Discrete Mathematics
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Lecture - 23
Handshake Problem

Welcome back. So, we have been looking at some problems and how to use various tools and techniques to solve the problems and also we started looking at Graph theory. So, we continue with our study of problem using Graph theory.

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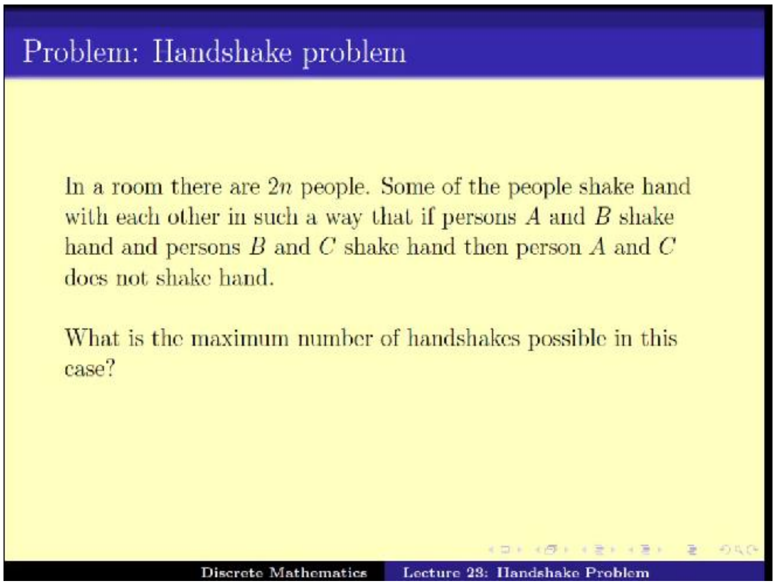

So, in this video we will be focusing on the handshake problem which basically says that if in a room you have 2 n people. Some of them shake hand with each other but the guaranties that if A shake hand with B and B shake hand with C, if A does not shake hand with C. If this is the condition, then what is the maximum number of handshake possible? Now, in last video we show that this particular problem can be modeled using Graph Theory.

So, let me quickly go for the introduction of Graph Theory once again.

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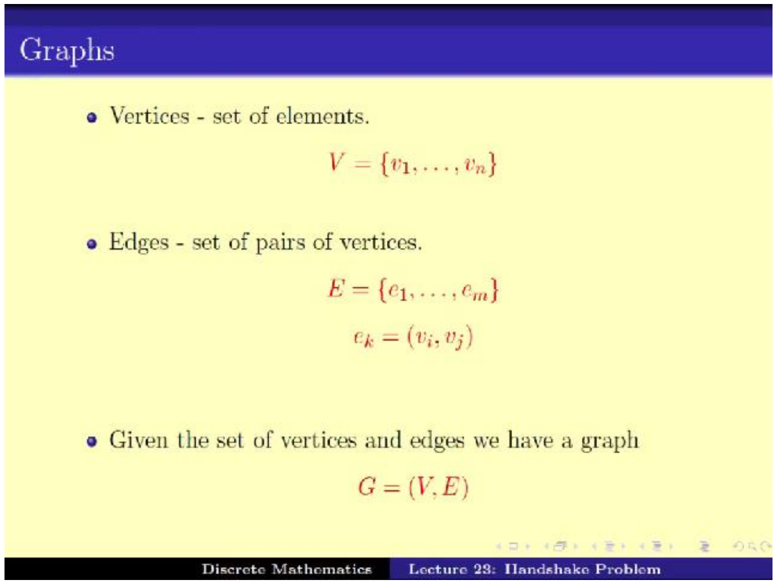

So, we would like to model this problem in using the language of Graph Theory. So, a graph consists of vertices or set of elements v 1 to v n and a set of edges which we call which are typically pairs of vertices. So, edges are e 1 to e m but e k is a pair of the form v i, v j. So, some particular pair of the vertices contributes to the edges. So, the graph is given as G, I as in two set the vertices set and the edge set. We usually did not test G equal to V, E.

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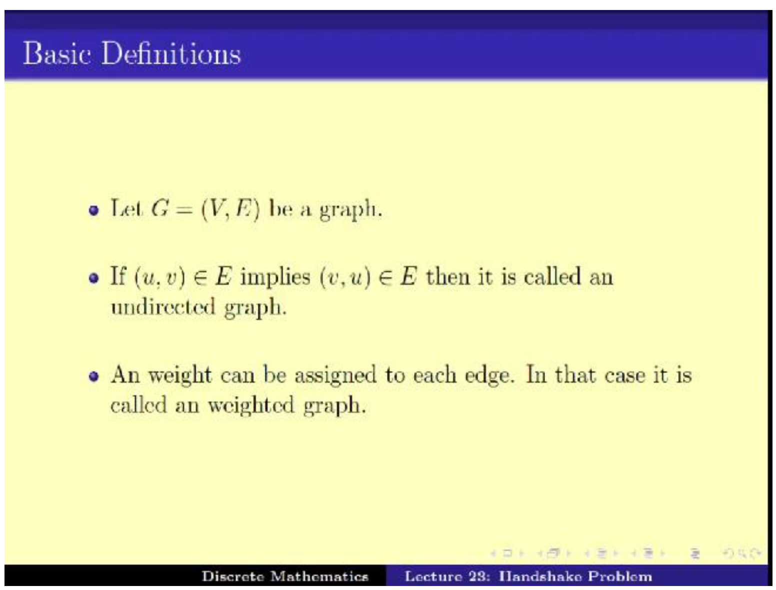

Now, there are some basic definitions. So, if G is a graph and if there is edge between u and p and then it implies that there is an edge between p and u that means the relation show u, v is a reflective relation in that case we call them graph an undirected graph. Sometimes for modeling purposes we will be assigning some weight to the edges and in that case we call them as a weighted graph.

