Discrete Mathematics
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Lecture - 24
Tournament Problem

Welcome back. So, we have been looking problems and how to solve them by using graph theory and induction and various other proof techniques that we have learned so far.

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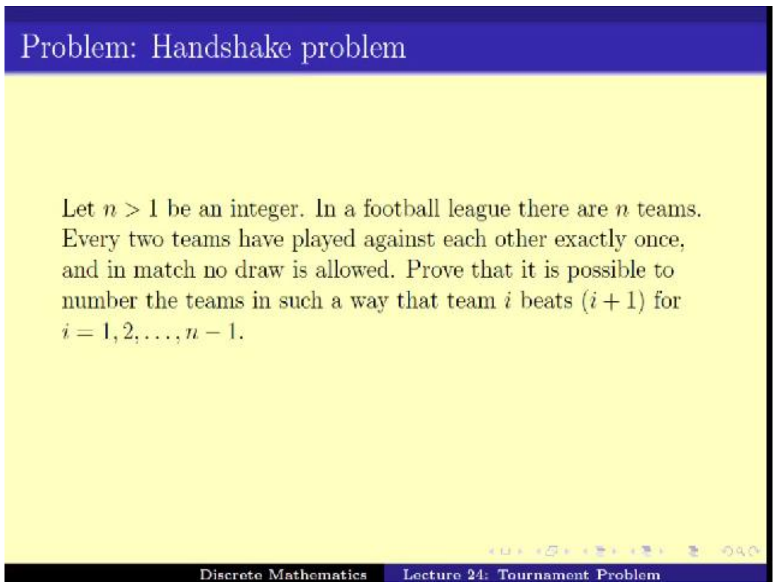

So, in this video lecture, we will be looking at this problem that we had discussed couple of videos ago. So, it says that in a football team there are n teams. Every two team has played with each other exactly once and there is no match indefinite draw. So, either of one of the team must have won. Prove that it is possible to number fifteen in such a way that team i with i + 1 for i equal to 1 to n minus 1.

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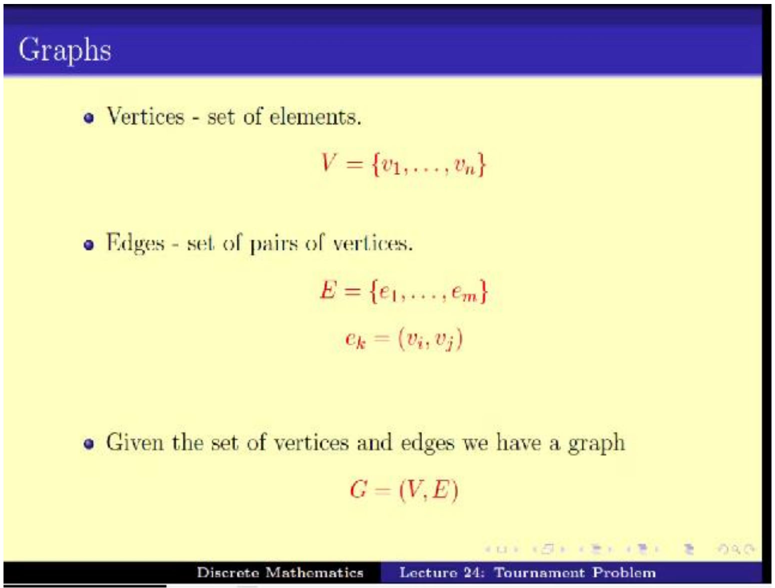

Now to solve this problem we would first like to use graph theory to model this problem and that would help us to solve it, visualize the problem. So, quick recap of graph theory. So, we have vertices and we have edges which are pairs of vertices and this constitutes of graph. The graph is give as the set of vertices and set of edges. Now, there are some basic definitions that are there and they are.

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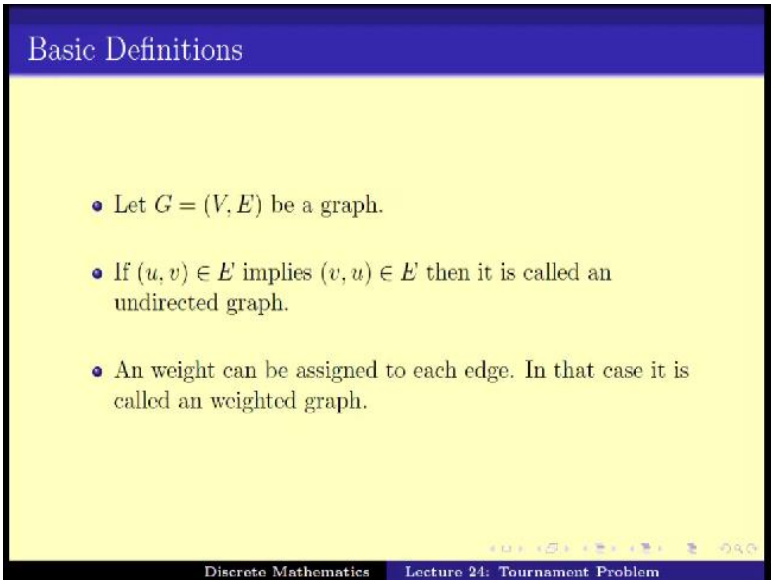

So, if u and v is in the edge implies v as u in the edge in other words that relationship is reflected we call it an undirected graph. Sometimes weights are assigned to the edges and in that case is a weighted graph.

