Discrete Mathematics
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Lecture - 25
Tournament Problem (Part 2)

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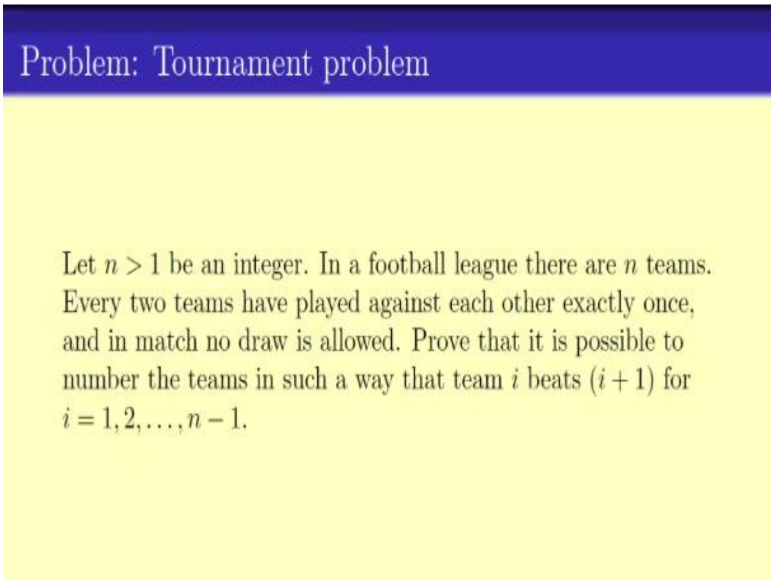

Welcome back, so we have been looking at the Tournament problem. So the problem states that in a football league there are n teams, any two team or every two team has played against each other exactly once, and either of the teams has won. So there is no draw. Prove that, it is possible to number the teams in such a way that team i defeats team i plus 1. That means team one defeats team two, team two defeats team three, team three defeats team four and so on. In the last video lecture, we saw how to model this problem as graph.

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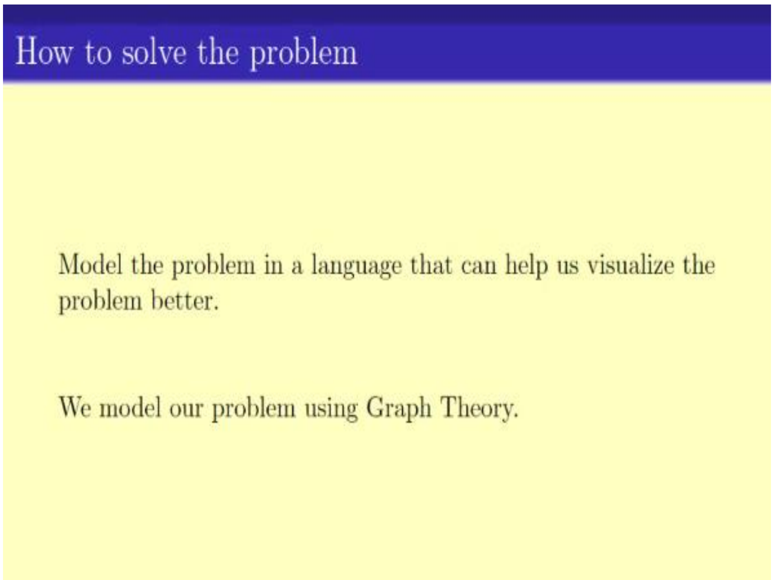

So as a quick recap let us go over it on.

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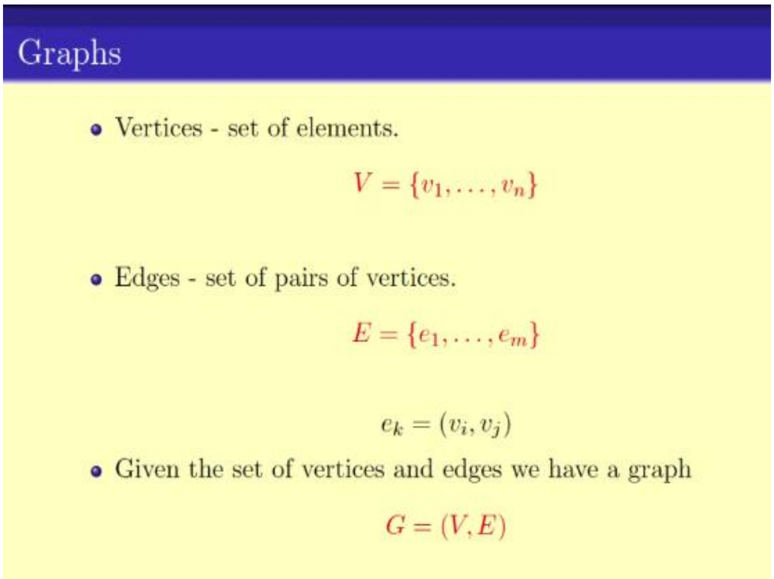

So, what is the graph, a graph is a set of vertices and a set of edges, which are basically pairs of vertices. A graph is given as v, e, where v is the set of vertices and e is the set of edges.

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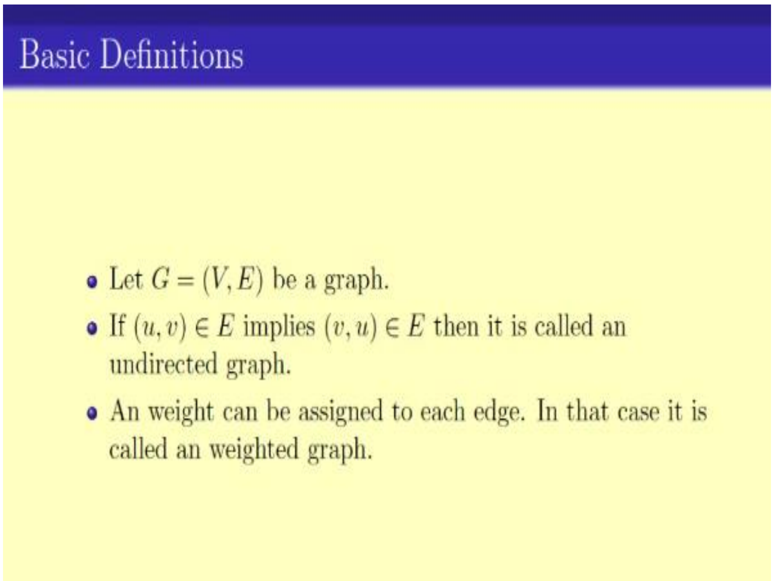

Now there is some basic definition of graphs. First of all, if u, v is an edge and that implies that v, u is an edge, then the graph is undirected and we can also have weights on the edges for our need as and when we will need.

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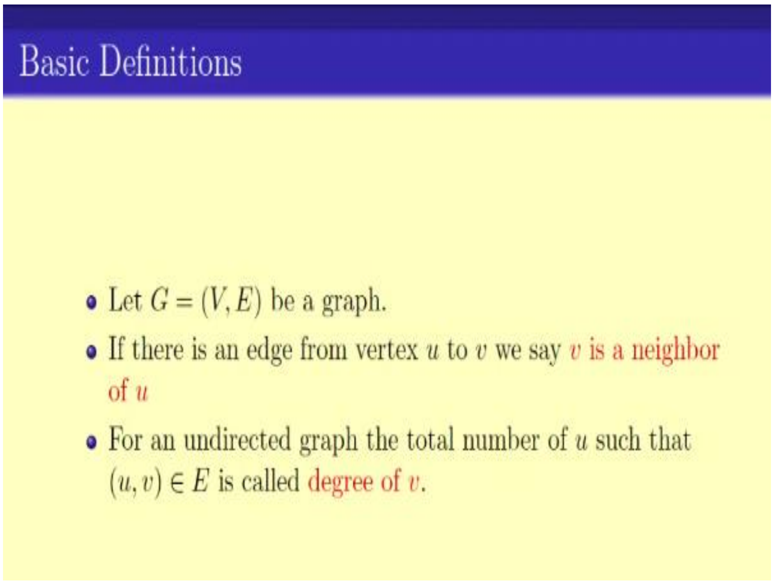

Also if there is an edge from u to v, then we call v is the neighbour of u and in an undirected graph, the number of neighbours that v has is called the degree of v.