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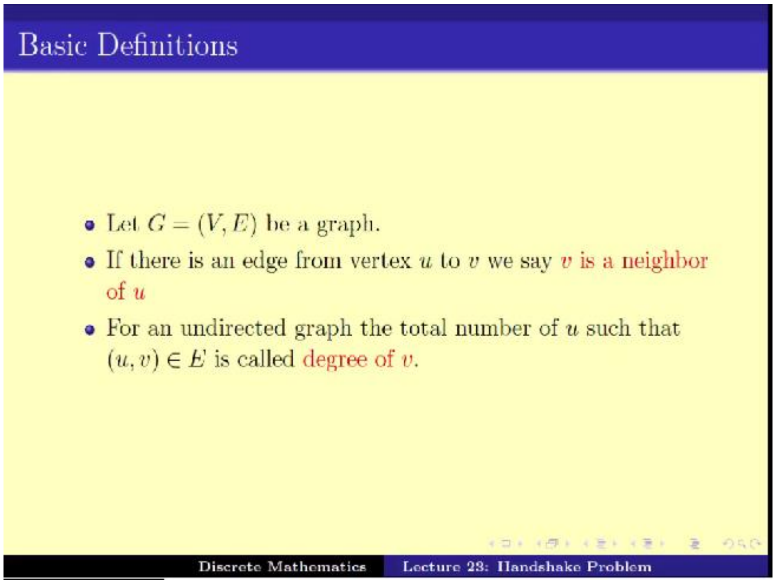

If there is an edge from u to v. We say that v is a neighbor of u and in an undirected graph the number of neighbors that is there of V is called E is called the degree of V. So, this number of edges that both out of v or both you know speaking to V since it is undirected they are same. So, these are the basic definition that we have.

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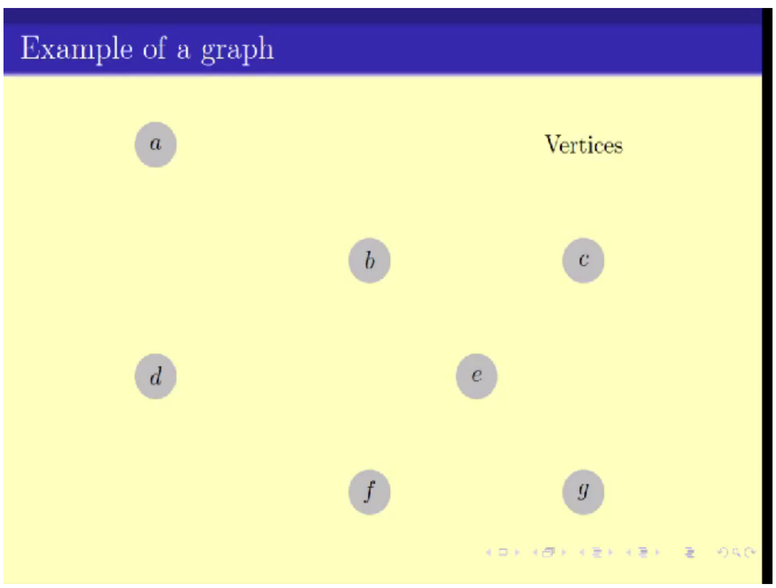

So, visually we usually dig out the graph using this block and line joining the block. So, these blocks are what we call as vertices. So, in this place we have vertex a b c d e f and g and the edges we denote it with the line.

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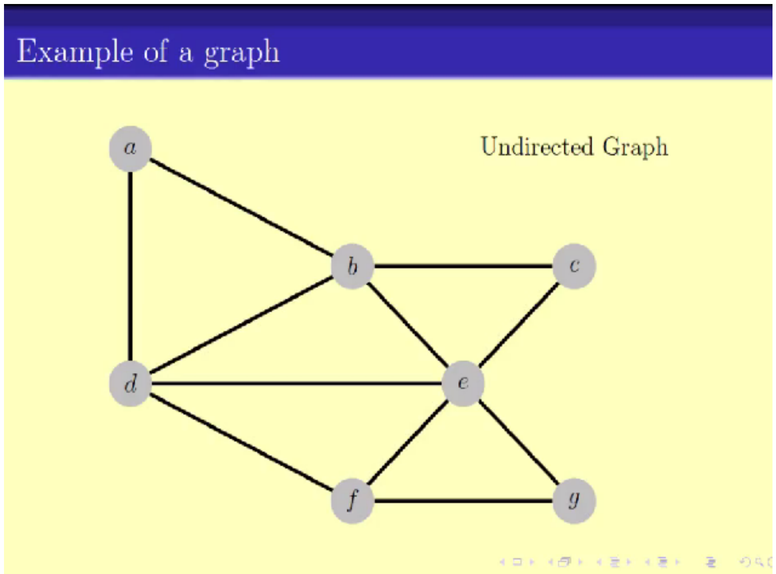

In the sense that if in the edge like this it means that a, d because there is a line here a, d is a pair similarly a, b is a pair in the edge set and so on. Now, this edge, no edge between a and c. So a, c is not in the edge set. So, usually we visualize this whole graph as something like this.