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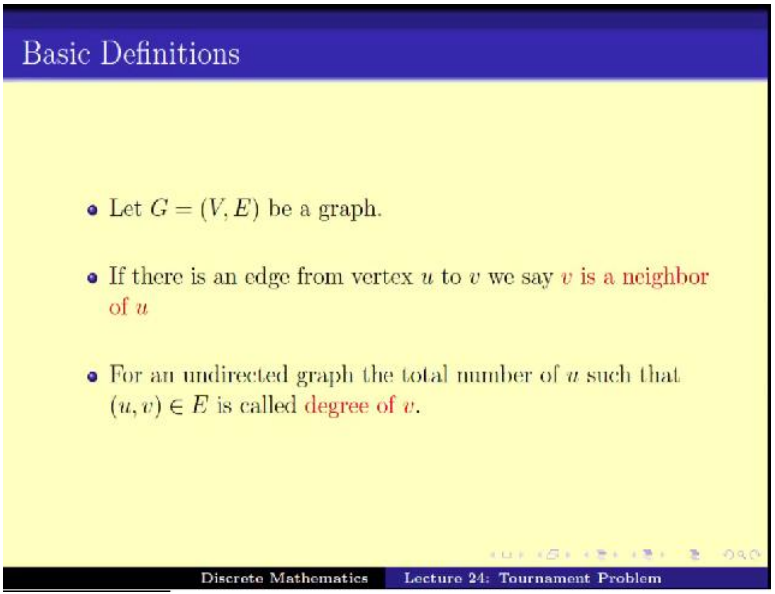

If there is edge between u to v we say v is neighbor of u and in an undirected graph the degree of v is the total number of edges that is going out of u now v.

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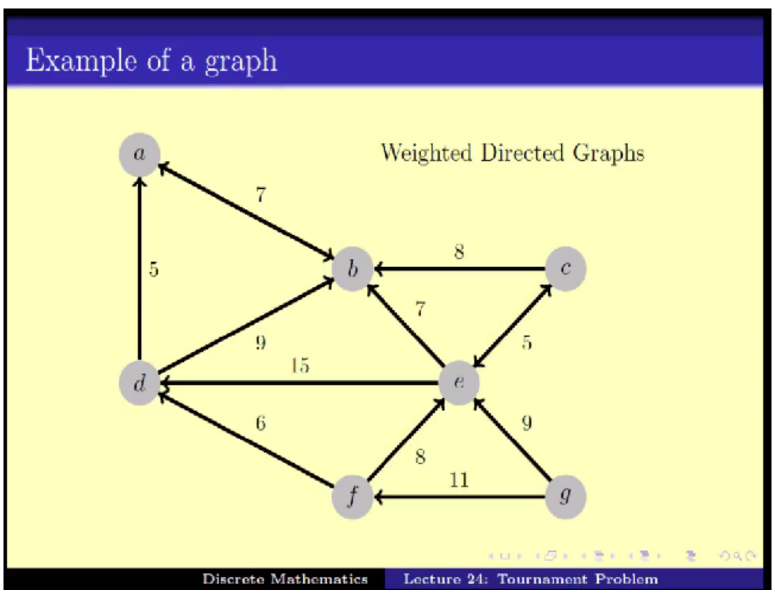

So pictorially we represent the graph as it is blocks so here the vertices are a, b, c, d, e, f and g and we can denote the edges or the relations by lines drawn from one block to the other So here for example this gives us the graph the edges are drawn between a and b, a and p, b and p and so on. There is no edge between a and c which means that a and c are not related. It is binary relation. Now there can be edges away from the edges and in that case, we call it weighted undirected graph and sometimes edges need not be undirected in the sense that d to a, and a to d might be different.

In that case, we get directed graph and if you notice using this arrow. So the arrow says that there is a graph, there is edge from d to a. This arrow is the arrow in both sides it means that there is edge from a to b and that edge from b to a. So, this helps us to represent graph or binary relations which are not reflexive.

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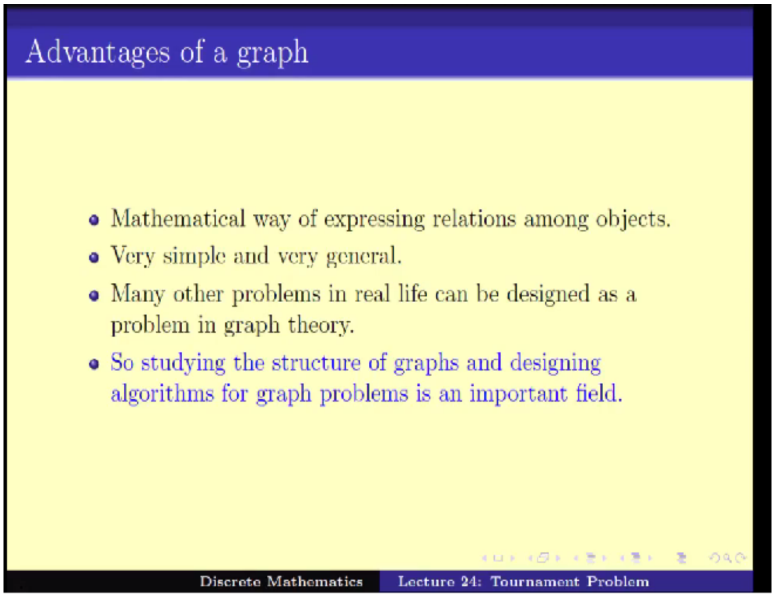

Now, there are various advantage of using a graph namely the graph are very simple objects and that they are very simple and very general. So, many problems in real life can be designed as problems in graph theory. So, starting the structure of graph and designing the algorithm for graph is an important field and hence studying graph in general with various properties can be a way of coming up with a uniform way of assessing various set of problems.

In this video, we will be looking at some more properties of graph or structure of graph that we will be studying.