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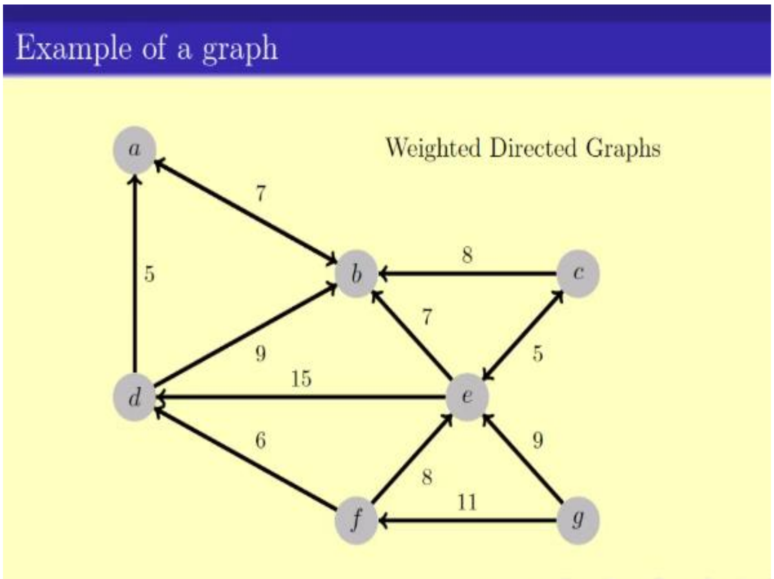

So pictorially, we draw the graphs as this kind of blocks representing the vertices a, b, c, d, e, f, g. The edges which are basically pairs of vertices are represented by line drawn between those two corresponding vertices. So that is here a, d or d, a is an edge. Well, a, c is not an edge. This is called an undirected graph.

Of course for our need, we might have weight from this edges and then we have all this as weighted undirected graphs and when the binary relation is not symmetric that means a, d is not same as d, a then we represent it using arrows like this. So arrows which represent here that as an edge from d to a, but there is no edge from a to d. So it is the non-symmetric binary relation In this case, we call it as a directed graph.

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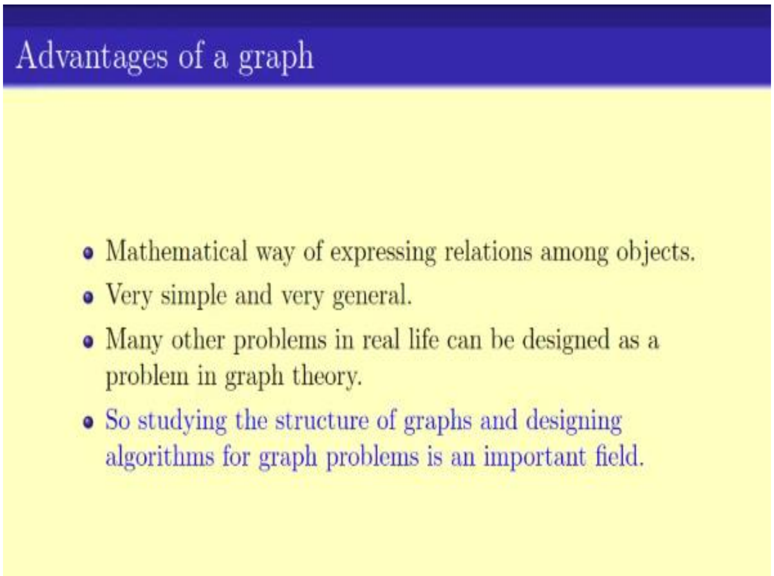

Now, we have already seen how a graph can help in visualising a problem. In general graphs are very useful for expressing relations, which is a very key component in many problems that we handle. So many problems in real life can be designed as problems in graph theory. We will see more of this in the coming weeks and so studying the structures of graph and designing algorithms is an important field in graph theory.

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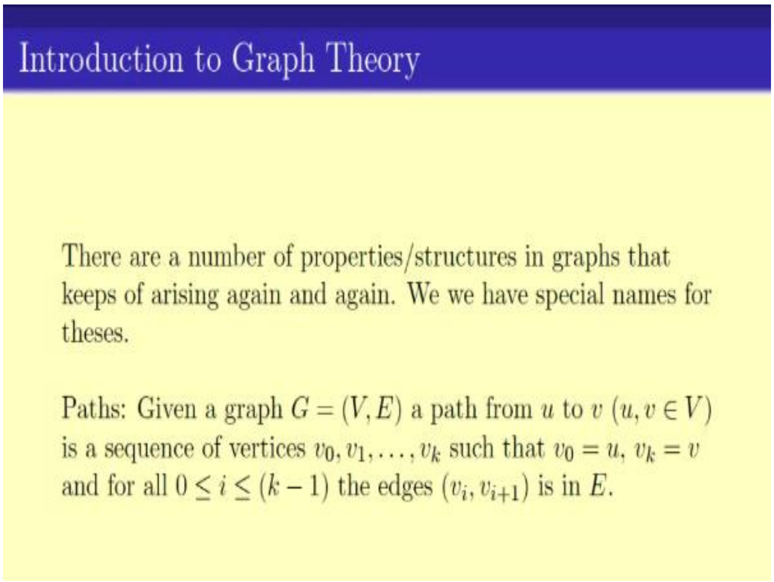

To some properties of graphs are used more often than not rather when we represent problems in graph, certain properties keeps coming up again and again. For them we have special names and this kind of start a new subject (()) (04:16), so one such thing is the path. So what is the path, a path is a set of sequence of vertices.

So the path from u to v, the sequence of vertices v0 v1 will vk by the first vertex v0 is u and the last vertex will be vk is v. And there is an edge between vi and vi plus 1, that means there is an edge between v0 and v1 from v1 to v2 and v2 to v3 and so on.

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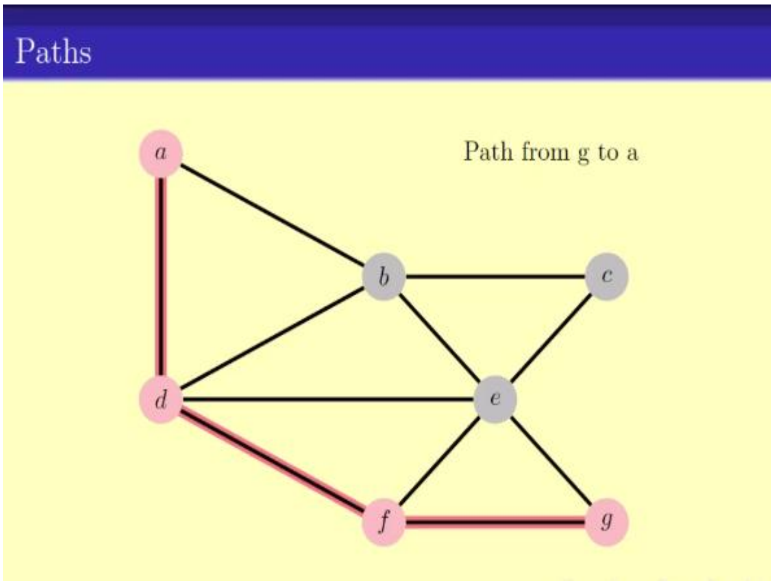

So for example, here if I want to draw a path between g to a, we can have this as a path, g, f, d, a, is the path.