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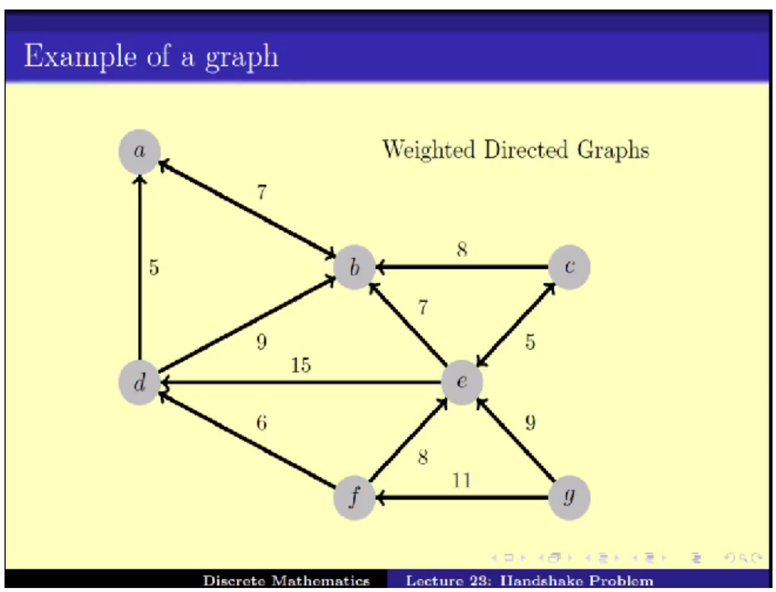

Now, we can have edges, we can put some weights that can be used for modeling of a problem during that case, we do not did like this or we draw it like this as a - and as we told they can - edges can need not be undirected in the sense that I can have direction meaning d, a is there in the edge set does not imply a, d is there. So, how do we denote that in the as a pictorial? So, we denote it by the arrows.

So, the arrow implies that there is an edge from d, d to a or in other worlds d, a is an edge. Similarly, there is an arrow like this which is between d, a and c. We have this arrow in both directions, so this physically means that both a, b and b, a are in the edge set. So, when you have something of this form we call it a Directed graph. Of course, in this direction we have weights on it so we call it as Weighted Directed Graph.

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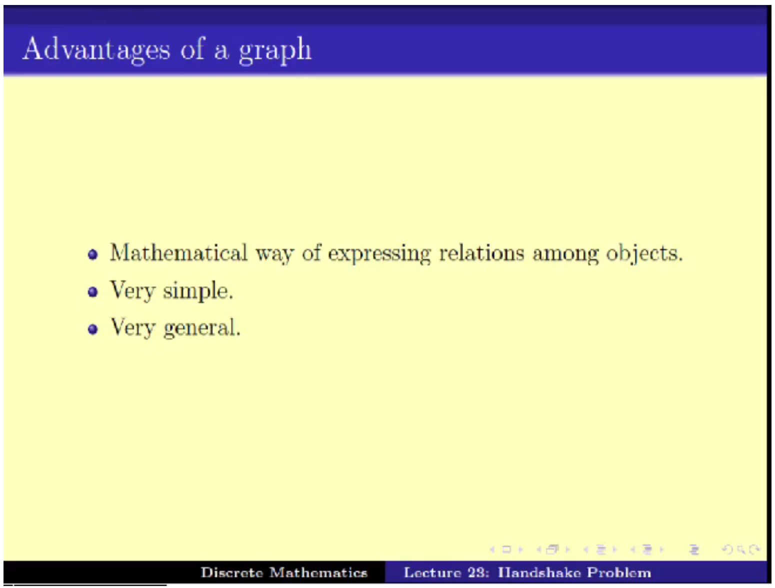

So, the Advantages of graph show that they are very nice or very useful data structure to represent binary relations. They are very simple and yet very general. They can be used in an enormous numbers of ways or called modeling in a number of problems.

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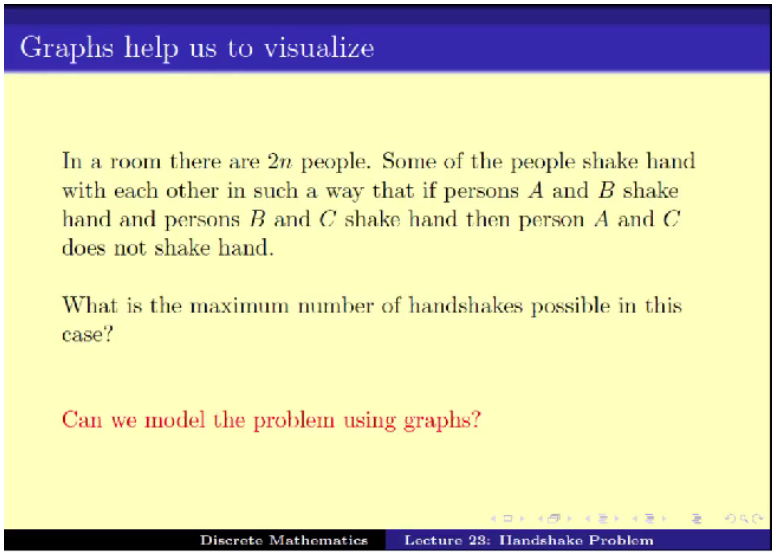