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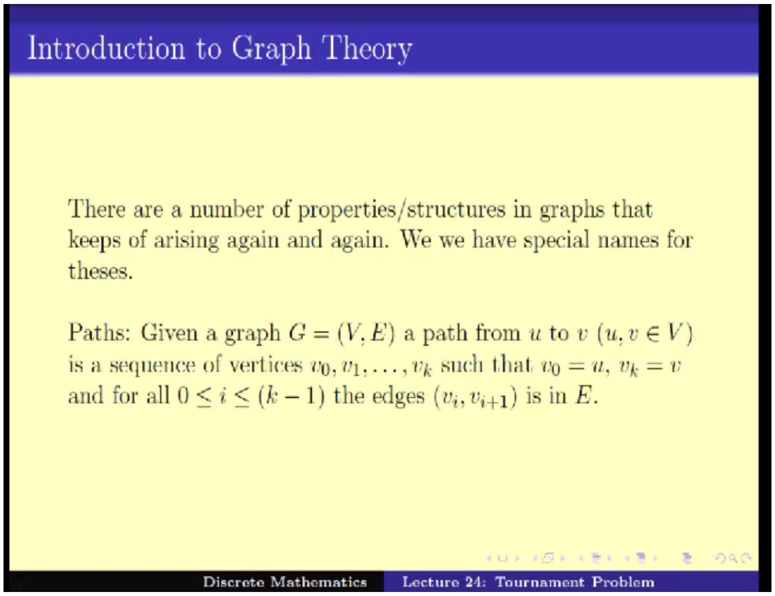

So, as an introduction of graph theory there are various properties and structure that keep arising again and again. We have special names for this kind of structure or property and you would like to study them. So here is one of them it is called the path. So, given a graph a path from u to v is a sequence of vertices v o to v k, k can be any number such that first one is u the last one is v and for all edges the edge demark edge from v i to v i +1 or v i from the v i + 1 is an edge should be edge second. So this is called a path.

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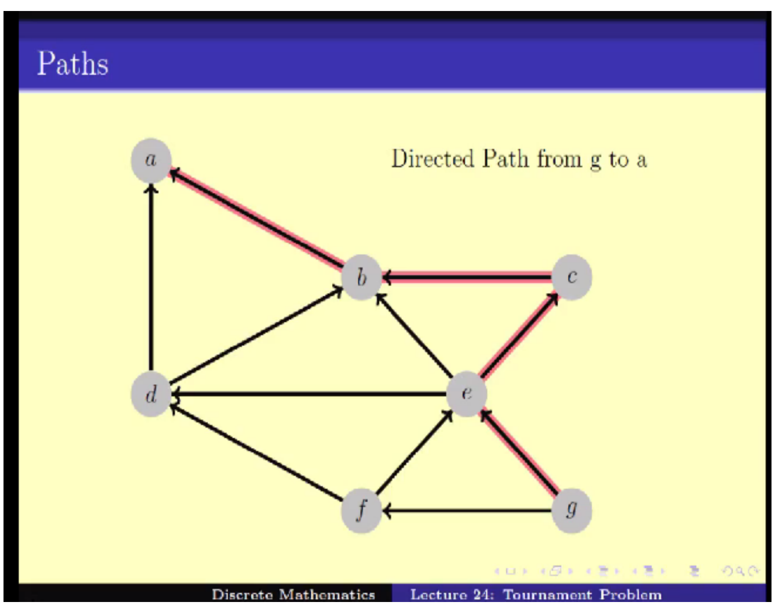

So, let's quickly see some example so if this is a graph and I want to see whether there is a path from g to a. So, this is a path there is path in the sense that I can go from g to f, from f to d and from d to a. So, in this case the path that we ride is g, f, d, a. So, this is v 0. This is v 1. This is v 2 and this is v 3. As you can see v 0 equal to g, v 3 is a and there is an edge between v 0 and v 1. There is an edge between v 1 and v 2 which is an f and d and there is an edge between v 2 and v 3 which is d x a.

There can be many paths of g to a. For example, this is a path, where as I can also have this path g to e, e to b and b to a is also a path from g to a. So, g e b a is a path from g to a. Similarly, this is another path from g to a, somewhat more coagulated path g e c b d a. Now, if graph is a under a is directed for example if this is a direct that I have denoted by the arrows then this is not a path from g to a. Why?

Because this a d to b is going to the opposite direction. If I have to go from g to a, then I have to go from g to e, e to c, c to b but I cannot go from b to d with edge is going in other direction. So, one good way of thinking about graph as if these are city and they are road connecting the city and there are roads on one way sometimes or maybe you can think of flight. There are flights from g to e. There is a flight from d to b but there is no flight from b to d.

But when we are asking from path from g to a it is basically saying can I go from if city is g to city a, by using a sequence of flights. But this is not a valid thing valid path from g to a. But I would like you to fly from g to e. I will be able to fly from e to c. I would limit to fly from c to b. But I cannot fly from b to d. This edge is in the opposite direction but we can come up with another path from g to a namely this one. And then we get a path from g to a.

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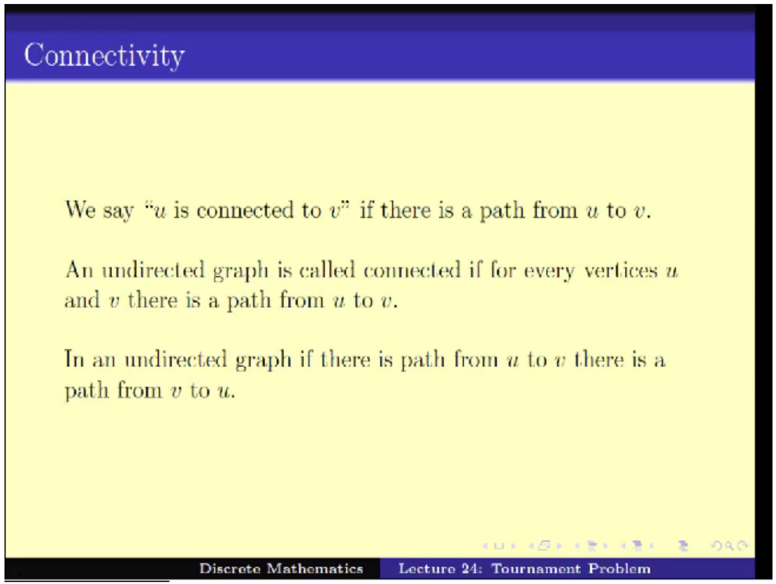