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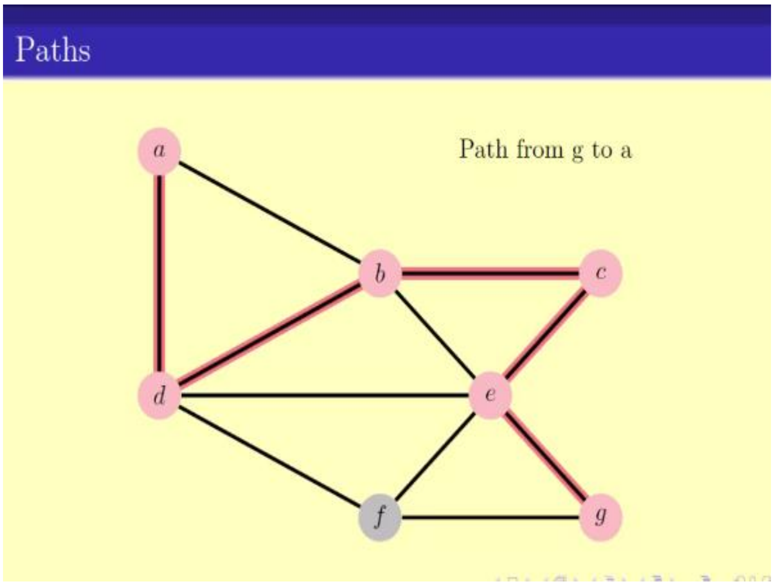

In a graph, one can have multiple path for example, where this is the path g, e, c, b, d, a, is the path. Now, we are looking at an undirected graphs so we do not care about the direction of the edges.

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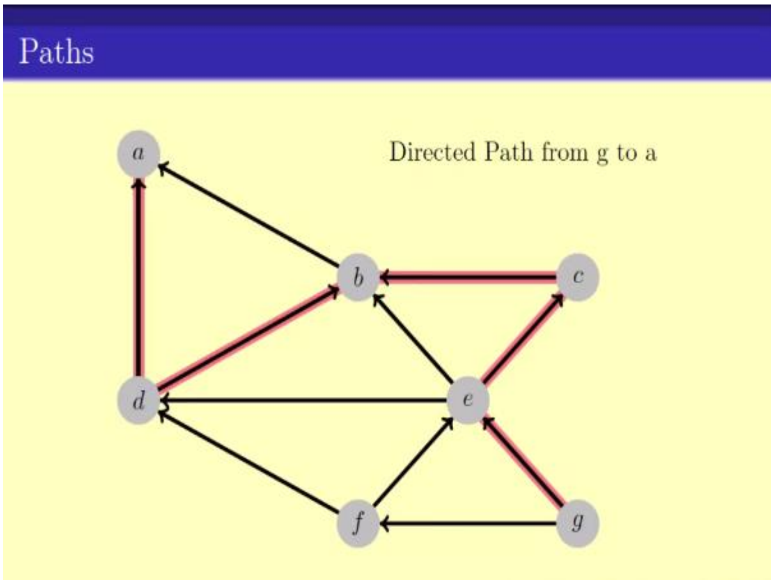

But if the graph is directed for example is this. In that case, as you can see this following the red path is not exactly a path because g, e is fine, we can have a path from g to e, a from e to c, c to b, but there is no edge from b to d. So this b to d is a problem. But there is a path from g to a in this graph, which is this one.

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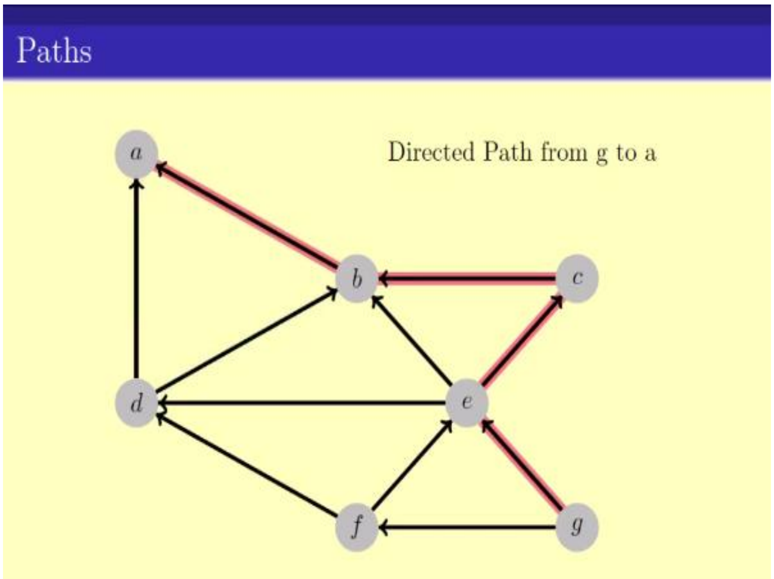

This is a path from g to a, g, e, c, b, a, right.

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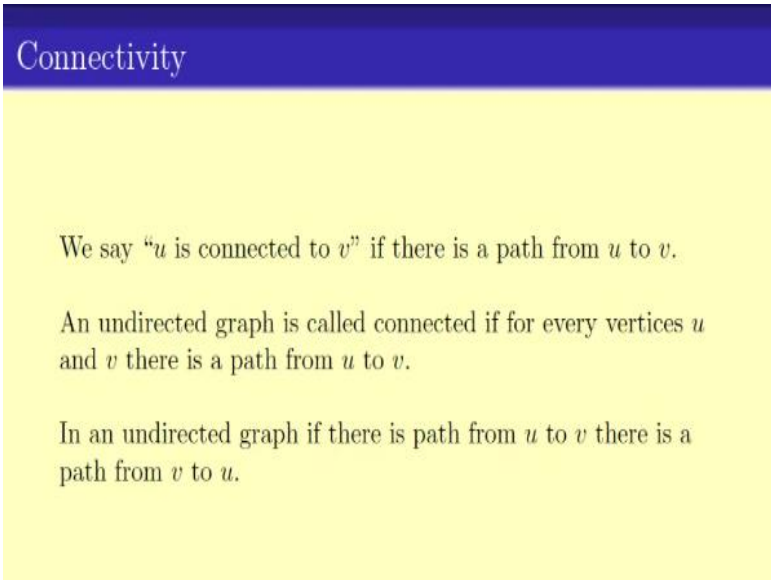

In an undirected graph, as in anywhere, in any graph, we say u is connected to v if there is a path from u to v. In and undirected graph, this concept of connectedness forms an equivalent relation and there is a path from u to v there is a part from v to u. An undirected graph is called connected, if every pairs of vertices in this graph is connected to each other by a path.

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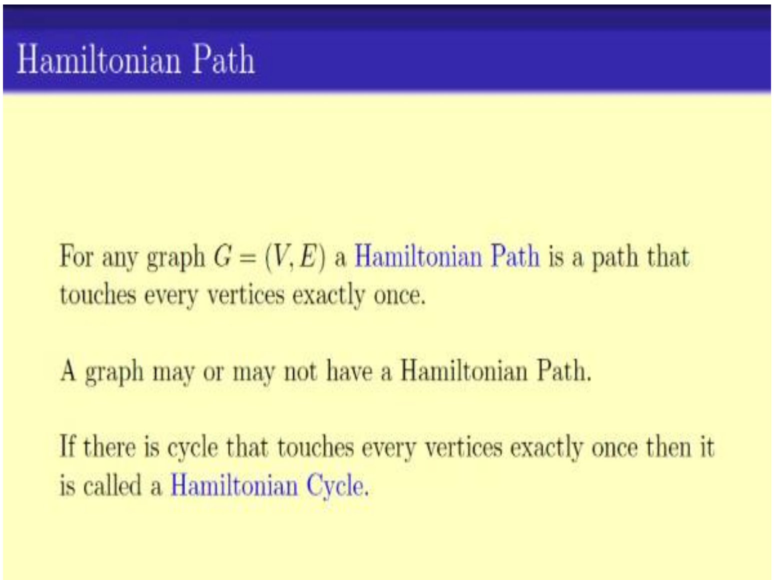

There are different kinds of paths that we also study, one of them which is important for us for this problem is the Hamiltonian path. Hamiltonian path, the path that touches every vertices exactly once. A graph may or may not have a Hamiltonian path and if I have a cycle that touches every vertex that we exactly want we call it as Hamiltonian cycle. So this Hamiltonian path or Hamiltonian cycle are of extreme important and has been studied quite thoroughly.