So, let's see how this graph can be used to visualize a particular problem. So, you recall this problem that we are at the handshake problem. So, how do we convert it to a problem or a problem on graph? So, first of all we have to define the graph. So to define the graph we start with

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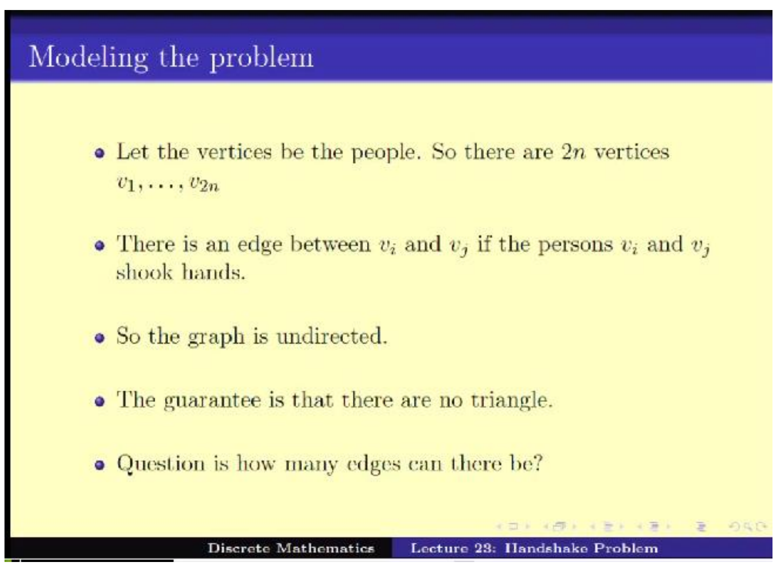

defining the vertices. So, we have 2 n people so for each people we denote a vertex. So, the vertices are the v 1 to v 2 n. Now, the relationship that we have to put or in another words the edges that we have to put should come from the problem. So, in this case we say that we draw an edge between v i and v j if the person v i and v j shook hand. Note that since v i section with v j also implies v j section with v i.

So, this graph is an undirected graph, meaning there is no direction. I do not need to differentiate between v i, v j and v j, v i, they are both same so they are undirected and now in this problem we have the condition that if a and b shakes hand and b and c shake hand. If a and c does not shake hand. What does it mean? So, if you think of it, it's like we if I have vertex, if I have the vertex A, vertex B and vertex C.

If A and B shakes hand and B and C shake hand then this vertex, A to C should not be there. So, in another words we call this kind of a, we call this of a thing at triangle. So, the guarantee should be that the graph does not have a triangle. So, there is no triangle in the graph. So, this is the initial set up and what the question saying? Question is asking how many edges can that be? So, the question is how many handshakes can there be?

Which implies that how many edges can there be? The problem was down to 9th problem on graph theory. A graph on two end vertices such that there does not exist a triangle, proof how many edges can there be?

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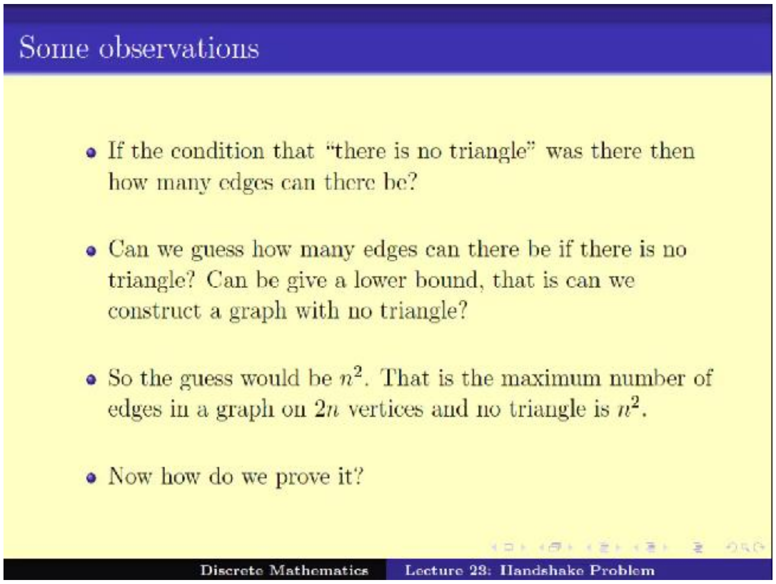

So, if I remove the condition that there is no triangle then how many edges can that be? It is like for any two of them we can have an A. So, it is of the form of n to 2. Now, if I have to guess what is the maximum number that can be? I have to first try out some examples of graph where the number of –there is no triangles but they have a high number of edges. Now, as we looked at last time.

We show that a bipartite graph when I partition the face of vertices into two-part L and R both of them having n vertices each and between them I jointed any two of them. So, we write V as L union R, size of L is equal to size of R equal to n so that is joined. And the edge set is the whole all possible pairs but one is coming from L and one is coming from R. So, observation is that this one does not have a triangle and number of edges is n square.

So, hence we get a graph which triangle free and it is a graph on two end vertices and the number of edges is m square. So, it's kind of guess that this possibly is the maximum number that we can have. And in fact that is what we should be trying to prove. So, we should be getting that if n square is the maximum number and that can be attained also. But question is how do you prove that statement? So this is where we were in the last class. Now, to prove the statement.

