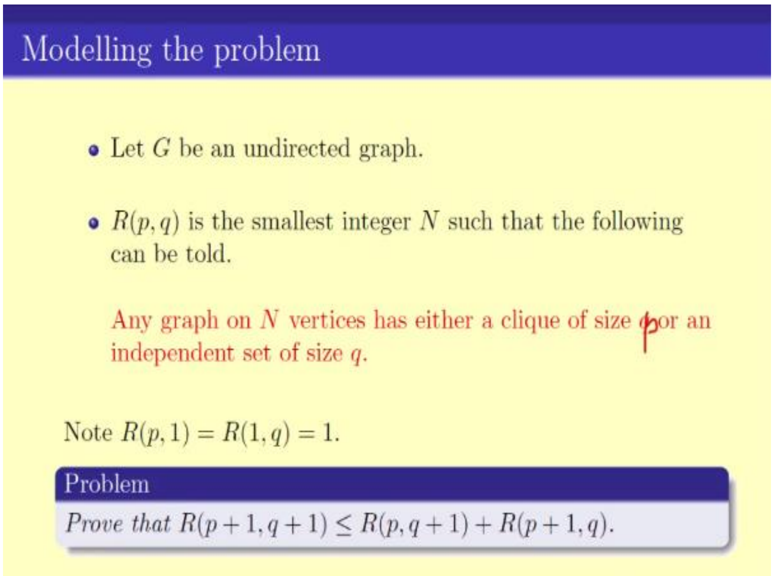

So in other words, we are asking that if G is an undirected graph, R p q is the smallest integer

N such that the following thing can be told. Any graph on N vertices has either a clique of size p or an independent set of size q and this is the small thing, note that R (p,1) and R (q,1) is 1. So the definition of R p q now becomes slightly more clear right. The problem is that prove that R p plus 1 q plus 1 is less than R p q plus 1 plus R p plus 1 q.

So this helps us to formulate this problem as the problem in graph theory in the next class, we will try to prove the following this part problem. Thank you.