Now, this also helps us defining the next big concept which is called connectivity, we say u is connected to v if there is a path from u to v if the graph is directed it means the directed from u to v. In an undirected graph since there is no one-way direction you can see for yourself that if there is path from u to v then there is path from v to u and if a graph any two vertices can be reached from one to the another then we call that graph a connected graph.

So, in another words the graph connected if I can go from any vertex to any other vertex may be through some direct complicated (()) (10:27). If something like you have a set of cities in India and there are flight networks that goes across the cities. Is it possible to go from any city in India to any other city in India using flights? And if it is so then we call it a connected network or a connected graph.

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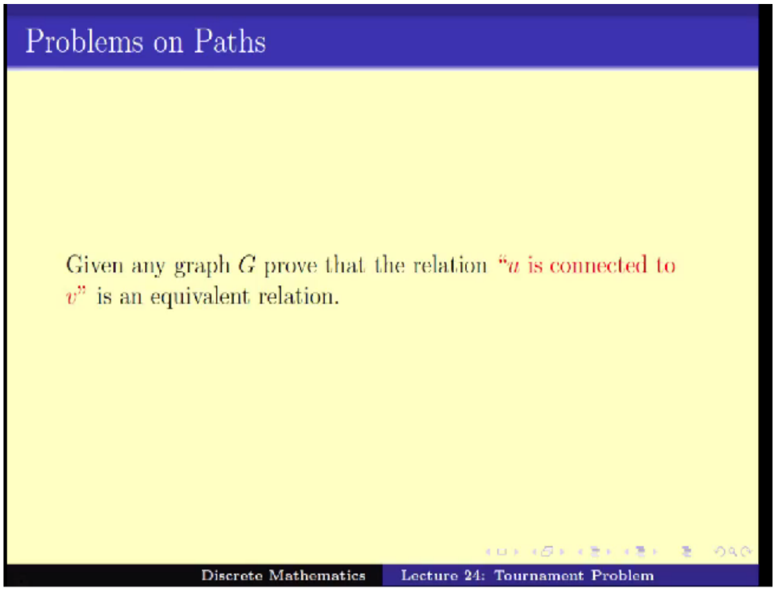

So, one problem that I would like to assign you is that prove that the relation u is connected to v is an equivalent relation. If we have to look at equivalent relation, means our first week and it is relation that means u is related to v if u is connected to v. They prove that it is equivalent relation.

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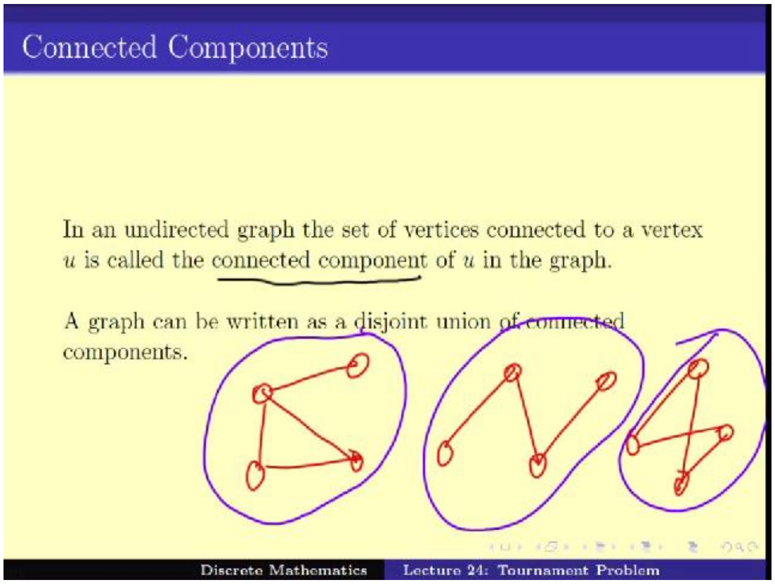

So, moving on once you have equivalent relation it is not hard to prove from the definition or from the properties of equivalent relation that the whole graph will split into equivalent classes we call them the connective components. So, connected components of a graph is the component where any two vertices are connected. The graph can be written as a disjoint union of connected components. For example, if I have a graph like this note that the connected component if this is the whole graph and the connect components are this is one connect component.

This is the other connect component and this is the other connected component. There are three connected components here.

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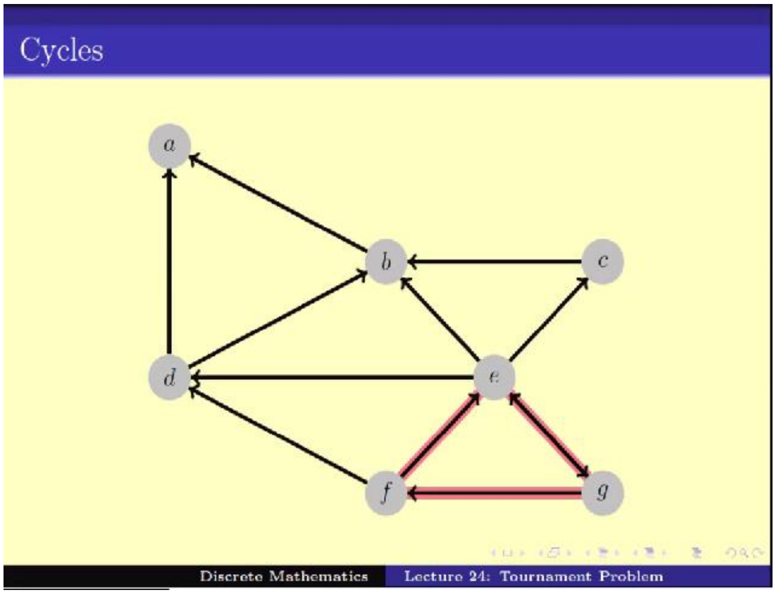