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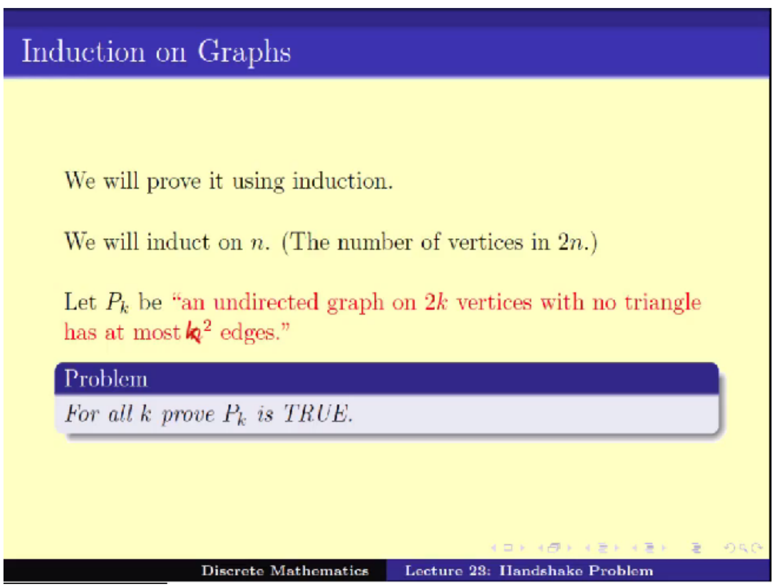

We will be using induction. Suppose that first of all we will be inducting on n, note that there is number of vertices in a graph with 2n and what is the induction, as we have seen in the induction. We have to break up the problem in two cases. So, let's P k which is thing saying that an undirected graph on 2 k vertices has no triangle has ordered so this should –it is a mistake here. This should be k square, has k square edges.

So, this is the P k problem and in that case we want to prove that for all case book that statement k is true. So, back to our induction basics we have to prove it using three steps mainly base case, induction hypothesis and inductive step.

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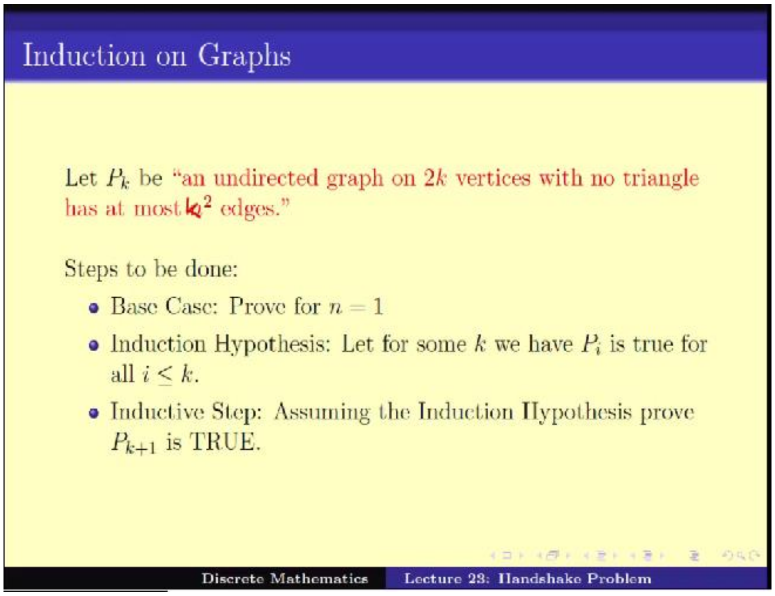

So, again here will be the P k and the step to be done of course the base case we have to prove this base case for n equal to one. We have the induction hypothesis. The induction hypothesis can be of course they can be different kind of induction hypothesis depending on what does the versions of induction that we are going to use. In this case, we will use the induction hypothesis square for some case, we have P i is true for all i less than or equal to k.

So, that means I have P 1 is true, P 2 is true, P 3 is true dot, dot, dot P k is true. Under the assumption that P 1 to P k is true can be proved P k + 1 is true. So, that is the induction version that we will be using P. So, using this induction we will try to solve the induction hypothesis.

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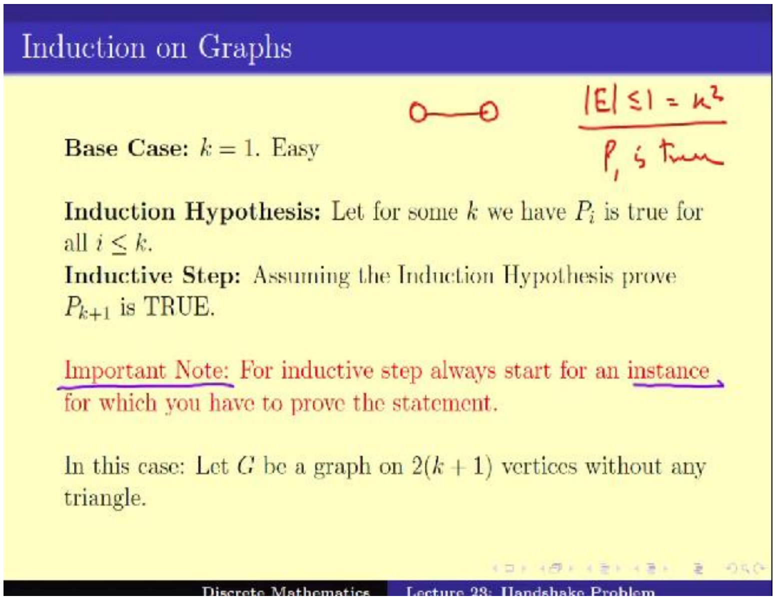

So, start with the base case. Now, can you pull the base case k equals to one. If the base of k equals to one, then in the graph there are two vertices and either they can have one H or zero H. So, number of H is less than or equal to one which is k square. Good that is what we wanted to prove. So, P 1 is true. So, this can be that easy to –not that hard to see that if base case k equal to one.