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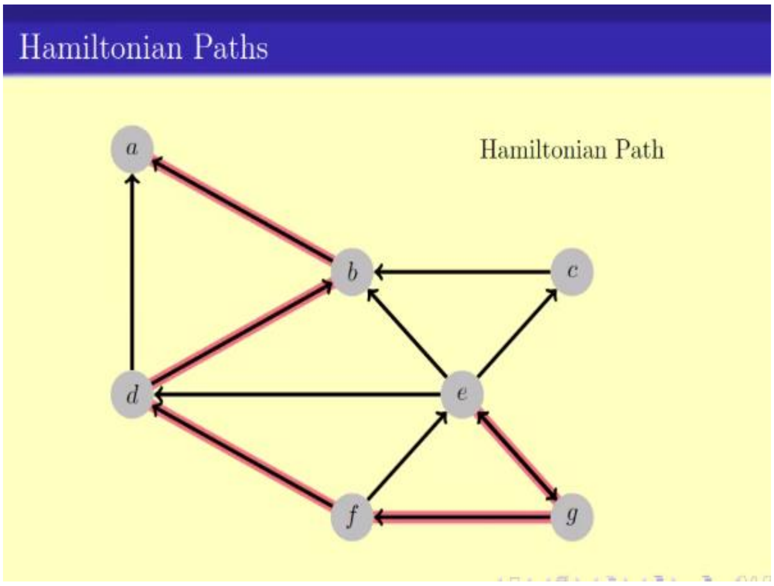

So let us quickly take a look at what does it look like. So in this problem, in a directed graph you can see that if this is the path from e to a, e, g, f, d, b, a, but it is not a Hamiltonian path because it does not touch every vertex. For example, it does not touch the vertex c. You can convince yourself by playing with this graph that this graph does not have a Hamiltonian path.

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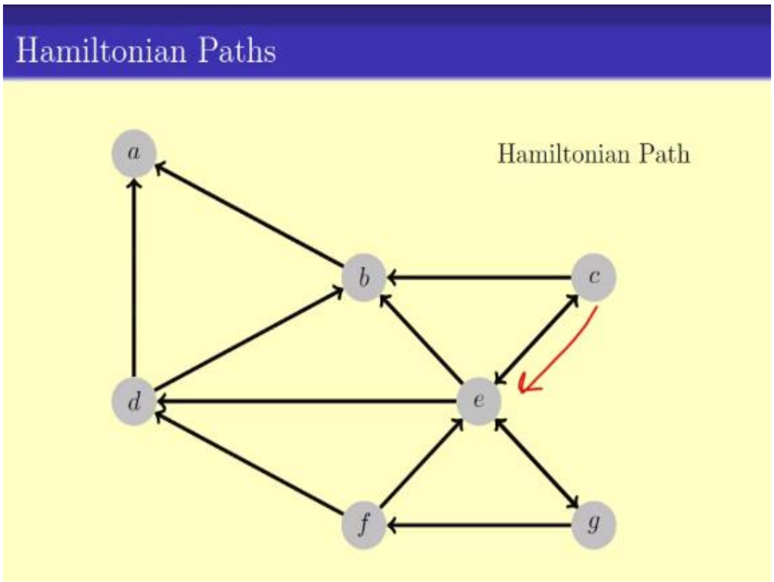

But if we grow at a from c to e, in that case, so if you have just now drawn this edge, this direction edge okay.

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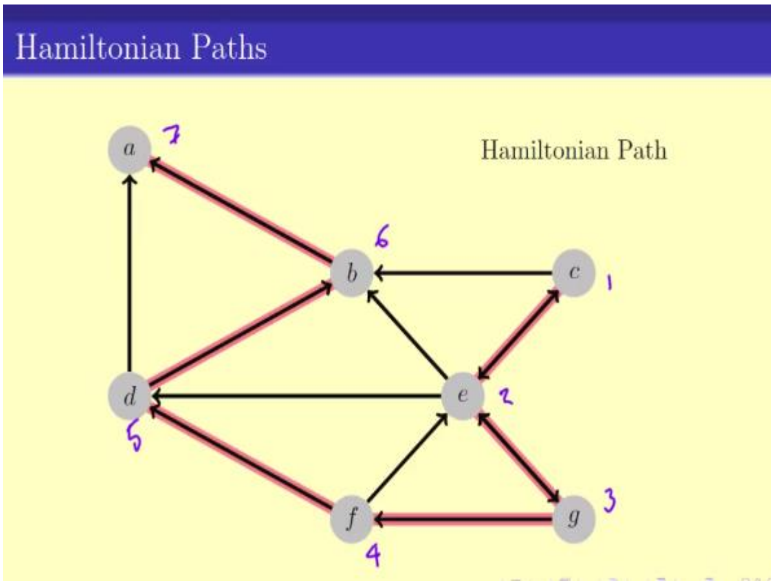

In that case, we can have a Hamiltonian path namely just once, c, e, g, f, d, b, a. Note that the original levelling of the vertices is just irrelevant. In the Hamiltonian path is just a path from some other vertex where every a vertex is appeared once. So if you are for example, we have marked this one at the first, second, third, fourth, fifth, sixth and seventh right. So the original levelling is not something of that importance once.

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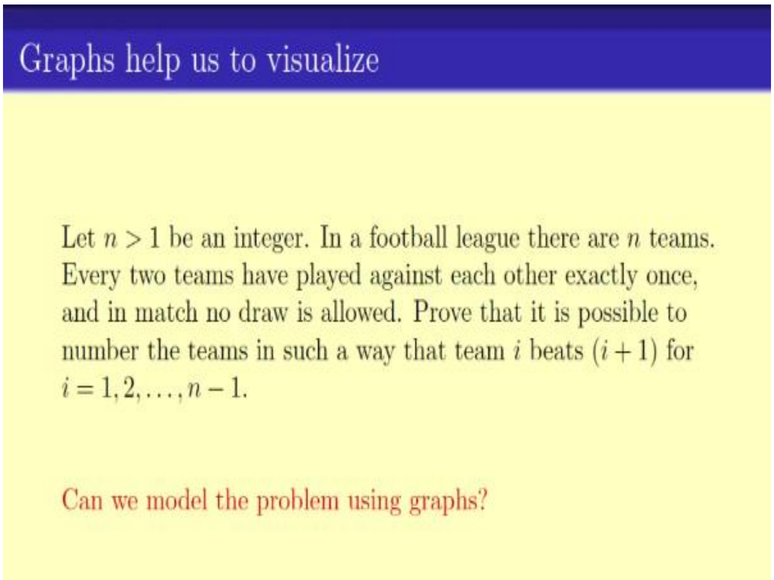

Now coming back to the problem that we have, we basically looked at this problem and converted this problem into a problem in graph theory, how.

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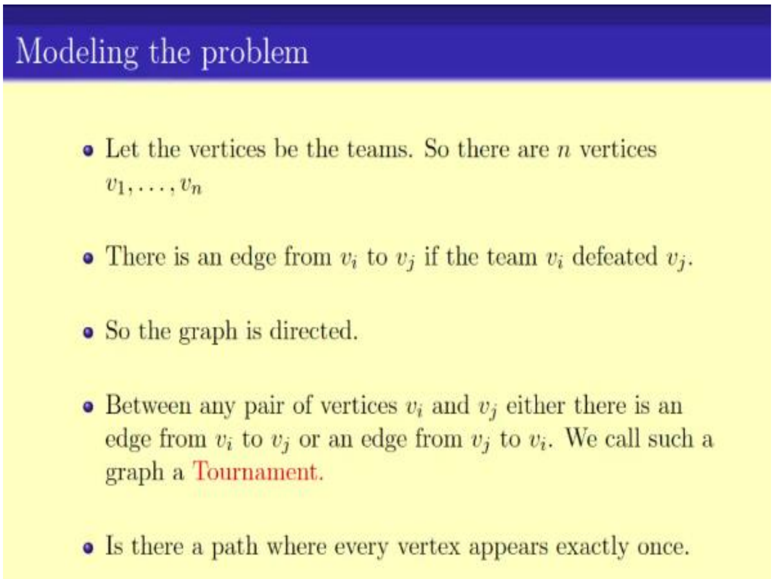

We consider the n vertices v1 to vn as the teams and edge drawn, view a edge from vi to vj if team vi defeated team vj. Now if you recall, if vi defeated vj then vj does not defeat vi, because they play only one game between them, which means that the graph is directed and the other thing is that between any two pairs of vertices since any two teams played at least one game and the game did not end in a draw, so there must be an edge one or the two directions.