Now sometimes we can have paths that end up in itself so then we call it as cycle. For example, this is a cycle. I can start from b I can go to a, d I go to f. I go to c, go to e and come back to d. So, cycle is a collection of edges that bring me back to my starting vertex. So, that is called a cycle. In an undirected graph again, we do not have to vary all direction of the edges but when the graph are directed then we have to worry about it. So this is another cycle that is there a b, d e but once we put the edges there we see that this is not exactly a directed cycle.

Why? Because I can start with e go to b if this is fine, b to a but I cannot go from a to d, neither can I go in the opposite direction. For example, I cannot start e and go into d first. So, irrespective of which way I go, I cannot come back to my place following the set of edges. So, this is not a cycle but this one is a cycle, direct cycle. Why? I can go from g to f, f to e and e to g because a, d to g exists. So this is a directed cycle.

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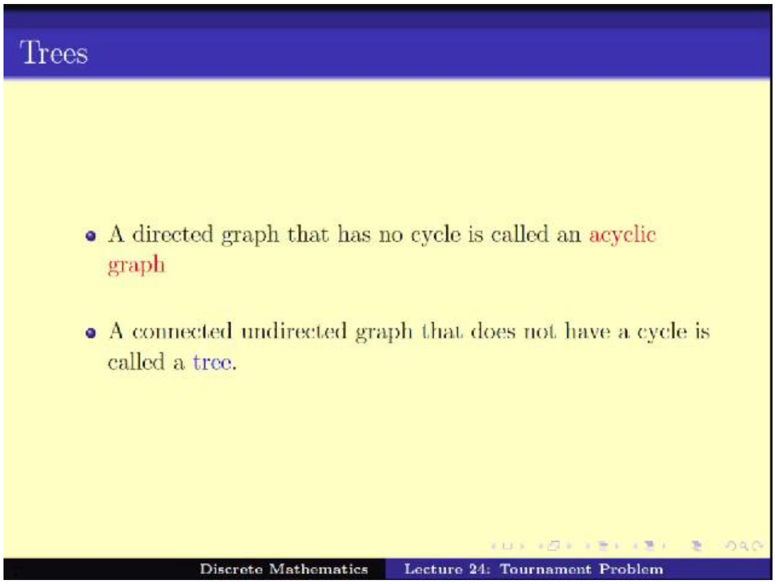

So, if a directed graph does not have a cycle because it has acyclic graph and in a connected under acyclic graph that has no cycle we call it a tree. This is a very important concept we will come back to this concept again and again in this next few videos. Trees are very important objects they are undirected acyclic graph or undirected graph that do not have a cycle.

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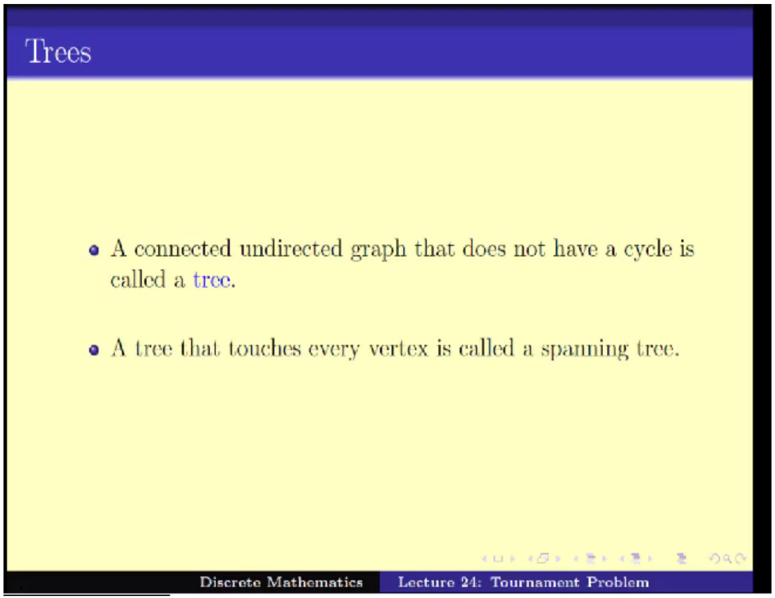

For example, here in this graph this is a tree, set of red edges forms a tree. Note, that by looking at the red edges, red edges were only part of the graph if there was not any cycle. So, red edges do not form a cycle at all. So, this is a tree. We can have many trees. This is another tree. So, I mean that the red edges are the tree edges. So, connected under the graph that does not have a cycle is called a tree. A tree that touches every vortex is called a spanning tree just like in this case.

If the original graph is this whole graph and it will red edges forms a tree note that it touches every vertex in this graph. So, this is a spanning tree. Now, there are few problems that I would like you guys think about we will come back next week solve many of the problems.