Now, the induction hypothesis of course that P 1 to P k is true and using this I want to prove that P k plus one is true. So, there is a very important note here whenever you have to prove induction you should always start with an instance of P k plus one what we want to prove. Not try to construct, not start with an instance of P k or something and also P k + 1. This is a statement.

This is a very important note and I will keep on repeating this thing for few times. You do not face this issue when you are solving using induction for number theory problems like you have seen till now. But whenever we apply induction for communitarial objects like graphs this is a very important statement. So, we have to prove P k + 1 so we should start with an instance of P k + 1.

So, in this case we should say that okay let G be a graph on 2 k + 1 vertices without any triangles and now we should reduce, use this graph and then use the induction hypothesis to prove that this graph has less k + one full square vertices.

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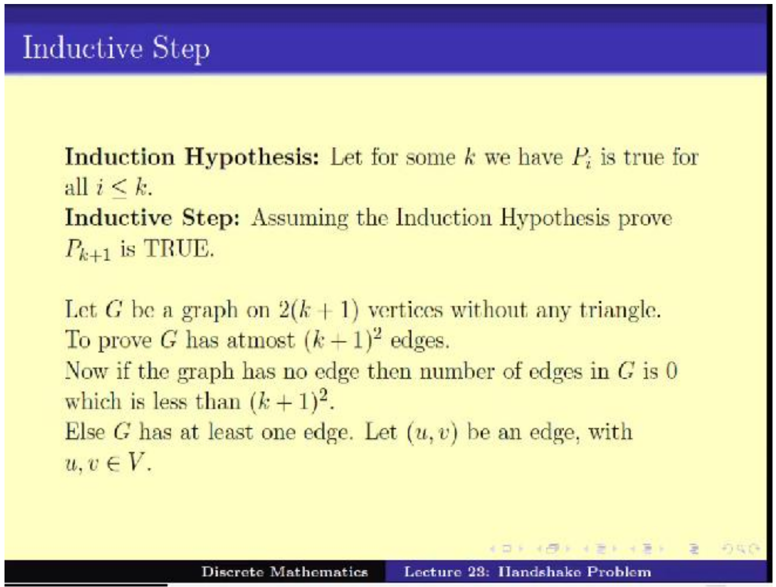

So, for the inductive step we have this induction hypothesis. You want to prove this inductive state. And we start with let a G be a graph on two times k + 1 vertices without any triangle. And we have to prove that in that case G has at most k + 1 whole square edges. Now, we somehow have to convert this G to a smaller new instance so that we can apply the induction hypothesis. So, let's see how we go about it.

So, first of all if the graph has no edge then the number of edges in this graph is zero which is clearly is less then k + 1 whole square. So, we can obviously assume that G has at least one edge. So, let's u, v be an edge in graph.

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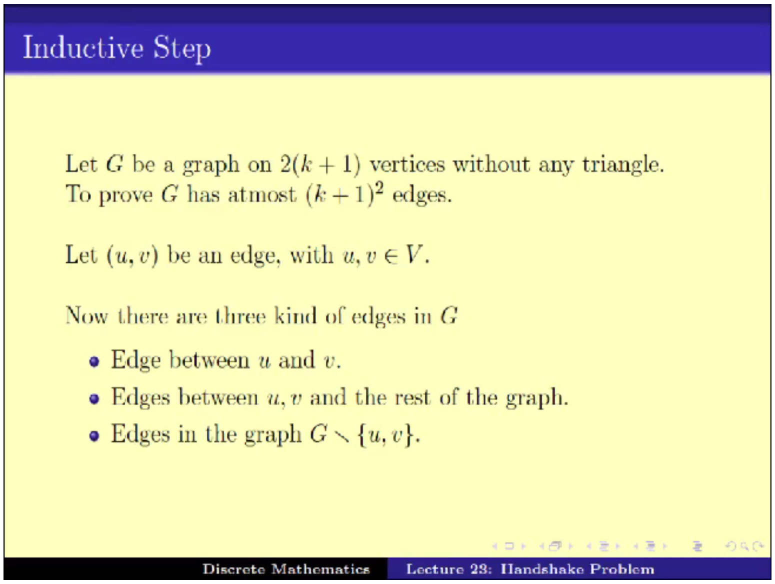

So, we have a graph on 2 k + 1 edges. We want to prove without any triangle, we want to prove that the graph has at most k + 1 whole square edges and we also assume that there is an edge in this graph. Now, there can be three kinds of edges in this graph. Number one, the edge between u and v there is only one such edge of force one of zero as it seems we have assumed that there is edge between u and v that is an edge.

There can be if I have this graph if this is the graph G. This is u, this is v. There can be three kind of edges, number one if this edge. Number two, if we look at all the vertices other than u, u and v there can be edges from u to some edge the vertices in the graph and v is the rest of the vertices. So, edges between u and v and the rest of the graph and the third one if the edges that are within vertices which does not even touch u v. So, the edges in the graph G minus v, u.