So that means, either there is an edge from vi to vj or a edge from vj to vi and what we have to find, we have to kind of find a number defeat vertices in such a way that the first vertex defeats second vertex, second vertex defeats third vertex, third vertex defeats fourth vertex in the n of vertices. In other words, I want to have a path from one vertex and some other vertex where the path is directed and every vertex appears exactly once.

So the question is that is there a path from where every vertex appears exactly once. By the way, this kind of a graph where between any two vertices there is an edge and it is oriented in one of the two direction is called a tournament. And as you have seen already, a path that touches every vertex is the Hamiltonian path.

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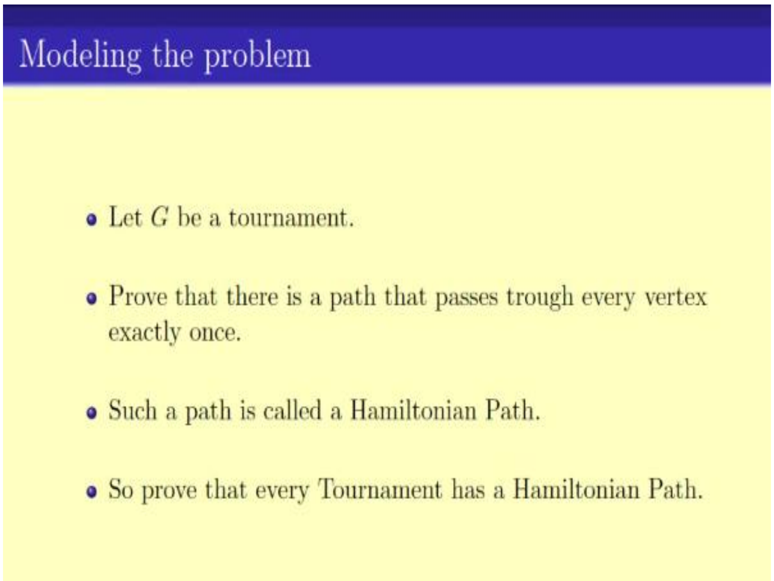

So in other words, what we are asking is that, if G is a tournament, prove that there is a Hamiltonian path right. So prove that every tournament has a Hamiltonian path. Recall that we just sometime we were told that not every graph have a Hamiltonian path. So this is the statement in graph theory (()) (12:03), which states that every tournament as a Hamiltonian path.

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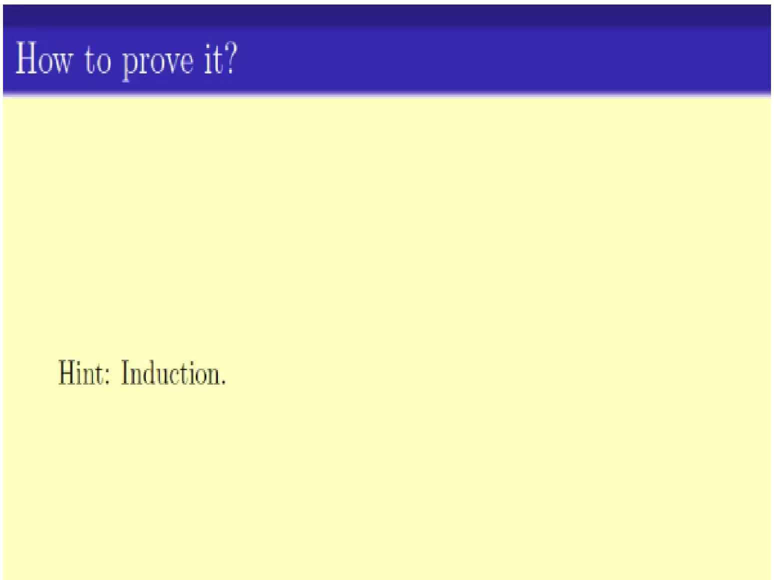

Now how do you prove it, of course the hint in induction.

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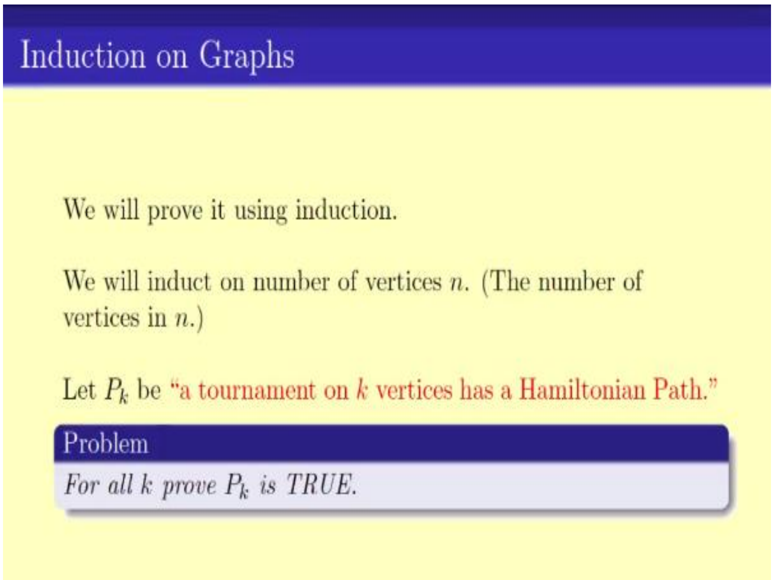

And how we go about it, so you prove it by induction. Now whenever we have a graph theory problem and we have a induction in her hand, we can induct on a number of things, we can induct on number of vertices, you can induct a number of edges, we can induct a number of cycles and so on and so far. In fact, we will show you a quite number of proofs using various kind of inductions.

But in this case, we will be inducting on the number of vertices. So here we have to split up into small parts as well do in any induction case and we have select pk be the statement that tournament on k vertices has a Hamiltonian path. And we have to prove that for all k let pk be true.

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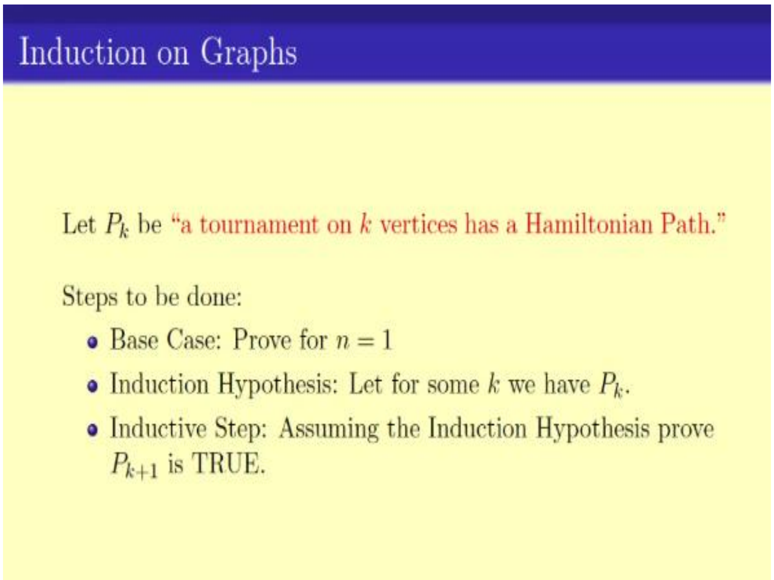

So of course, if this is the pk, that the tournament on k vertices has a Hamiltonian path. We have to do three steps. First of all, base case, induction hypothesis which where we say that okay let it be true for some case and then assuming that we know that it is true for some k, prove this induction, case for k equal to pk plus 1 right.