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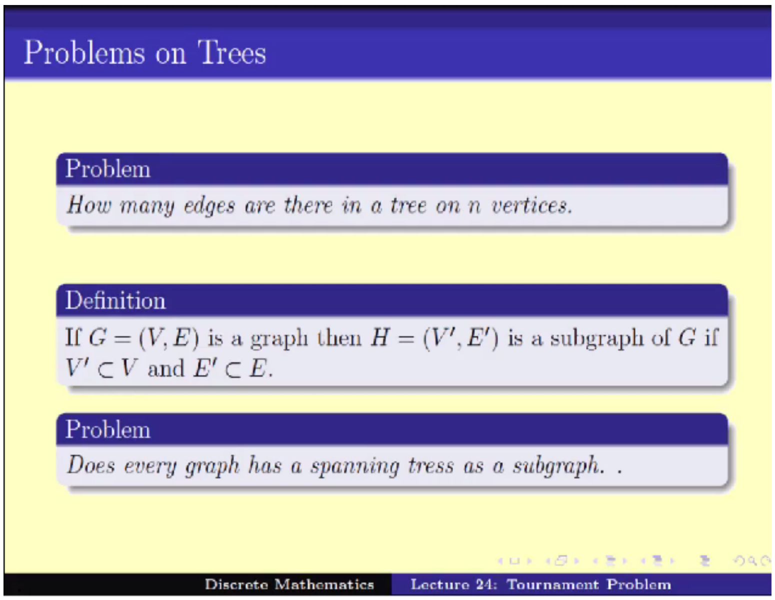

First of all, how many edges are there in a tree on n vertices? Can you give me a number and secondly just like you have been talking about this spanning tree or the example if g is a graph and H is an another graph we call H is a subgraph of G if the vertex set is the subset of the original vertex set and edges set and H is a subset of H set in G. So, in another words if I give you a G and if I remove some of the edges till I get a subgraph of G.

A question is that does every graph has a spanning tree as a subgraph. So, this kind quick introduction to graph theory.

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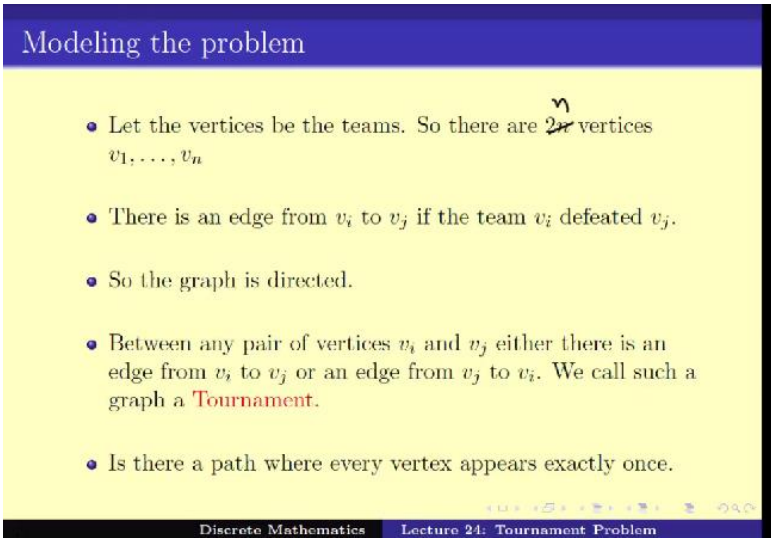

And now using this set of graphs that we have learned. A various graph that you have seen can we at least formalize this statement of this problem? So, here again there are n teams. Any two teams play with each other one of them win other one loses, prove that it is possible to number the teams in such a way that the team i repeats team i + 1. So, to formalize as a graph theory problem you have to first set up the graph. So, of course let there be n teams.

So, this is not n, this will be just not 2 n but just n. There are n teams so let the n vertices be v 1 to v n and we draw edge from u v i to v j if v i defeated the team v i defeated team v j. If v i defeated v j it does not implies j defeated v i. So in fact the graph is directed. So, here it is not an undirected graph but we have a directed graph and more over with the any two pair since they have played with each other v i and v j there must be an edge and there must be exactly one edge meaning there is edge either to v i and v j or edge from v j to v i not both.

Such a graph is called a Tournament. A tournament is a graph where between any two vertices there is an edge but edge is oriented or in another words the edge is either in one direction or the other not (()) (19:26). Question is that, is there a path where every vertex appears exactly once. Now, why is it the right question?

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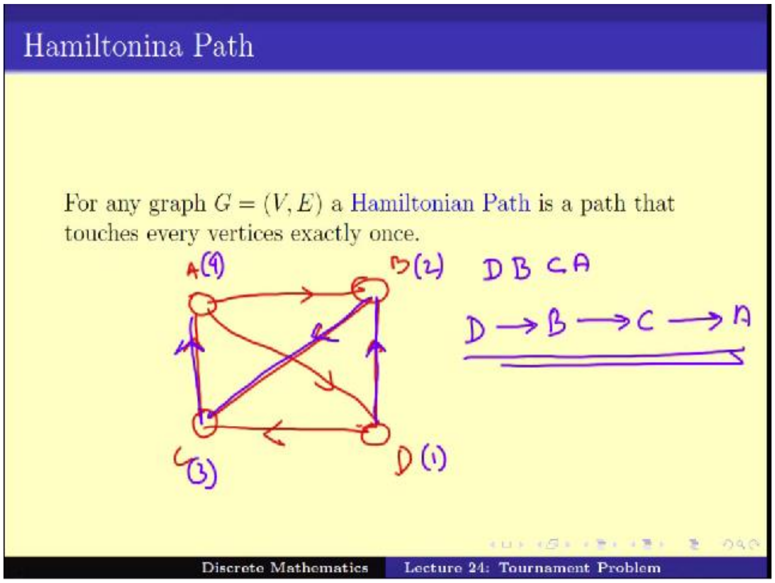

Before, I could assign the value to be in path. Let me quickly try to understand is an example of it. So, say I have this team. So, here are four teams A B C D and say A has defeated B, C has defeated A, B has defeated C, D has defeated C. But D has been defeated by A and B has been defeated by D. So, this is arbitrary graph of a tournament. Now, what do I mean by, what do I have to do? I have to order them in such a way that the i team defeats the i + 1 team.