So, by this notation I mean that I take the graph we remove the vertices u and v and all the edges connecting to the u and v. So, I get rest of the graph and at that graph. So, any graph can be split up into this way. Edges, between one v, edges between u, v and the rest of the graph and edges in the graph g minus v, u. Now, I will doubt these edges in each of the three categories one by one. So, for the first category is related with c that is just one what about the second category?

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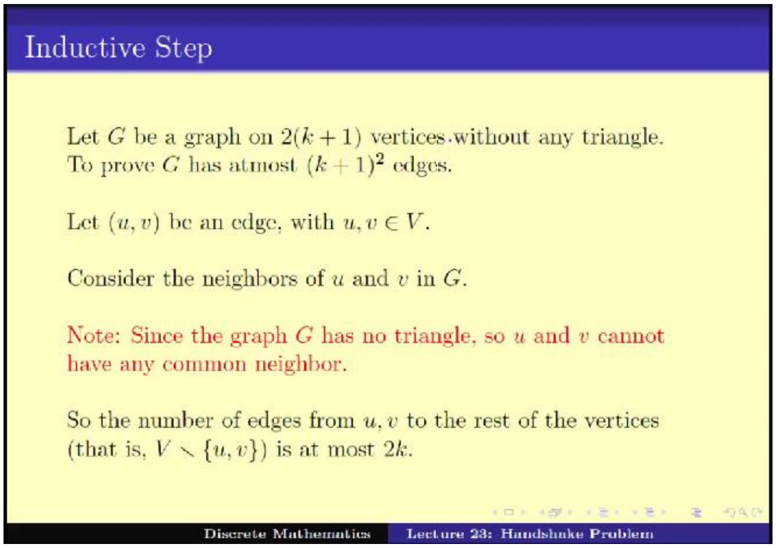

So, here we have assumed there is an edge between u and v, now consider this neighbors of u and v in this graph. So, if this is the graph here and I have taken out two vertices u and v from here and I know there is edge between u and v. Now, I claim that there cannot be any vertex here which has neighbors with both u and v. If it has we have got a triangle but because we have assumed that G does not have any triangle so no neighbor of u, no neighbor vertex and G can have neighbor both u and v.

In other words, u and v cannot have any common neighbors. So, how does it look like therefore so let me just rewrite, redraw with the ink. So, here is the graph, here is u, here is v. u and v there is an a some of the vertices has edges with u. Some of the vertices has edges with v, someone of them might not has not any age at all that we do not know, may or may not have but more importantly every vertex here in t minus u can have an edge with at most u and v.

Number of vertices here is how much? Is two k because initially was two k minus one. I have taken out two so this has k vertices. So, every vertex in e minus v, u can have edge either to u or to v. So, in another words the number of edges from u v to the rest of the graph can actually most v 2 k. So, in the (()) (21:14) that we did the second category that is the number of edges between u v to the rest of the graph can have at most two k edges.

This is where we have used with property that the G is tangential free very crucially.

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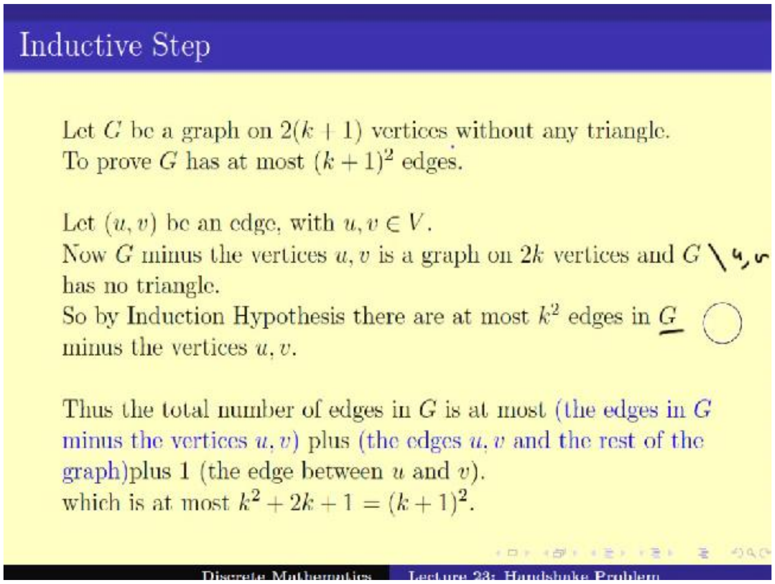

Now, to complete the proof the third step. Here again we have G which was the graph of 2 k + 1 vertices without any triangle. I picked out u v which has an edge now look at G minus the vertex u, v that is a graph on 2 k vertices and since G did not have a triangle, it's new G it should be G minus u v also does not have a triangle. With the triangle cannot be produce by removing two vertices.

So, the new graph which is g minus u, v is a graph on 2 k vertices without a triangle so by induction hypotheses there can be at most at most k square graph edges in the graph e minus the vertices u, v. This is where we crucially use the induction hypothesis. I take to start with G remove to vertices to get a smaller graph. In the smaller graph I see that, that smaller graph has same property.

In the other sense that the smaller graph is the graph on smaller number of vertices and it has no triangle. So, we can then apply induction hypothesis to claim that G minus v, u does not have more than k square edges. So, finally the total number of edges in G is at most the edges in G minus the vertices v u plus edges between u v and the rest of the graph plus one which is the edge between u and v. Let us put them back.