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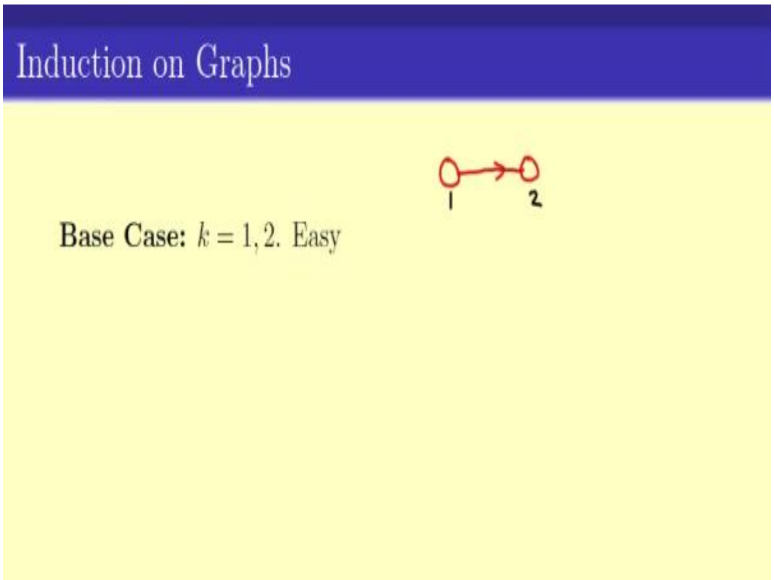

Now the base case, so k equals to one and two is just for simply sitting as second both the cases. Now what does the case, so k equals to one, I have only one vertex. It does not make any sense. K equals to two, I have two vertex and of course there is a edge from one to the other and hence I have a Hamiltonian path right. So if again it is a separate easy thing. So the base case here in both the cases are pretty easy.

So this is a Hamiltonian path, so I just have to number this one as one and number this one two and we get this one.

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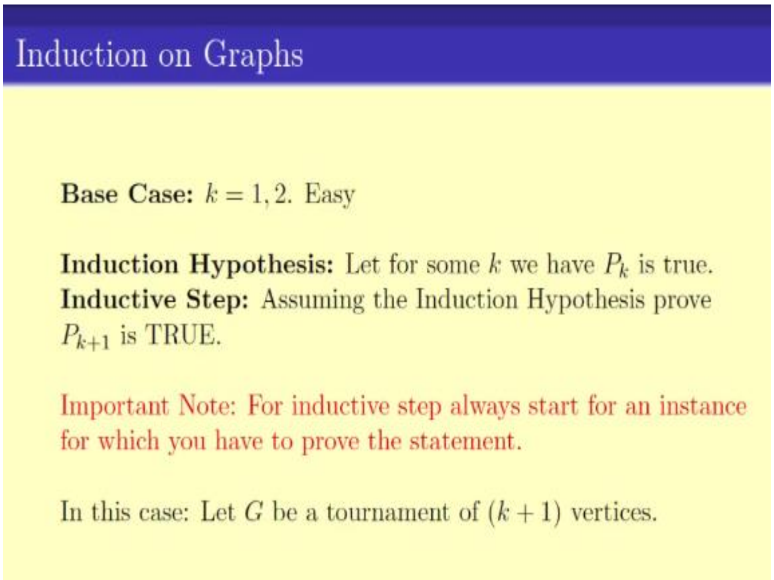

Now the induction hypothesis says that for some k, we have pk is true and you will be using that we have to prove that it is true for pk plus 1. Now whenever we do graph induction on graph, this is something that very crucial that we should always start from an instant for which we have to prove a statement. So here we have to prove the statement for pk plus 1 which means that you start with the tournament on k plus 1 vertices.

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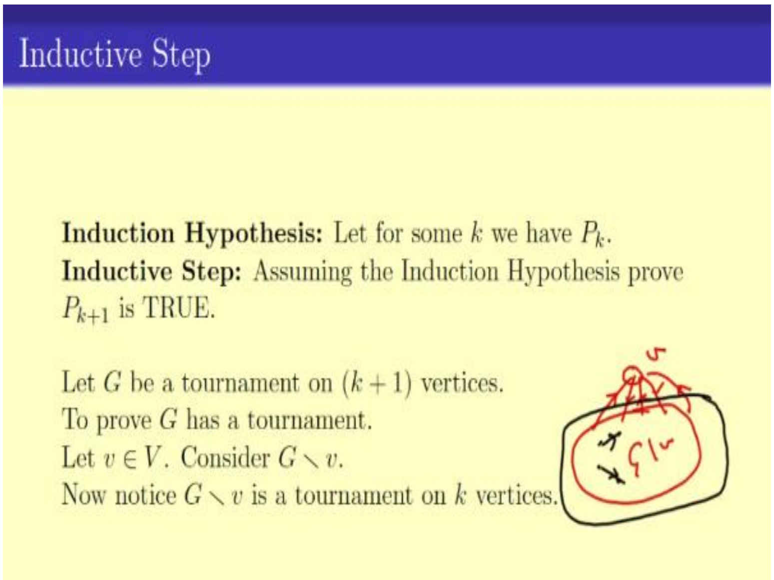

So you start with the tournament of pk plus 1 vertices. If we have to use the induction hypothesis, we have to reduce the problem into a smaller case, so how do we do it. If you remember in the last problem that we did, we do all this hand shake problem, we picked up a pair of vertex of a particular kind and then inducted on that. So here if you have to prove that G is a tournament, we start with any vertex V.

Now how does it look like, so here with a graph G and I picked up a vertex out of it. So this is V and this is G minus V right and there are edges from V to all the vertices in G minus V and they are oriented in some direction, if we do not know. Now one thing to notice that, this graph G minus V, if in fact, a tournament on k vertices. Original graph or G or the tournament on k plus 1 vertices,

I remove one vertex so the number of vertices in G minus V is k and why is G minus V a tournament because any edge between any two vertex in G minus V there must be an edge and is direct in one of the (()) (16:44) that is the definition of tournament right. And since G was the tournament originally so this must also be a tournament between a to vertex there must be an edge. So I can apply induction hypothesis on this problem right.

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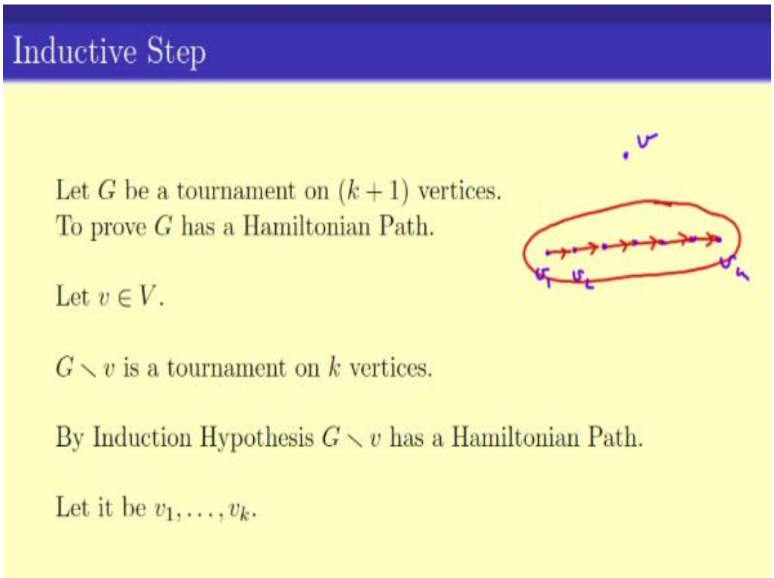

So assume that, so we start with the V, we look at G minus V that the tournament on k vertices and the induction hypothesis, it has a path right. Now what does it mean by that so let us try to draw it again once again very carefully. So G has a Hamiltonian path so that means G must have some set of vertices, a Hamiltonian path, say this is the Hamiltonian path. Well we drawn the vertices in blue.