So, in this particular graph think of this way that if I numbered D to be equals to 1, B to be number 2, c to be number 3 and A to number 4. What do I have? I get DB CA and note that D repeated B because of the edge, B defeated C because of the edge again and C defeated A. So, I get a path here like this where every edge, every vertex is coming once. So, this is what I have to get. So, I have to get a path like this which touches every vertex, exactly want.

Such kind of an edge or such kind of path is called a Hamiltonian Path.

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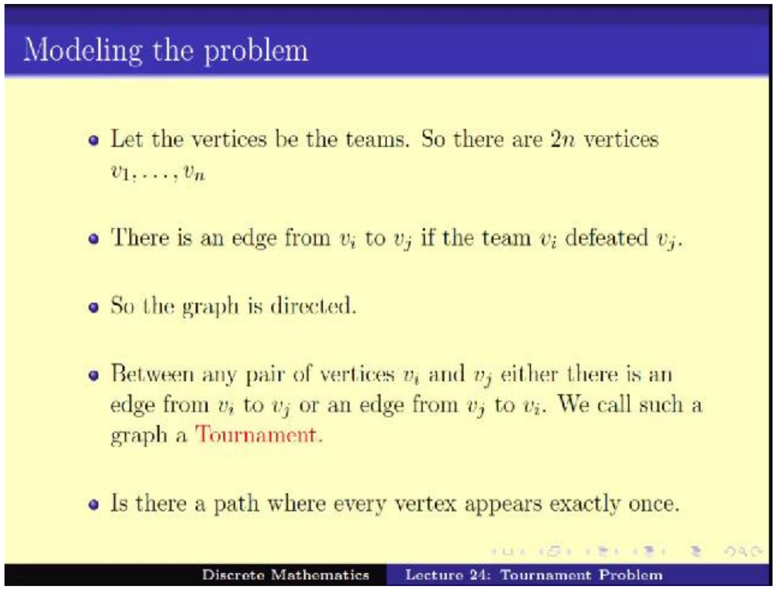

Going back to the problem so our formulation seems to be right in the sense that what we have been asked to do is find a path where every vertex appears to be exactly once.

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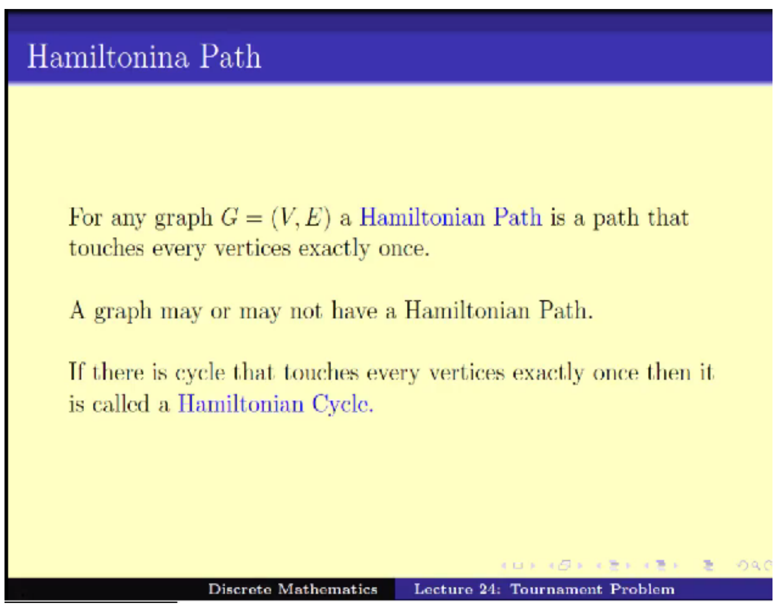

So, such a path is called a Hamiltonian Path. A graph in general may or may not have a Hamiltonian Path. It may or may not have a –and if there is a cycle of that form it touches every vertex exactly you want. We call it as a Hamiltonian Cycle. So, graph may or may not have a Hamiltonian Cycle.

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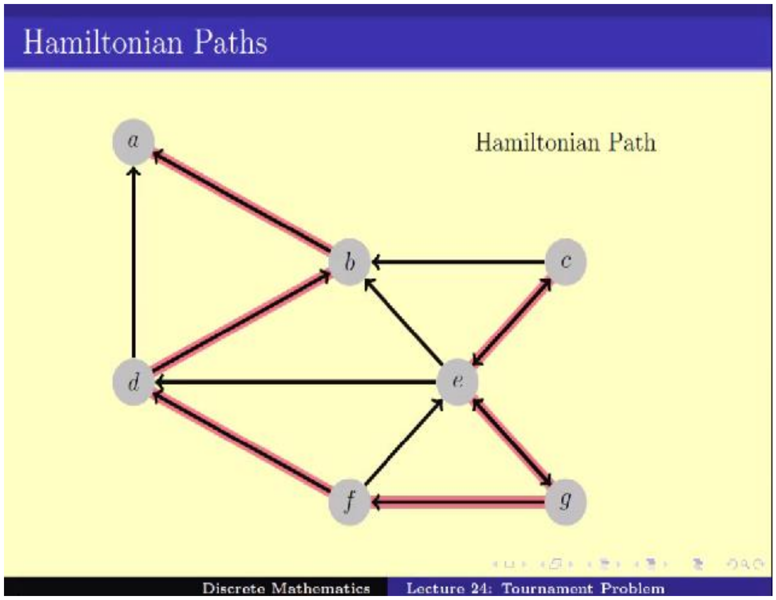

So, let's look at some of the examples. So if this is a graph now let's try to see whether this graph has a Hamiltonian Path. How about this one? Now this is a path it form proper path because I can go from e to g, from g to f, f to d, d to b and b to k. But I cannot extend it to go to c if then you convince yourself that this particular graph does not have a Hamiltonian Path. But if I had to flip this, a into c and made it into a two way path.