So, this is k square, edges between G minus this one this we just now proved with k square. The edges with u v and the rest of the graph is the earliest like we proved it is 2 k so we have different at most k square plus 2 k + 1 which is k + 1 whole square and hence we have proved what we wanted to prove. That the graph G —so wrapping up what we have.

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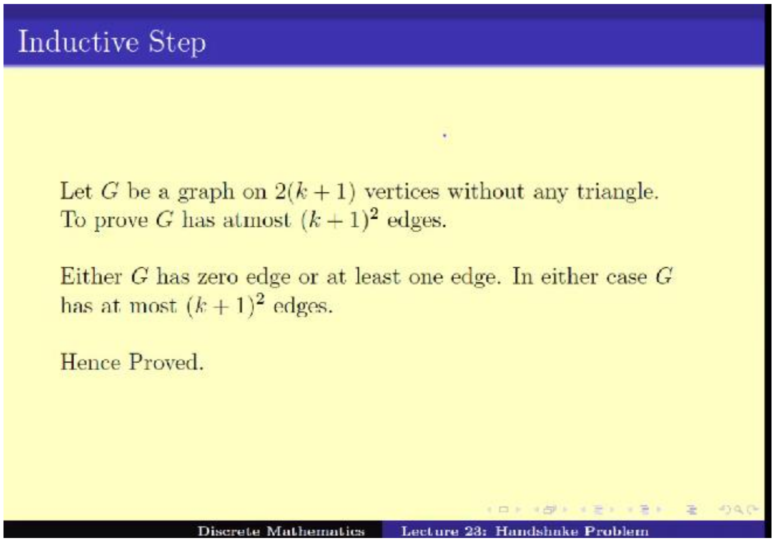

So we started with the graph on two times k + 1 vertices without any triangle, we wanted to prove that the graph has at most k + 1 square edges. Either the graph has zero edges or at least one edge in either case we prove the number of vertices in graph is –number of edges in graph left both k + 1 whole square hence proved.

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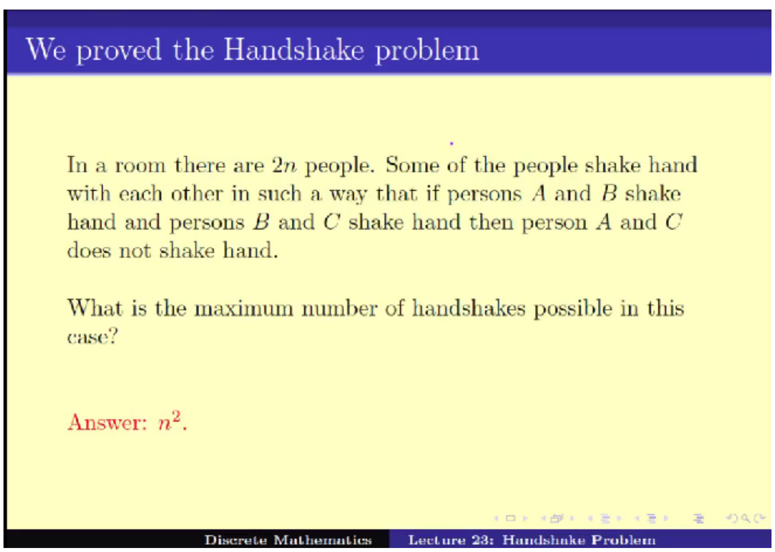

So, we have used first of all graph theory to represent this problem so that we can visually work on it, also we have used induction of graph to solve this problem then. And it proves that a graph on two end vertices without a triangle has at most n square edges which is same as the hand shake problem.

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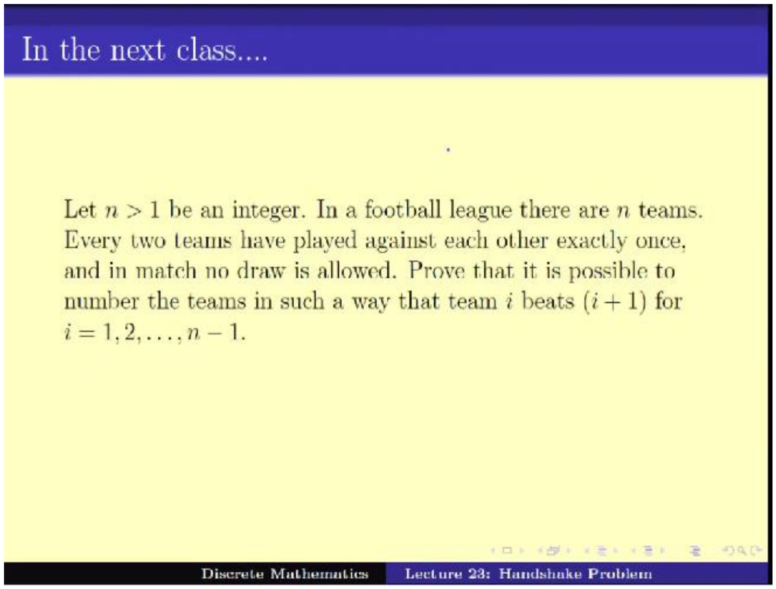

In the next video, we will be looking at other problem that we have talked about which was the problem on tournament. Thank you.