So we have somehow renamed the whole set of vertices and we have got things like a is the one is V1, second one is V2, till the last one is Vk and I got a Hamiltonian path. That means there is a age from V1 to V2, V2 to V3 and so on till Vk. And I have the V sitting over here, this is the V. Question is that how is the V connected to rest of them. So let V1 to Vk be the Hamiltonian path.

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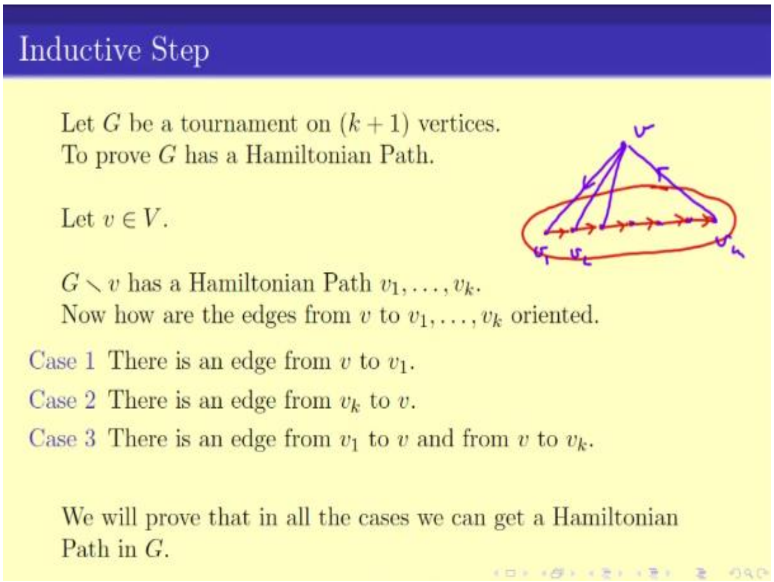

So we have this G minus V is a Hamiltonian path and this has the path V1 to Vk. Now how are the edges in V1 to Vk oriented. So look at the edge from V1 to V, V2 to V, V3 to V and by going so till the end. Now we have split the whole cases into three cases. The first case, there is an edge from V to V1 something like this. Second case, is there is a age from Vk to V and in the third case, neither of them happened that means there is an edge in opposite direction V1 to V and V2 to Vk.

Now in each of these three cases, I will show that I can extend this Hamiltonian path V1 to Vk to a Hamiltonian path on G (()) (19:49) one V1 to Vk and V. Although note that this three cases are covers all the cases okay because this is one case V2 to V1, Vk to V and the third case is neither of the above two happens. For any of the cases we will show that we can extend it. So we will prove that in all the cases we can get a Hamiltonian path on G.

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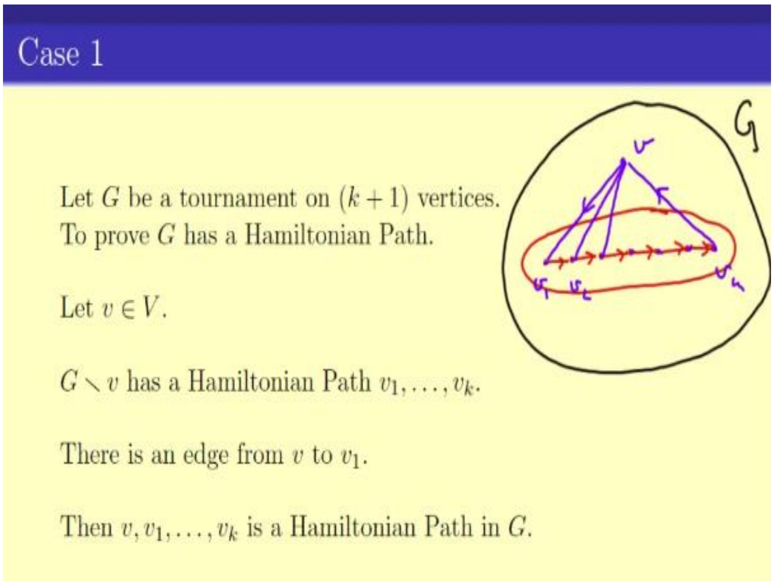

Now let us start with the first case. In the first case there is an edge from V2 to V1, if that is the case can you see a Hamiltonian path here. Hamiltonian path is a path on k plus 1 vertices. And yes there is the Hamiltonian path. V, V1, V2, V3 till Vk. There is a edge from V2 to V1 great. So in the case one, there is a Hamiltonian path on the whole graphs. Note that the whole graph, this whole graph is the G graph right.

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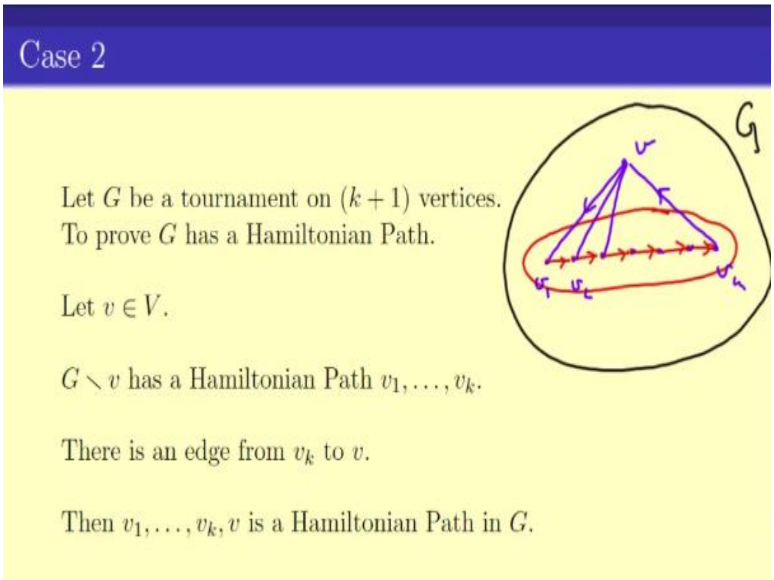

Now let us look at the case two. Case two is that there is a edge from Vk to V. If there is an edge from Vk to V, can you see V Hamiltonian path. Again yes, I have V1, V2, V3 till Vk and then to V. So V1 to Vk, V is a Hamiltonian path in G great. Now we come to the third case.

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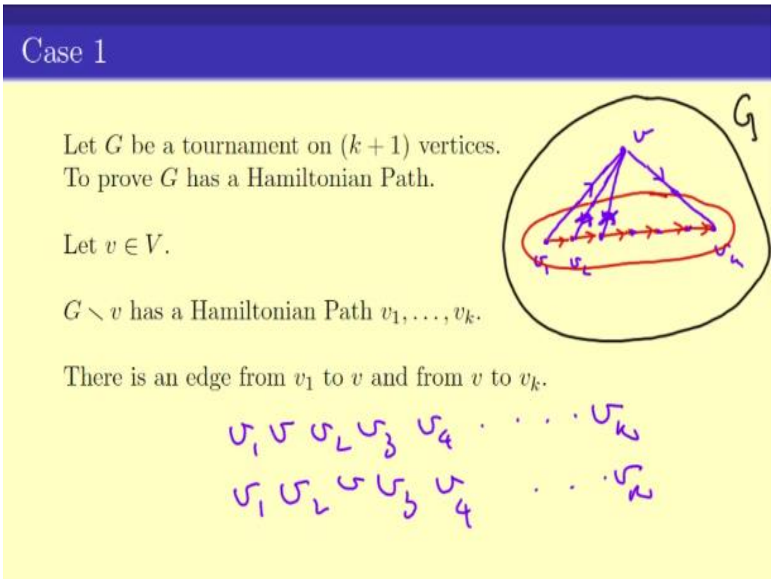

Third case is neither of this two happens right. So there is an edge from V1 to V, we will draw them, there is an edge from V1 to V and the edge from V2 to Vk. Now one this is the case unfortunately, we cannot obviously extend it a Hamiltonian path. There is no Hamiltonian path can be seen here in this picture right now. But what happened if there is a edge from V2 to V2 like this, in that case I will then plot an Hamiltonian path which is V1, V, V2, V3, V4, V5, so I first start with V1, V, V2, then V3, then V4 and so on.