Then I could have brought a Hamiltonian Path of this kind. C to e, e to g, g to f, f to d, e to b, b to a. So, this gives us a Hamiltonian Path. Note that this graph does not have a Hamiltonian Cycle. Also note that this graph is not at a tournament because there is no edge between a and c either this way or that way and so on. I do not have an edge between a and d also. So, in a general graph we can have a Hamiltonian Path, need not have a Hamiltonian Path.

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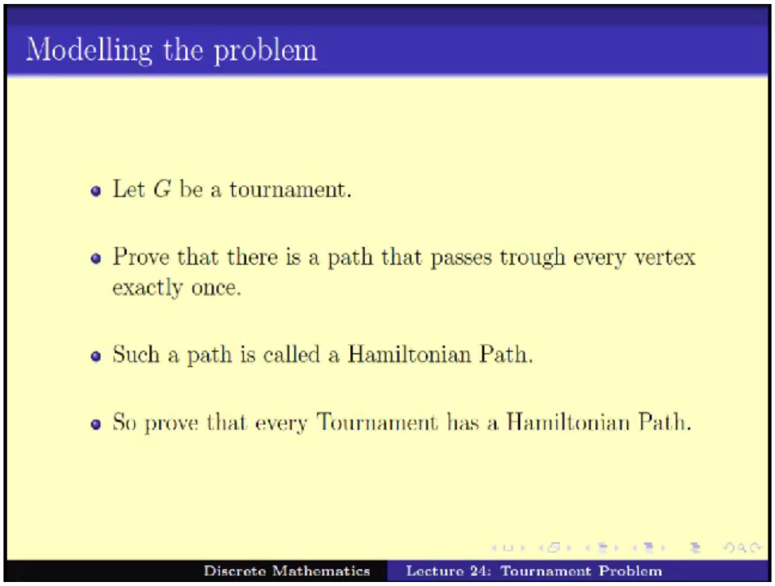

But the problem now says that if G is a tournament that means between any pairs of vertices there is edge going from u to v or v to u proves that there is a Hamiltonian Path. So, you prove that a Tournament has a Hamiltonian Path, and this is the formulation of the problem that we have in terms of graph theory. Now, how to prove it?

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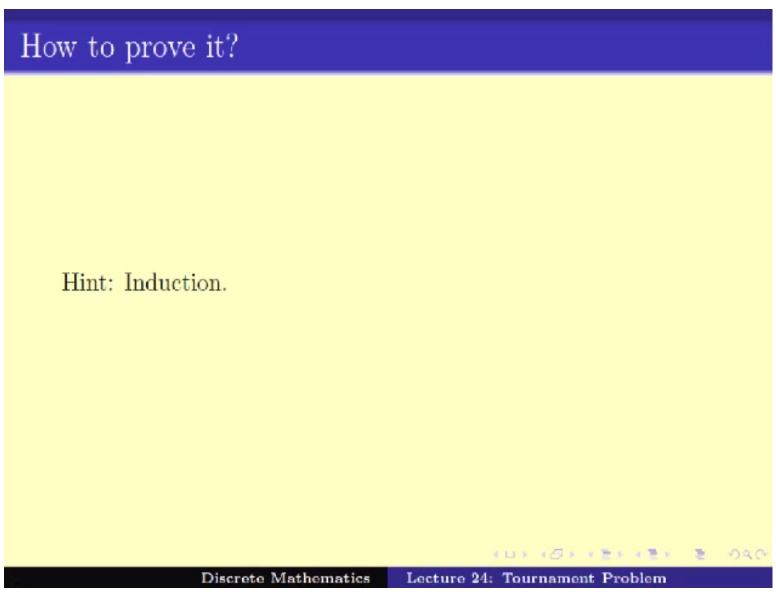

Again the hint is induction. We will come back next video and prove this problem using induction. We have now converted the problem into a proven in

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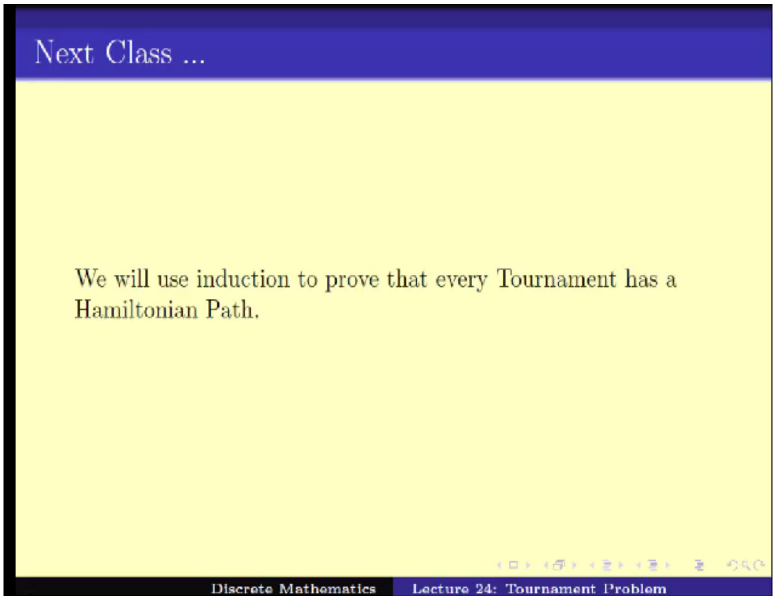

Next class we will use induction to prove that every tournament has a Hamiltonian Path. Thank you.