So V1, V, V2, V3, V4 till Vk would have been a Hamiltonian path. But again then there is no guarantee of that also right. What is the guarantee that the edge between V2 goes in this direction, downward direction, it might be that edge is actually going upward. In that the (()) (23:16) if I have a edge going from V2 to V3, I would again get a path which is in the case V1, V2 then V then V3, then V4 till Vk okay.

But problem is that even then there is no guarantee that I would have got the, I could have this path from V3 to V going the opposite direction like this. Now question is that, can I always have path going in the upward direction, answer is no, because all the parts are going in the opposite direction but at the end V2 to V1 that V2 to Vk goes in the downward direction.

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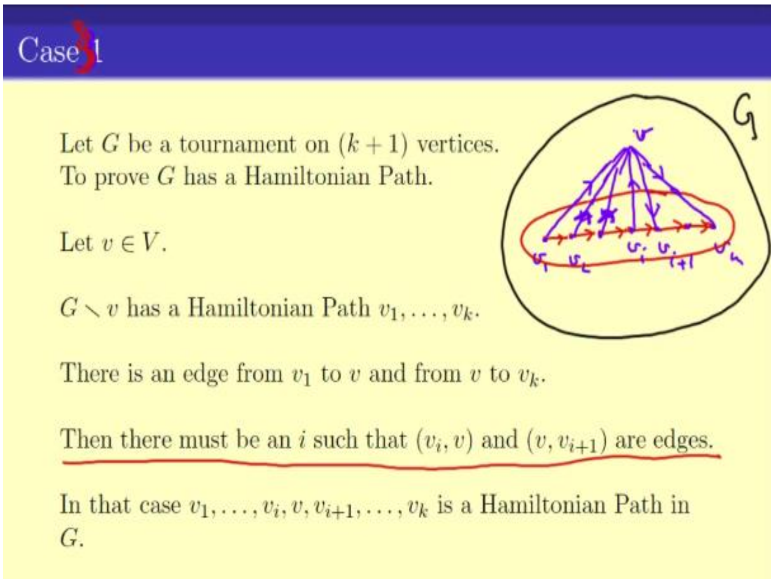

So there must be an i where I have a from Vi I will have a edge going up and the Vi plus 1 as a edge going down. Vi to V, this is the i and this is the i plus 1. Vi to V and V2 to the i plus 1 and in that case we will have a Hamiltonian path which is of course V1 to Vi then Vi from there will go up to V then V2 Vi plus 1 and then completed to Vk.

So thus in this case also we will get the solution. Of course this is something I made a mistake, this should be three, this is the case number three. So in a crucial part, here was understanding this statement that there exist an i such that Vi to V and V to Vi plus 1 edges. So we have proved that for all the three cases, we can state the Hamiltonian path on G minus V to the Hamiltonian path of G.

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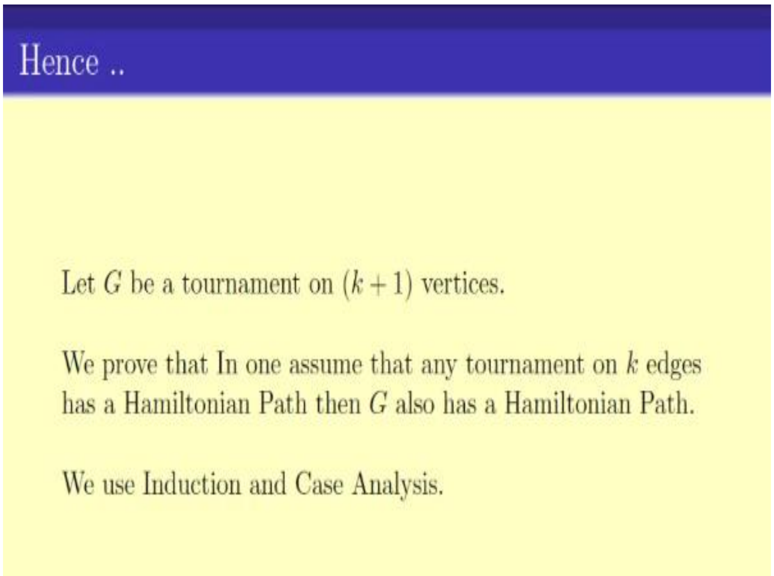

And hence by induction hypothesis, we have the proof that any Hamiltonian, any tournament has a Hamiltonian path.

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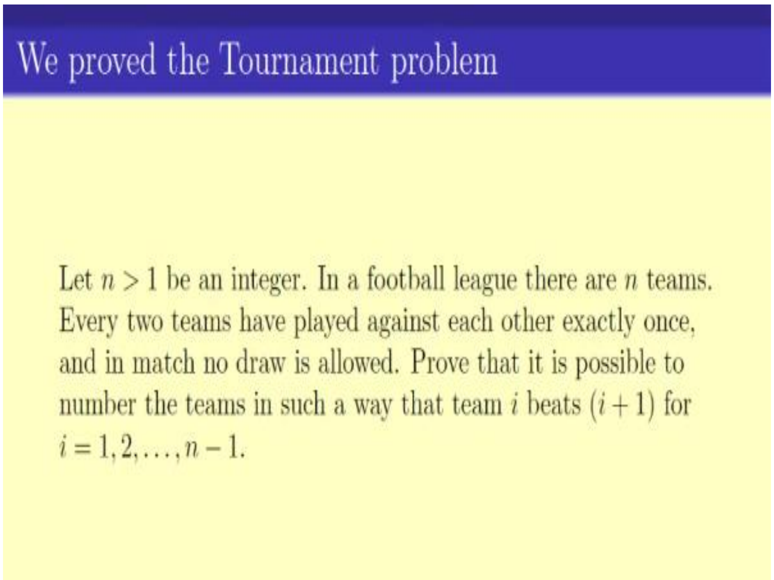

In the next video lecture, we will be, so we have proved this particular big tournament problem in this video lecture.

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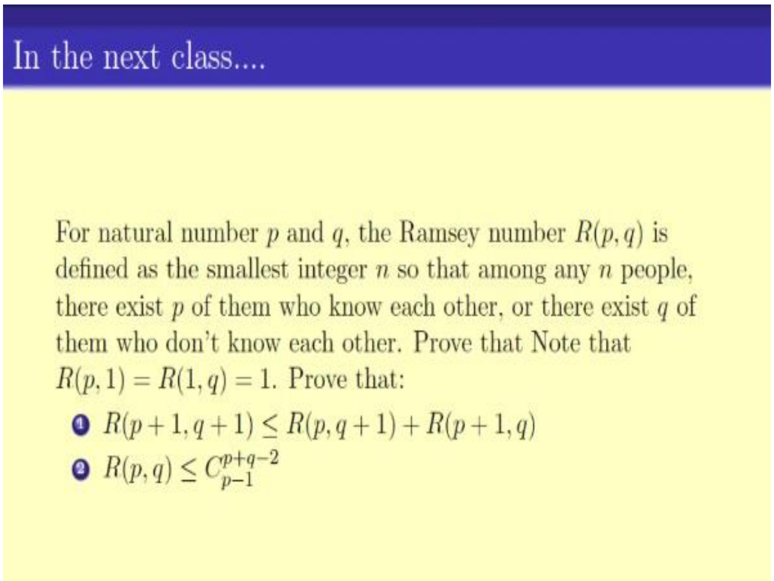

The next video lecture, we will be looking at the Ramsey number problem and a new understanding on graph theory, new concept of graph theory will be useful. Thank you.